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Actuarial valuation under interest rate uncertainty via A-linearly correlated fuzzy processes

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Abstract. Interest rate assumptions play a fundamental role in actuarial valuation, since even small variations may significantly affect premiums and benefits. In many practical situations, however, interest rate uncertainty is epistemic rather than stochastic, arising from imprecision in expert judgment and incomplete market information. To address this issue, this paper applies the theory of A-linearly correlated fuzzy processes (fuzzy processes in which the dependence structure is governed by a linear correlation-type operator A acting on fuzzy-valued functions) to the valuation of life insurance contracts. The contracted benefit is modeled as a fuzzy number, leading to a fuzzy present value represented by a fuzzy initial value problem with an explicit analytical solution. The A-linearly correlated fuzzy process is formally defined and employed to characterize the dependence structure underlying the fuzzy present value. By interpreting this value as a fuzzy function of the insured's lifetime, a closed-form expression for the fuzzy actuarial present value is derived through α-level sets. It is shown that the resulting actuarial present value is a triangular fuzzy number whose bounds coincide with the classical actuarial present values obtained under extreme deterministic interest rate scenarios. The proposed framework provides a transparent and computationally efficient way to quantify the impact of interest rate imprecision on premiums and benefits, complementing traditional actuarial models without requiring probabilistic assumptions.

Keywords: Life insurance benefits, insurance market, fuzzy sets, A-linearly correlated fuzzy processes
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1 Introduction

The pricing of financial and actuarial products, such as insurance policies and pension plans, plays a central role in risk management and in ensuring the sustainability of long-term contracts. To achieve

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accurate valuations, actuaries rely on financial and biometric assumptions, among which the interest rate is one of the most influential. Even small variations in this rate can lead to significant changes in premiums, benefits, and technical reserves, thereby affecting pricing decisions and long-term solvency. Consequently, the interest rate is a key component in the calculation of the actuarial present value, which serves as the foundation for premium determination and reserve valuation.

In this context, fuzzy set theory has emerged as a promising framework for addressing uncertainty in the pricing of financial and actuarial products [5,6], enabling actuarial assumptions to be represented in a more flexible and realistic manner. In particular, fuzzy differential equations have proven to be effective tools for modeling the variability inherent in dynamic systems affected by imprecise information [10]. Several approaches have been proposed to address differential and integral equations under uncertainty [2–4, 19, 20]. Nevertheless, there remains a significant need for studies that explicitly incorporate dependency structures among fuzzy variables arising from initial conditions or model parameters [1,9,14,21,24,26,27]. Such dependencies, characterized through joint possibility distributions [11], naturally lead to the concept of A-linearly correlated fuzzy processes, which parallels the notion of autocorrelation widely used in time-series analysis.

Recent advances in uncertainty theory have also found important applications in financial and energy markets. For example, uncertain energy models have been developed for electricity and natural gas futures, with applications to spark-spread option pricing [17]. Multi-factor uncertain processes have been employed to model electricity spot prices, including empirical studies of the Nordic market [18]. Moreover, uncertain fractional differential equations have been used for parameter estimation and the valuation of multi-asset options, providing a flexible framework for capturing financial uncertainty [28]. These developments highlight the growing relevance of uncertainty-based methodologies in financial modeling and further motivate the actuarial framework proposed in this paper.

The main contribution of this study is the systematic application of the theory of A-linearly correlated fuzzy processes to a relevant actuarial problem: the valuation of life insurance benefits under interest rate uncertainty. Rather than proposing a purely conceptual framework, we derive explicit analytical expressions for fuzzy actuarial present values that can be computed through α-level sets. These representations make it possible to quantify the impact of interest rate imprecision on premiums and benefits in a transparent and operational manner, rather than merely describing it qualitatively.

Although stochastic and classical actuarial approaches for modeling interest rate uncertainty are well established, the objective of this work is not to replace them, but rather to complement them by addressing a different form of uncertainty frequently encountered in actuarial practice. In many real-world applications, the interest rate used in the pricing of insurance and pension products is not generated by a statistically calibrated stochastic process. Instead, it is often selected within a prescribed range based on regulatory requirements, market conventions, or expert judgment. In such cases, uncertainty is predominantly epistemic, reflecting imprecision arising from limited information rather than randomness supported by sufficient historical data.

The decision to treat interest rate uncertainty as a predominantly epistemic phenomenon, rather than a purely stochastic one, is grounded in actuarial practice and contemporary regulatory frameworks. Over the long horizons typical of life insurance and pension contracts, the calibration of stochastic models based solely on historical data may fail to adequately capture structural changes and regulatory uncertainties. In practice, regulatory authorities (such as SUSEP in Brazil) and international standards (such as IFRS 17 and Solvency II) often establish technical interest rates and risk margins based on prudential criteria and expert judgment. In these settings, uncertainty does not arise from directly measurable random fluctuations but rather from the inherent imprecision involved in selecting long-term parameters. Consequently, fuzzy processes provide a valuable complementary tool for formally modeling this regulatory subjectivity and for generating valuation bands that more faithfully reflect the degree of prudence required for actuarial solvency.

Within this framework, fuzzy set theory offers a mathematically consistent approach for modeling interest rate uncertainty without imposing potentially unjustified probabilistic assumptions. It is important to emphasize that the fuzzy valuation bands obtained in this model should not be interpreted as probabilistic confidence intervals. Whereas stochastic interest rate models quantify uncertainty through probability distributions calibrated from historical data, the proposed fuzzy framework represents uncertainty through membership functions and possibility levels. Consequently, the proposed approach is particularly suitable for situations in which interest rate uncertainty stems from expert judgment, regulatory prudence, or incomplete information rather than from statistically observable randomness.

The use of A-linearly correlated fuzzy processes allows uncertainty to be incorporated directly into the valuation framework while preserving analytical tractability. In particular, by working in the space $\mathbb{R}_{\mathscr{F}(A)}$, the fuzzy valuation problem can be reduced to computations in $\mathbb{R}^2$, yielding explicit closed-form expressions and providing a structured representation of dependencies among uncertain quantities.

An important feature of the proposed methodology is that the resulting fuzzy actuarial present value explicitly encompasses, through its α-level sets, the classical actuarial present values obtained under different deterministic interest rate scenarios. This provides a natural benchmark with traditional actuarial models, demonstrating that the fuzzy formulation does not replace established methods but rather integrates their outcomes into a unified framework. As a result, the well-known sensitivity of premiums and benefits to interest rate variations is not merely acknowledged but explicitly quantified.

Although this paper does not aim to perform empirical calibration or validation using real-world data, it provides rigorous theoretical results and numerical illustrations based on standard actuarial assumptions. These results demonstrate how interest rate imprecision propagates through benefit and premium calculations, producing actionable information in the form of interval-valued estimates and associated membership levels. Consequently, the contribution of this work is both methodological and operational, providing a concrete advancement in actuarial valuation under epistemic uncertainty.

The remainder of this paper is organized as follows. Section 2 reviews the fundamental concepts of fuzzy set theory, with particular emphasis on A-linearly correlated fuzzy processes. Section 3 introduces the actuarial present value and its calculation. Section 4 presents the fuzzy modeling of life insurance benefits and discusses the implications of the results obtained. Finally, Section 5 concludes the paper.

2 Preliminaries

This section provides a brief review of the fundamental concepts of fuzzy set theory that are required for the developments presented in this paper.

Let $X \neq \emptyset$ be a universe of discourse. A fuzzy subset A of X is defined by a membership function $A : X \longrightarrow [0,1]$ where $A(x)$ represents the degree of membership of an element $x \in X$ in the fuzzy set A [7].

Associated with any fuzzy subset A of X is a family of classical subsets of X, called the α-level sets (or α-cuts) of A. For each $\alpha \in [0,1]$, the α-level of A is defined by

$$[A]_\alpha = \begin{cases} \{x \in X\,;\, A(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ \overline{\{x \in X\,;\, A(x) > 0\}}, & \text{if } \alpha = 0, \end{cases} \tag{1}$$

where $\{x \in X\,;\, A(x) > 0\}$ is the support of A, denoted by $\mathrm{supp}(A)$, and $\overline{(\cdot)}$ denotes the closure of a subset of X [7].

A fuzzy subset A of $\mathbb{R}$ is called a fuzzy number if, for every $\alpha \in [0,1]$, its α-level set is a nonempty, bounded, and closed interval of $\mathbb{R}$. In this case, the α-level set of A can be written as $[A]_\alpha = \big[a_-(\alpha), a_+(\alpha)\big] = \big[a^-_\alpha, a^+_\alpha\big]$, for all $\alpha \in [0,1]$, see [7]. The set of all fuzzy numbers is denoted by $\mathbb{R}_{\mathscr{F}}$. A commonly used example is the triangular fuzzy number. Its α-level sets are given by $[A]_\alpha = \big[(m - a^-_0)\alpha + a^-_0, (m - a^+_0)\alpha + a^+_0\big]$ for all $\alpha \in [0,1]$ where $[A]_0 = [a^-_0, a^+_0]$ and $[A]_1 = \{m\}$. Such a fuzzy number is usually denoted by the triple $(a^-_0; m; a^+_0)$. The diameter of a fuzzy number A is defined by $\mathrm{diam}(A) = a^+_0 - a^-_0$. For triangular fuzzy numbers, this quantity provides a natural measure of the uncertainty or fuzziness associated with A. A fuzzy number $A \in \mathbb{R}_{\mathscr{F}}$ is said to be symmetric with respect to a point $x \in \mathbb{R}$ if $A(x - \zeta) = A(x + \zeta)$, for all $\zeta \in \mathbb{R}$. If no such point exists, then A is called non-symmetric [13].

2.1 A-Linearly correlated fuzzy space ($\mathbb{R}_{\mathscr{F}(A)}$)

In [13], an operator $\psi_A : \mathbb{R}^2 \to \mathbb{R}_{\mathscr{F}}$ was proposed, which associates to each pair $(u,v) \in \mathbb{R}^2$ a fuzzy number whose α-level sets are defined by

$$[\psi_A(u,v)]_\alpha = u + v[A]_\alpha, \tag{2}$$

for every $\alpha \in [0,1]$. The image of ψ_A, denoted by $\mathbb{R}_{\mathscr{F}(A)}$, consists of all fuzzy numbers that are linearly generated by a fixed fuzzy number A, that is

$$\mathbb{R}_{\mathscr{F}(A)} = \big\{\psi_A(u,v) = u + vA\,;\, (u,v) \in \mathbb{R}^2\big\}$$

and is referred to as the space of A-linearly correlated fuzzy numbers [13].

Proposition 2.1. ([13]) Let $A \in \mathbb{R}_{\mathscr{F}}$. The mapping $\psi_A : \mathbb{R}^2 \to \mathbb{R}_{\mathscr{F}}$, defined by $\psi_A(u,v) = u + vA$, is injective if and only if the fuzzy number A is non-symmetric.

Throughout the sequel, unless explicitly stated otherwise, we assume that the fuzzy number A is non-symmetric.

Corollary 2.2. ([13]) Define the operations

(i) $B +_{\psi_A} C = \psi_A\big(\psi_A^{-1}(B) + \psi_A^{-1}(C)\big) = (u_B + u_C) + (v_B + v_C)A$,

(ii) $\gamma \cdot_{\psi_A} B = \psi_A\big(\gamma \psi_A^{-1}(B)\big) = \gamma u_B + (\gamma v_B)A$,

for all $B = u_B + v_B A$, $C = u_C + v_C A$ in $\mathbb{R}_{\mathscr{F}(A)}$ and $\gamma \in \mathbb{R}$. Then the triple $(\mathbb{R}_{\mathscr{F}(A)}, +_{\psi_A}, \cdot_{\psi_A})$ is a real vector space of dimension two.

Moreover, equipped with the norm

$$\|B\|_{\psi_A} = \|\psi_A^{-1}(B)\|_\infty, \qquad B \in \mathbb{R}_{\mathscr{F}(A)},$$

this space is complete, and therefore constitutes a Banach space.

It should be emphasized that, for $u, v \in \mathbb{R}$, the relation

$$u +_{\Psi_A} v \cdot_{\Psi_A} A = u + vA$$

holds, where $+$ and $\cdot$ denote the usual addition and scalar multiplication of fuzzy numbers. Furthermore, the norm $\|\cdot\|_{\Psi_A}$ induces a metric d_{Ψ_A} on $\mathbb{R}_{\mathscr{F}(A)}$ given by

$$d_{\Psi_A}(B,C) = \|B -_{\Psi_A} C\|_{\Psi_A}.$$

Definition 1. *([13]) Let $A \in \mathbb{R}_{\mathscr{F}}$ be fixed and define*

$$\mathscr{LCF}_A([a,b]) = \left\{\omega : [a,b] \to \mathbb{R}_{\mathscr{F}(A)}\ ;\ \omega(t) = u(t) + v(t)A\right\},$$

where $u, v : [a,b] \to \mathbb{R}$. Any function $\omega \in \mathscr{LCF}_A([a,b])$ is called an A-linearly correlated fuzzy process *on $[a,b]$.*

Remark 1. *The assumption that the fuzzy number A is non-symmetric is adopted to ensure the injectivity of the mapping ψ_A and to allow $\mathbb{R}_{\mathscr{F}(A)}$ to be identified with $\mathbb{R}^2$. Symmetric fuzzy numbers may naturally arise when uncertainty around a central actuarial estimate is balanced, for example when benefits or premiums vary symmetrically around a best-estimate value. However, in many actuarial applications uncertainty is inherently asymmetric, reflecting regulatory constraints, conservative pricing policies, or different risk perceptions for upward and downward deviations. If A were symmetric, the mapping ψ_A would no longer be injective, implying that distinct pairs (u,v) could generate the same fuzzy number. In this case, the representation of fuzzy processes in terms of real-valued components would not be unique, and the reduction of fuzzy differential equations to classical systems in $\mathbb{R}^2$ would no longer hold. Consequently, the construction of derivatives, integrals, and the analytical solution of the actuarial valuation problem would require modifications. Therefore, the non-symmetry assumption is mainly technical and ensures analytical tractability without significantly restricting practical actuarial applications.*

Corollary 2.3. ([13]) Let $\omega \in \mathscr{LCF}_A(\mathbb{R})$. Then ω is Fréchet differentiable at a point $t \in \mathbb{R}$ if and only if the composition $\Psi_A^{-1} \circ \omega$ is Fréchet differentiable at t.

The following result shows that for A-linearly correlated fuzzy processes Fréchet derivative is well defined.

Theorem 2.4. ([13]) Let $\omega = \Psi_A \circ p : [a,b] \to \mathbb{R}_{\mathscr{F}(A)}$ such that $\omega(t) = \Psi_A(u(t), v(t))$ for all $t \in \mathbb{R}$. The function ω is Fréchet differentiable at $t \in \mathbb{R}$ if and only if $p'(t) = (u'(t), v'(t))$ exists. In addition, $\omega'(t)(h) = \Psi_A(u'(t)h, v'(t)h)$ for all $h \in \mathbb{R}$. Furthermore,

$$\omega'(t) = \Psi_A(u'(t), v'(t)) = u'(t) + v'(t)A.$$

The previous result provides a straightforward procedure for constructing such fuzzy functions. In particular, it avoids the need for case-by-case analysis or concerns regarding the occurrence of switching points.

Definition 2.5. ([12, 27]) Let $\omega \in \mathscr{LCF}_A([a,b])$. The function ω is said to be Ψ-*differentiable* at a point $t_0 \in [a,b]$ if the limit

$$\lim_{h \to 0} \frac{1}{h}\left(\omega(t_0 + h) -_{\Psi_A} \omega(t_0)\right)$$

exists in $\mathbb{R}_{\mathscr{F}(A)}$ and coincides with a fuzzy number $\omega'_\Psi(t_0) \in \mathbb{R}_{\mathscr{F}(A)}$. Moreover, ω is said to be Ψ-differentiable on $[a,b]$ if $\omega'_\Psi(t)$ exists for every $t \in [a,b]$.

Theorem 2.6. ([12, 27]) Let $\omega \in \mathscr{LCF}_A([a,b])$. The function ω is Ψ-differentiable at $t_0 \in [a,b]$ if, and only if, $u, v : [a,b] \to \mathbb{R}$ are differentiable at t_0. Moreover,

$$\omega'_\Psi(t_0) = \Psi_A\left(u'(t_0), v'(t_0)\right) = u'(t_0) + v'(t_0)A.$$

Corollary 2.7. ([12, 13]) Let $\omega \in \mathscr{LCF}_A([a,b])$. The function ω is Ψ-differentiable at $t_0 \in [a,b]$ if and only if ω is Fréchet differentiable at $t_0 \in [a,b]$. Additionally, Ψ-derivative coincides with the Fréchet derivative of ω.

From now on we will no longer differentiate the notations of the Fréchet derivative and the Ψ-derivative, since, by Corollary 2.7, they coincide.

Definition 2.8. ([12]) Let $\omega \in \mathscr{LCF}_A([a,b])$. We say that ω is Ψ-*differentiable of order* $n \geq 2$ at a point $t \in [a,b]$ if and only if the classical derivatives $u^{(n)}(t)$ and $v^{(n)}(t)$ exist. In this case, the nth Ψ-derivative of ω at t is defined by

$$\omega^{(n)}(t) = u^{(n)}(t) + v^{(n)}(t)A.$$

Furthermore, ω is said to be Ψ-differentiable of order n on $[a,b]$ whenever $\omega^{(n)}(t)$ exists for all $t \in [a,b]$.

Since $\mathbb{R}_{\mathscr{F}(A)}$ forms a Banach space when A is non-symmetric, one can define integral for A-linearly interactive fuzzy process by means of the definition of Riemann integral on Banach spaces [25].

Definition 2.9 (Ψ-Riemann integral)**.** ([25]) Let $\omega \in \mathscr{LCF}_A([a,b])$. We say that ω is *fuzzy interactive Riemann integrable* (or Ψ-Riemann integrable, for short) if there exists a fuzzy number $S \in \mathbb{R}_{\mathscr{F}(A)}$ such that, for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying: for any partition

$$a = t_0 < t_1 < \cdots < t_n = b \quad \text{with} \quad t_i - t_{i-1} < \delta, \quad i = 1, \ldots, n,$$

and any choice of points $\xi_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$, the following holds:

$$\left\| (\Psi_A) \sum_{i=1}^{n} \omega(\xi_i)(t_i - t_{i-1}) -_{\Psi_A} S \right\|_{\Psi_A} < \varepsilon, \tag{3}$$

where the sum is taken with respect to the addition $+_{\Psi_A}$. In this case, S is called the *fuzzy interactive Riemann integral* of ω over $[a,b]$ and is denoted by

$$(\Psi) \int_a^b \omega(t)\, dt.$$

Theorem 2.10. ([25]) Let $\omega \in \mathscr{LCF}_A([a,b])$. We say that ω is Ψ-Riemann integrable, if, and only if, the real-valued functions $q, r : [a,b] \to \mathbb{R}$ are Riemann integrable. Moreover

$$\begin{aligned} (\Psi) \int_a^b \omega(t)dt &= \Psi_A\left(\int_a^b u(t)dt, \int_a^b v(t)dt \right) \\ &= \left(\int_a^b u(t)dt \right) + \left(\int_a^b v(t)dt \right) A. \end{aligned} \tag{4}$$

The results presented next play a fundamental role, since they provide the basis for defining the fuzzy improper Riemann integral of a fuzzy-valued function ω through the improper Riemann integrals of its associated real-valued components u and v.

Definition 2.11. ([25]) Let $\omega \in \mathscr{LCF}_A([a,b])$. We say that ω is said to be improperly Ψ-Riemann integrable on the interval $[a,+\infty)$ if, for every $b > a$, the function ω is Ψ-Riemann integrable on $[a,b)$ and, in addition, the limit

$$\lim_{b\to+\infty} (\Psi) \int_a^b \omega(t)\,dt$$

exists with respect to the norm $\|\cdot\|_{\Psi_A}$. In this situation, the improper Ψ-Riemann integral of ω over $[a,+\infty)$ is defined by

$$(\Psi) \int_a^{+\infty} \omega(t)\,dt := \lim_{b\to+\infty} (\Psi) \int_a^b \omega(t)\,dt. \tag{5}$$

When this limit exists, the improper interactive integral is said to be convergent; otherwise, it is called divergent.

Theorem 2.12. ([25]) Let $\omega \in \mathscr{LCF}_A([a,b])$. Then ω is Ψ-Riemann integrable on every compact subinterval of $[a,+\infty)$ if and only if the real-valued functions u and v are Riemann integrable on every compact subinterval of $[a,+\infty)$. Furthermore, whenever the improper integrals exist, one has

$$(\Psi) \int_a^{+\infty} \omega(t)\,dt = \int_a^{+\infty} u(t)\,dt + \left(\int_a^{+\infty} v(t)\,dt\right) A.$$

Remark 2. *Working in $\mathbb{R}_{\mathscr{F}(A)}$ transforms the calculus of fuzzy functions into ordinary calculus on $\mathbb{R}^2$, preserving the structure of uncertainty while ensuring analytical simplicity. This is the key reason for adopting this framework in the modeling that follows.*

3 Interest and financial update

Interest can be understood as the cost of money over time. Several forms of interest exist, depending on the context, among which compound interest is particularly important. Under compound interest, interest is calculated not only on the principal amount but also on the interest accumulated in previous periods. When the interest rate varies continuously over time, capital growth occurs at every instant. Thus, if $C(t)$ denotes the capital at time t, then

$$C'(t) = \delta(t)C(t), \tag{6}$$

where $\delta(t)$ denotes the instantaneous capitalization rate at time t.

The solution of (6) is given by

$$C(t) = C(s)e^{\int_s^t \delta(y)dy}, \tag{7}$$

where s and t are two instants of time such that $s < t$.

In most applications of this capitalization system in the financial market, the constant interest rate is considered, that is, $\delta(t) = \delta$. Then, we have

$$C(t) = C(s)e^{\delta(t-s)}, \tag{8}$$

where $C(t)$ corresponds to the future value of capital in t as a function of capital in s.

3.1 Present value and expected present value

In many financial applications, the objective is to determine the present value at time $s = 0$ of a given future amount a, assuming that a constant interest rate δ has been specified. That is,

$$Z(t) = ae^{-\delta t}, \tag{9}$$

where $Z(0) = a$. That is, the current value of an amount of money that is expected to be available at a given future time t, taking into account that money earns interest over time. Note that (9) is solution of the initial value problem

$$\begin{cases} Z'(t) &= -\delta Z(t), \\ Z(0) &= a. \end{cases} \tag{10}$$

The concept of present value is fundamental to the evaluation of financial projects and many other financial applications. The expected present value, also known as the actuarial present value (APV), is a key tool in financial decision-making and plays a central role in risk management and the pricing of actuarial products, such as life insurance policies and pension plans. It incorporates both the time value of money and the uncertainty associated with future cash flows by discounting future payments according to the probability of their occurrence.

Formally, the APV is defined as the expected value of expression (9), where t is regarded as a random variable representing the time of occurrence of a future event that triggers the payment of an amount a. Let T denote the random variable associated with the payment time of the amount a, such as the time of death of an insured individual. Consequently, $Z(T)$ is a random variable, and therefore

$$Z(t) = ae^{-\delta t}, \quad T = t. \tag{11}$$

Consequently

$$APV = a\int_0^{\infty} e^{-\delta t} f(t)dt, \tag{12}$$

where $f(t)$ corresponds to the probability density function associated with t, and APV is equivalent to $E(Z(T))$. In actuarial sciences, more precisely in the insurance sector, it is common to establish a priori the value of a in the form of a benefit to be paid as a result of a certain event.

A major challenge in financial valuation and discounting is the uncertainty associated with future interest rates, which directly affects loans, investments, and actuarial contracts, among other financial arrangements. Such uncertainty may arise from a variety of sources, including monetary policy decisions, inflation dynamics, macroeconomic conditions, credit risk, and fluctuations in financial markets. Variations in interest rates can complicate financial planning, influence investment profitability, affect decisions regarding new investments or business expansion, and alter the cost of public debt, among other consequences.

More generally, interest rate uncertainty creates an environment of increased risk and may contribute to greater volatility in financial and economic markets. This motivates the adoption of flexible and transparent methodologies capable of formally and consistently addressing situations in which the available information is imprecise or incomplete.

Considering that the time until the occurrence of this event is modeled by a random variable, expression (12) represents the average amount required in the current date to guarantee payment of the benefit as events occur.

4 Fuzzy modeling of the benefit contracted in insurance life

Insurance is a financial contract that provides protection against financial losses in exchange for the payment of a premium. A fundamental characteristic of insurance is its dependence on future and uncertain events. In life insurance, for example, contracts are based on the risk of the insured's death and are designed to guarantee the payment of a specified benefit to the beneficiary upon the occurrence of that event.

The valuation of life insurance contracts relies on survival models that describe the future lifetime of the insured. Several methods can be employed to determine the premium associated with a contract. From an actuarial perspective, the minimum premium that can be charged is based on the actuarial present value (APV) of the insured benefit. This quantity represents the expected present value of the future benefit payable under the contract, excluding additional technical loadings such as administrative expenses, taxes, contingency margins, and profit provisions.

Let us consider an individual aged x who takes out lifetime life insurance which stipulates the payment of a benefit upon his death in the amount a, in which his survival can be modeled by the Weibull probability model[1] with two parameters, whose density function is given by

$$f(t) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-\left(\frac{t}{\lambda}\right)^k}, \quad t > 0, \tag{13}$$

where k corresponds to the shape parameter (related to the model's mortality rate) and λ represents the average age at death. This way, we have to:

$$APV = a\int_0^\infty e^{-\delta t}\frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-\left(\frac{t}{\lambda}\right)^k}\,dt.$$

In Brazil, insurance companies historically use interest rates that vary from 3% to 6% per year. We will adopt the average value of these fees, that is, $i = 4.5\%$. The interest rate range from 3% to 6% per year is based on historically observed levels in Brazil. According to official macroeconomic data reported in the *Panorama Macroeconômico* of the Brazilian Ministry of Economy [16], the benchmark Selic interest rate presented an average of 5.90% in 2019, decreased to 2.64% in 2020, reached 3.91% in 2021, and recorded an average of 12.34% in 2022. Therefore, the selected interval from 3% to 6% represents a plausible range of low interest-rate scenarios observed in the Brazilian economy during the period from 2019 to 2022. It is important to highlight that this value does not correspond to the instantaneous interest rate δ. The relationship between the rate δ and the rate i is given by $\delta = \ln(1+i)$. Therefore, the interest rates adopted by insurance companies correspond to values of δ that vary between $\delta_1 = \ln(1+0.03) \approx 0.02956$ and $\delta_2 = \ln(1+0.06) \approx 0.05827$. With this, we will consider, $\delta = 0.04402$.

The values of the parameters k and λ were chosen so that the insured's life expectancy was close to 50 years, as[2] $E(T) = \lambda\Gamma(1+1/k)$.

[1] Mortality laws describe the mortality behavior of a population, representing the probability of survival or death of individuals at different ages. Weibul's Law has its origins in reliability engineering and is useful for describing mortality in young populations.

[2] The symbol Γ denotes the Gamma function, i.e., $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$, $x > 0$.

Thus, for $a = 100000$, an annual interest rate $\delta = 0.04402$, $\lambda = 55$ and $k = 1.5$ we have

$$\begin{aligned} APV &= \frac{1.5}{55} 10^5 \int_0^\infty e^{-0.04402t} \left(\frac{t}{55}\right)^{0.5} e^{-\left(\frac{t}{55}\right)^{1.5}} dt \\ &\approx 22942.06. \end{aligned}$$

The value 22942.06 represents the average present value needed to guarantee the payment of 100000 to beneficiaries as policyholders in this profile die.

In practice, insurance contracts are sometimes designed from a predetermined premium amount. In this setting, the insurer must determine the benefit level that can be supported by the proposed premium under its own actuarial assumptions.

Assume that the actuarial present value is computed using the force of interest $\delta = 0.04402$ and a benefit amount $A = 100000$. Then,

$$APV = \frac{3}{110} \left(\int_0^\infty \left(\frac{t}{55}\right)^{\frac{1}{2}} e^{-\left(\frac{t}{55}\right)^{\frac{3}{2}} - 0.04402t} dt \right) 100000,$$

which yields

$$APV \approx 0.2294206 \times 100000 = 22942.06.$$

Assume that $APV = 22942.06$ represents the premium amount available for the purchase of life insurance coverage. If the insurer adopts a different interest rate, namely $\delta_1 \approx 0.02956$, then the corresponding insured benefit is obtained by solving

$$22942.06 = \frac{3}{110} \left(\int_0^\infty \left(\frac{t}{55}\right)^{\frac{1}{2}} e^{-\left(\frac{t}{55}\right)^{\frac{3}{2}} - 0.02956t} dt \right) a_1.$$

Therefore,

$$\begin{aligned} a_1 &= \frac{22942.06}{\frac{3}{110} \int_0^\infty \left(\frac{t}{55}\right)^{\frac{1}{2}} e^{-\left(\frac{t}{55}\right)^{\frac{3}{2}} - 0.02956t} dt} \\ &\approx 69162.86. \end{aligned}$$

Consequently, if the insurer works with the interest rate $\delta_2 = 0.05827$, we will have $a_2 \approx 134966.4$. Examples like this illustrate that small changes in the interest rate initially used to calculate the premium can result in a benefit value completely different from that initially projected.

To address interest rate uncertainty, as discussed above, insurers often adopt prudential measures such as incorporating additional margins into the premium, commonly referred to as safety loadings, in order to cover potential adverse deviations. In this work, however, we focus on the fuzzy uncertainty associated with the contracted benefit. As previously observed, uncertainty in the interest rate δ may implicitly translate into uncertainty regarding the benefit amount itself. Consequently, the presence of uncertainty in the initial condition of problem (10) provides a natural justification for the use of fuzzy sets to describe the dynamics of the system.

Let us now consider determining the present value function $Z(t)$, assuming that the contract benefit $Z(0)$ is uncertain and modeled through a fuzzy set to deal with the situation described in the previous example. The fuzzy present value function has been widely investigated within the framework of fuzzy differential equations, and multiple approaches for obtaining its solutions have been proposed in the literature [8,12,15,22,23]. In this type of valuation model, it is reasonable to assume a linear dependency over time, meaning that the system's state at a given moment t influences and shapes the evaluation at the subsequent instant.

So, we consider the fuzzy initial value problem (FIVP) given by

$$\begin{cases} Z'(t) &= -\delta Z(t), \\ Z(0) &= u + vA \in \mathbb{R}_{\mathscr{F}(A)}, \end{cases} \tag{14}$$

where the real number $\delta \in \mathbb{R}_+$, $u, v \in \mathbb{R}$ and $Z \in \mathscr{LCF}_A([a,b])$ with $A \in \mathbb{R}_{\mathscr{F}}$ non-symmetric.

Now, considering that $Z(t) = u(t) + v(t)A$, where $u, v : \mathbb{R} \longrightarrow \mathbb{R}$, we have that $Z'(t) = u'(t) + v'(t)A$. In this way, we obtain two real systems, given by

$$\begin{cases} v'(t) &= -\delta v(t), \\ v(0) &= v_1, \end{cases} \tag{15}$$

and

$$\begin{cases} u'(t) &= -\delta u(t), \\ u(0) &= u_1. \end{cases} \tag{16}$$

By solving (15) and (16), we obtain

$$v(t) = v_1 e^{-\delta t} \text{ and } u(t) = u_1 e^{-\delta t}.$$

Therefore, the fuzzy present value function, solution of (14) is given by

$$Z(t) = u_1 e^{-\delta t} + v_1 e^{-\delta t} A.$$

Remark 3. *The use of A-linearly correlated fuzzy processes is particularly relevant in actuarial valuation problems involving dynamically propagated uncertainty. In classical fuzzy arithmetic based on independent extensions, repeated occurrences of the same uncertain quantity are treated independently, which may artificially enlarge uncertainty bands due to overestimation effects. In contrast, the A-linearly correlated framework preserves the intrinsic dependence structure associated with the original uncertain parameter throughout the entire evolution of the system. In the present model, the uncertainty associated with the contracted benefit propagates through the differential equation*

$$Z'(t) = -\delta Z(t),$$

where the same fuzzy quantity influences the complete trajectory of the solution. The representation

$$Z(t) = u_1 e^{-\delta t} + v_1 e^{-\delta t} A$$

preserves this dependence structure explicitly, avoiding artificial interval inflation and yielding a coherent family of actuarial valuation scenarios linked by a common source of uncertainty. Consequently, the lower and upper bounds of the fuzzy actuarial present value coincide with the classical actuarial present values obtained under extreme deterministic interest-rate assumptions.

In view of the benefit values discussed before, we will assume $A = (69162.86; 100000; 134966.4)$. Therefore

$$Z(t) = e^{-0.04402t} A. \tag{17}$$

It is easy see that if the initial condition of the Problem (14) is a real number a, then (17) become the classical solution given by (11), whose graph can be seen in the Figure 1.

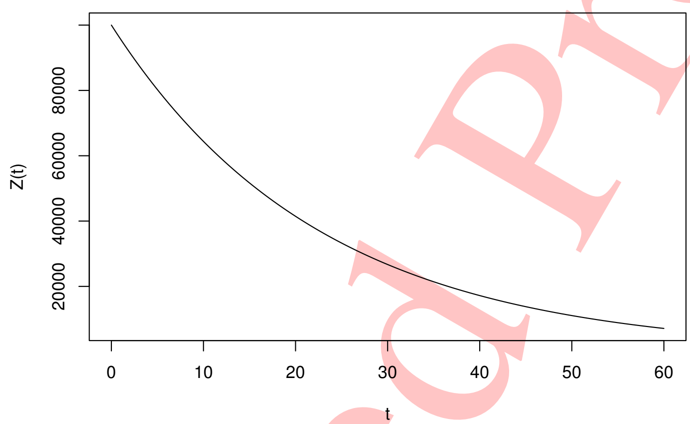

Figure 1: Graphical representation of the present value given by (17), for $v_1 = 0$, $u_1 = a = 100000$ and $\delta = 0,04402$

On the other hand, if $[A]_\alpha = [a^-_\alpha, a^+_\alpha]$ for all $\alpha \in [0,1]$, the diameter of (17), given by $diam(Z(t)) = e^{-0.04402t}(a^+_0 - a^-_0)$ is decreasing for all $t > 0$, i.e, the fuzzy solution (17) is decreasing fuzziness.

Figure 2 presents the graphical representation of the solution (17), considering $\delta = 0.04402$ and $A = (69162.86; 100000; 134966.4)$. This graph allows us to observe the behavior of the present value $Z(t)$ over time, taking into account the uncertainty in the value of the benefit. The gray scale, from white to black, represents the α-levels of the $Z(t)$ function, as defined in (17), ranging from 0 to 1. It is possible to note that uncertainty in the condition initial tends to have a decreasing influence on $Z(t)$.

Figure 2 illustrates the temporal dynamics of the fuzzy present value function $Z(t)$, providing a visual representation of how epistemic uncertainty associated with interest rate assumptions propagates through the valuation of life insurance benefits. By modeling the contracted benefit as an A-linearly correlated fuzzy process, the figure displays a risk-sensitive valuation band in which the gray-scale layers correspond to α-level sets, ranging from maximum uncertainty at $\alpha = 0$ to the core deterministic scenario at $\alpha = 1$.

From an actuarial perspective, the progressive narrowing of these bands over time reflects a reduction in fuzziness, indicating that the sensitivity of the present value to uncertainty in the initial benefit amount—which implicitly captures uncertainty in the interest rate assumption—decreases exponentially as the time until the insured event increases. This framework enables insurers to quantify a structured range of admissible actuarial values, thereby facilitating more robust solvency assessments and capital allocation decisions. Moreover, it aggregates multiple interest rate scenarios into a single analytical representation that remains computationally tractable and easy to interpret.

Moreover, as illustrated in Figures 1 and 2, the present value required to finance a benefit amount a decreases exponentially as time increases. In addition, the uncertainty associated with the initial condi-

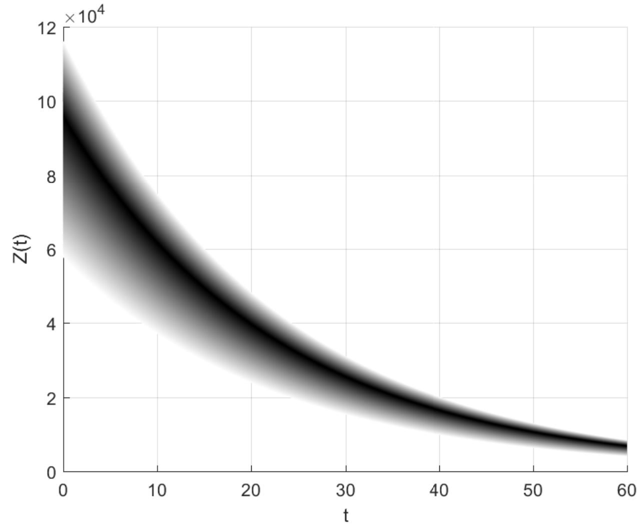

Figure 2: Graphical representation of the fuzzy present value function given by (17) for $A = (69162.86; 100000; 134966.4)$, in gray-scale (from white to black represent the α-levels of (17) varying from 0 to 1)

tion in (14) has a progressively smaller influence on $Z(t)$ over time. Since the capitalization period is related to the longevity of the insured, longer time horizons become increasingly unlikely. Consequently, the behavior of $Z(t)$ is consistent with the probabilistic structure of human survival and longevity.

These observations motivate the consideration of an average value of $Z(t)$ weighted by the probabilities associated with the occurrence times. Therefore, if $Z(t)$ is a fuzzy-valued function of the random variable T, then

$$APV_{fuzzy} = \left(\int_0^\infty u_1 e^{-\delta t} f(t)dt\right) + \left(\int_0^\infty v_1 e^{-\delta t} f(t)dt\right) A. \tag{18}$$

Note that $v_1 = 1$, $u_1 = 0$, $\delta = 0.04402$ and f is given by (13) with $\lambda = 55$, $k = 1.5$. Thus

$$\begin{aligned} APV_{\text{fuzzy}} &= \frac{3}{110}\left(\int_0^\infty \left(\frac{t}{55}\right)^{\frac{1}{2}} e^{-\left(\frac{t}{55}\right)^{\frac{3}{2}} - 0.04402t}\, dt\right) A \\ &\approx 0.2294206A. \end{aligned}$$

Remark 4. *The triangular form of the fuzzy actuarial present value observed in Table 1 follows from assuming that the contracted benefit is modeled as a triangular fuzzy number. The proposed α-level formulation is not restricted to this choice and can be extended to trapezoidal or more general fuzzy numbers. In such cases, the resulting fuzzy actuarial present value preserves the same analytical structure, although its membership function may not be triangular. The triangular assumption is adopted for analytical simplicity and interpretability.*

According to the ordinal structure of APV_{fuzzy}, represented by $(b; u; c) = (15867.38; 100000; 30964.07)$, the values b and c correspond, respectively, to the classical APV values associated with benefits equal to

69162.86 and 134966.4. Table 1 presents the α-levels associated with the fuzzy numbers A and APV_{fuzzy}. For $\alpha = 0$, the intervals $[A]_0 = [69162.86, 134966.40]$ and $[APV_{fuzzy}]_0 = [15867.38, 30964.07]$ describe the full range of possible benefit and actuarial present value outcomes, respectively, corresponding to scenarios of maximum uncertainty with respect to the interest rate used in premium valuation.

It is worth noting that the width of the interval $[APV_{fuzzy}]_0$ reflects the variation induced by the extreme deterministic interest-rate scenarios considered in the model, rather than a probabilistic dispersion measure such as a standard deviation. As α-levels increase, the intervals become progressively narrower, indicating reduced uncertainty in the values of A and APV. For $\alpha = 1$, the crisp values $A = 100000$ and $APV = 22942.10$ are obtained, corresponding to a scenario without uncertainty in the interest rate assumption.

Table 1: α-levels associated with the fuzzy numbers A and APV_{fuzzy}

α-level	$[A]_\alpha$	$[APV_{fuzzy}]_\alpha$
0.0	$[69162.86, 134966.40]$	$[15867.38, 30964.07]$
0.2	$[75330.29, 127973.12]$	$[17282.32, 29359.67]$
0.4	$[81497.72, 120979.84]$	$[18697.25, 27755.27]$
0.6	$[87665.14, 113986.56]$	$[20112.19, 26150.86]$
0.8	$[93832.57, 106993.28]$	$[21527.12, 24546.46]$
1.0	$[100000.00, 100000.00]$	$[22942.06, 22942.06]$

4.1 Practical interpretation of fuzzy actuarial present values

From a practical perspective, the fuzzy actuarial present values reported in Table 1 may be interpreted as a risk-sensitive valuation band for long-term financial liabilities. In practice, life insurance contracts and pension products are often priced under interest rate assumptions that are affected by model uncertainty, regulatory requirements, and expert judgment. The fuzzy framework adopted in this study captures such imprecision by replacing a single discount rate with a structured family of admissible values, thereby producing a range of actuarial present values rather than a single point estimate. These valuation bands are particularly relevant for asset–liability management, solvency assessment, and capital allocation, as they provide insurers and financial institutions with a transparent representation of the potential impact of interest rate uncertainty without requiring the specification of a fully stochastic interest rate model.

From this perspective, the widths of the resulting α-level sets should be viewed as complementary to, rather than substitutes for, confidence intervals derived from stochastic models. Whereas stochastic confidence intervals quantify uncertainty arising from probabilistic variability and random fluctuations, fuzzy actuarial present values capture imprecision associated with incomplete information, expert judgment, and other forms of epistemic uncertainty. Consequently, the two approaches address different dimensions of uncertainty and can coexist as complementary tools within the actuarial valuation framework.

5 Conclusions

In this work, we investigated fuzzy uncertainty in life insurance valuation, with particular emphasis on the actuarial present value of a whole-life insurance benefit. To this end, we employed the theory of A-linearly correlated fuzzy processes. Specifically, we analyzed a present value model under the assumption that the contracted benefit is uncertain and represented by a fuzzy number. The resulting solution naturally takes the form of an A-linearly correlated fuzzy process, since the state of the system at time t influences its evolution at subsequent times.

Furthermore, we assumed that the fuzzy present value function is a fuzzy-valued function of the random variable T, from which the actuarial present value is obtained. It was shown that the resulting fuzzy actuarial present value is a triangular fuzzy number of the form $(b;u;c)$, where b and c correspond to the actuarial present values associated with the lower and upper bounds of the contracted benefit, respectively. This result provides an important operational advantage, as it considerably simplifies the calculations when compared with approaches that introduce uncertainty directly into the interest rate.

In addition, the results indicate that the fuzzy actuarial present value exhibits a relatively limited spread around its core value, suggesting that the impact of the considered interest rate uncertainty on premium valuation remains controlled. This finding is particularly noteworthy because, although the actuarial literature consistently identifies the interest rate as one of the assumptions with the greatest influence on premium calculations, moderate variations in the rate translate into only limited uncertainty in the contracted benefit and, consequently, a relatively small effect on the resulting premium. In this regard, the proposed fuzzy framework should be viewed as a complementary actuarial tool, especially in situations where interest rate uncertainty arises from imprecision, expert judgment, or incomplete information rather than from stochastic variability.

Future research may explore methodological comparisons with alternative uncertainty-based approaches, including interval, robust, and stochastic frameworks. Such investigations may contribute to a deeper understanding of the relationships between probabilistic and possibilistic methodologies and their respective roles in actuarial valuation under uncertainty.

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Conflict of Interest

The authors declare no competing financial interests or personal relationships that could have influenced this work.

References

[1] M.Z. Ahmad, B. De Baets, *A predator-prey model with fuzzy initial populations*, in Proc. Joint 2009 Int. Fuzzy Syst. Assoc. World Congr. and Eur. Soc. Fuzzy Logic Technol. Conf. (IFSA/EUSFLAT), (2009) 1311–1314.

[2] Z. Alijani, U. Kangro, *Numerical solution of a linear fuzzy Volterra integral equation of the second kind with weakly singular kernels*, Soft Comput. **26** (2022) 12009–12022.

[3] Z. Alijani, U. Kangro, *On the smoothness of the solution of fuzzy Volterra integral equations of the second kind with weakly singular kernels*, Numer. Funct. Anal. Optim. **42** (2021) 819–833.

[4] Z. Alijani, B. Shiri, I. Perfilieva, D. Baleanu, *Numerical solution of a new mathematical model for intravenous drug administration*, Evol. Intell. (2023) 1–17.

[5] J. De Andrés-Sánchez, L.G.V. Puchades, *Using fuzzy random variables in life annuities pricing*, Fuzzy Sets Syst. **188** (2012) 27–44.

[6] J. De Andrés-Sánchez, L.G.V. Puchades, *Some computational results for the fuzzy random value of life actuarial liabilities*, Iran. J. Fuzzy Syst. **14** (2017) 1–25.

[7] L.C. Barros, R.C. Bassanezi, W.A. Lodwick, *First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics*, Springer-Verlag, Berlin Heidelberg, (2017).

[8] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer-Verlag, Berlin Heidelberg, (2013).

[9] V.M. Cabral, L.C. Barros, *Fuzzy differential equation with completely correlated parameters*, Fuzzy Sets Syst. **265** (2015) 86–98.

[10] Y. Chalco Cano, H. Román-Flores, *On new solutions of fuzzy differential equations*, Chaos Solitons Fractals **38** (2018) 112–119.

[11] C. Carlsson, R. Fullér, P. Majlender, *Additions of completely correlated fuzzy numbers*, in Proc. 2004 IEEE Int. Conf. Fuzzy Syst. **1** (2004) 535–539.

[12] E. Esmi, B. Laiate, F. Santo Pedro, L.C. Barros, *Calculus for fuzzy functions with strongly linearly independent fuzzy coefficients*, Fuzzy Sets Syst. **436** (2021) 1–31.

[13] E. Esmi, F. Santo Pedro, L.C. Barros, W. Lodwick, *Fréchet derivative for linearly correlated fuzzy function*, Inf. Sci. **435** (2018) 150–160.

[14] E. Esmi, G. Barroso, L.C. Barros, P. Sussner, *A family of joint possibility distributions for adding interactive fuzzy numbers inspired by biomathematical models*, in Proc. Joint 2015 Int. Fuzzy Syst. Assoc. World Congr. and Eur. Soc. Fuzzy Logic Technol. Conf. (IFSA/EUSFLAT), (2015).

[15] N.A. Gasilov, S.E. Amrahov, *A survey and comparative analysis of different approaches to fuzzy differential equations modelling dynamics with uncertain parameters of deterministic character*, Kybernetika **1** (2025) 289–347.

[16] Brazil, Ministry of Economy, Secretariat for Economic Policy, *Panorama Macroeconômico: novembro de 2025*, SPE/ME, Brasília, 2025. Available at: https://www.gov.br/fazenda/pt-br/central-de-conteudo/publicacoes/conjuntura-economica/panorama-macroeconomico/2025 (in Portuguese).

[17] F. Mehrdoust, I. Noorani, W. Xu, *Uncertain energy model for electricity and gas futures with application in spark-spread option price*, Fuzzy Optim. Decis. Mak. **22** (2023) 123–148.

[18] I. Noorani, F. Mehrdoust, W. Lio, *Electricity spot price modeling by multi-factor uncertain process: a case study from the Nordic region*, Soft Comput. **25** (2021) 13105–13126.

[19] B. Shiri, I. Perfilieva, Z. Alijani, *Classical approximation for fuzzy Fredholm integral equation*, Fuzzy Sets Syst. **404** (2021) 159–177.

[20] B. Shiri, Z. Alijani, Y. Karaca, *A power series method for the fuzzy fractional logistic differential equation*, Fractals **31** (2023) 10.

[21] M. Shahidi, L.C. Barros, E. Esmi, *Linear fuzzy partial differential equations for A-linearly correlated fuzzy processes*, Inf. Sci. **691** (2025) 121629.

[22] S. A. B. Salgado, L. C. Barros, E. Esmi, D. E. S. Ibanez, *Solution of a fuzzy differential equation with interactivity via Laplace transform*, J. Intell. Fuzzy Syst. **37** (2019) 2495–2501.

[23] S.A.B. Salgado, O.J.R. Silveira, S.M. Souza, O. Rojas, *Modeling the linear drag on falling balls via interactive fuzzy initial value problem*, Comput. Appl. Math. 41 (2022) 43.

[24] S.A.B. Salgado, O.J.R. Silveira, S.M. Souza, O. Rojas, *Fuzzy modeling of a class of linear oscillators and its application to electrical circuits*, Comput. Appl. Math. **43** (2024) 1–18.

[25] F. Santo Pedro, B. Laiate, E. Esmi, L.C. Barros, *Interactive Riemann integral in the space of the linearly correlated fuzzy numbers* $\mathbb{R}_{F(A)}$, in Proc. Braz. Conf. Fuzzy Syst., (2021).

[26] F. Santo Pedro, S.A.B. Salgado, E.S. Daniel, E. Esmi, L.C. Barros, *On fuzzy Laplace transform in linearly correlated fuzzy space*, Soft Comput. **27** (2023) 1425–1438.

[27] Y. Shen, *Calculus for linearly correlated fuzzy number-valued functions*, Fuzzy Sets Syst. **429** (2022) 101–135.

[28] Y. Xin, Y. Zhang, I. Noorani, F. Mehrdoust, J. Gao, *Estimation of parameters and valuation of options written on multiple assets described by uncertain fractional differential equations*, Appl. Math. Comput. **487** (2025) 129109.