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On α-prime ideals in the ternary semiring of non-positive integers

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Abstract. In this paper, we introduce the concept of α-prime ideals in a commutative ternary semiring with identity element and obtain the characterizations of α-prime ideals in the ternary semiring of non-positive integers.

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1 Introduction

Theory of semirings is given by Golan [9] and theory of ideals in the semiring of non-negative integers is studied by Allen and Dale [1]. Generalizing the notion of ternary ring introduced by Lister [12], Dutta and Kar [6] introduced the notion of ternary semiring. A non-empty set R together with a binary operation called addition $(+)$ and a ternary operation called ternary multiplication $(\cdot)$ is called ternary semiring, if it satisfies the following conditions for all $a, b, c, d, e \in R$;

1) $(a+b)+c = a+(b+c)$;
2) $a+b = b+a$;
3) $(a \cdot b \cdot c) \cdot d \cdot e = a \cdot (b \cdot c \cdot d) \cdot e = a \cdot b \cdot (c \cdot d \cdot e)$;
4) there exists $0 \in R$ such that $a+0 = a = 0+a; a \cdot b \cdot 0 = a \cdot 0 \cdot b = 0 \cdot a \cdot b = 0$;
5) $(a+b) \cdot c \cdot d = a \cdot c \cdot d + b \cdot c \cdot d$;

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6) $a \cdot (b + c) \cdot d = a \cdot b \cdot d + a \cdot c \cdot d$;

7) $a \cdot b \cdot (c + d) = a \cdot b \cdot c + a \cdot b \cdot d$.

Clearly, every semiring is a ternary semiring. Denote the sets of all non-positive integers, negative integers and positive integers respectively by $\mathbb{Z}_0^-$, $\mathbb{Z}^-$ and $\mathbb{N}$. The set $\mathbb{Z}_0^-$ is a ternary semiring under usual addition and ternary multiplication of non-positive integers but it is not a semiring. If there exists an element e in a ternary semiring R such that $eex = exe = xee = x$ for all $x \in R$, then e is called the identity element of R. A ternary semiring R is said to be commutative, if $abc = acb = cab$ for all $a, b, c \in R$. The ternary semiring $(\mathbb{Z}_0^-, +, \cdot)$ is commutative with the identity element -1. A non-empty subset I of a commutative ternary semiring R is called an ideal of R, if the following conditions are satisfied;

1) $a, b \in I$ implies $a + b \in I$;

2) $a \in I, r, s \in R$ implies $rsa \in I$.

If R is a commutative ternary semiring with identity element, then a proper ideal I of R is called prime, if $abc \in I$, $a, b, c \in R$ implies $a \in I$ or $b \in I$ or $c \in I$. If $n \in (\mathbb{Z}_0^-, +, \cdot)$ and $n \leq -2$, then n can be written as,

$$\begin{aligned} n &= (-p_1)^{r_1}(-p_2)^{r_2} \dots (-p_k)^{r_k}(-1)^{r_1}(-1)^{r_2} \dots (-1)^{r_k}(-1) \\ &= (-p_1)^{r_1}(-p_2)^{r_2} \dots (-p_k)^{r_k}(-1)^{(\sum_{i=1}^{k} r_i)+1} \end{aligned}$$

where $p_1, p_2, \dots, p_k \in \mathbb{N}$ are pairwise distinct prime numbers and $r_i, k \in \mathbb{N}$. For $a, b \in (\mathbb{Z}_0^-, +, \cdot)$ and $a \neq 0$, we define $a \mid b$ if and only if $b = \alpha\beta a$ for some $\alpha, \beta \in \mathbb{Z}_0^-$. An ideal I of $(\mathbb{Z}_0^-, +, \cdot)$ is said to be generated by a subset $A = \{a_1, a_2, \dots, a_n\}$ of $\mathbb{Z}_0^-$, if for every $x \in I$, there exist $\alpha_i, \beta_i \in \mathbb{Z}_0^-$ such that $x = \sum_{i=1}^{n} \alpha_i\beta_i a_i$. If $A = \{a\}$, then $\mathbb{Z}_0^-\mathbb{Z}_0^- a$ is called the principal ideal generated by a. For $a_1, a_2, \dots, a_k \in \mathbb{Z}_0^-$, we denote (i) $\langle a_1, a_2, \dots, a_k \rangle$ = the ideal generated by $a_1, a_2, \dots, a_k$ in the ternary semiring $\mathbb{Z}_0^-$; (ii) $(a_1, a_2, \dots, a_k)$ = g.c.d. of $a_1, a_2, \dots, a_k$. Two elements $a_1, a_2 \in \mathbb{Z}_0^-$ are said to be relatively prime, if $(a_1, a_2) = 1$. Furthermore, (i) for $n \in \mathbb{Z}^-$, we denote $2n = n + n = (-1)(-2)n$. Also, $-2p = (-1)(-2)(-p)$ for any prime number p. (ii) for $n \in \mathbb{Z}^-$, we denote $n + 1$ as the immediate successor of n in $\mathbb{Z}_0^-$; (iii) for $n \in \mathbb{Z}^- - \{-1\}$, we denote $n + 2$ as the immediate successor of $n + 1$ in $\mathbb{Z}_0^-$. For example, $-5 = (-6) + 1$ is the immediate successor of -6 and $-4 = (-6) + 2$ is the immediate successor of $(-6) + 1$ $(= -5)$.

Dutta and Kar [7, 8] have characterized respectively the prime k-ideals and semiprime k-ideals of the ternary semiring of non-positive integers. Theory of ideals in the ternary semiring of non-positive integers is studied by Kar [10]. Further, prime ideals, semiprime ideals and irreducible principal T-ideals in the ternary semiring of non-positive integers are characterized by Chaudhari and Ingale [4]. Recently, 3-absorbing principal T-ideals in the ternary semiring of non-positive integers are characterized by Chaudhari and Ingale [5].

The concept of an α-prime submodule of a module over a commutative ring R and an α-prime ideal in a commutative ring is introduced by Khumprapussorn [11]. Further, it is generalized to a commutative semiring with identity element by Bonde and Chaudhari [2,3]. In this paper, we introduce the concept of α-prime ideals in a commutative ternary semiring with identity element which is a generalization of an α-prime ideals in a commutative semiring with identity element. We characterize all α-prime ideals in the ternary semiring of non-positive integers.

The following results will be used to prove our results.

Lemma 1 ([10]). *Let $I = \langle a_1, a_2, \ldots, a_n\rangle \subseteq \mathbb{Z}_0^-$. If $(a_1, a_2, \ldots, a_n) = d$, then there exists a largest $t \in \mathbb{Z}_0^-$ such that $(-1)(-d)r \in I$, for all $r \leq t$.*

Theorem 1 ([10]). *Every ideal of the ternary semiring $(\mathbb{Z}_0^-, +, \ldots)$ is finitely generated.*

Theorem 2 ([4]). *A non-zero ideal I of the ternary semiring $\mathbb{Z}_0^-$ is prime if and only if $I = \langle -p\rangle$, for some prime number $p \in \mathbb{N}$ or $I = \langle -2, -3\rangle$.*

2 α-prime ideals in the ternary semiring $\mathbb{Z}_0^-$

In this section, we introduce the concept of α-prime ideals in a commutative ternary semiring with identity element and characterize all principal α-prime ideals as well as all non-principal α-prime ideals in the ternary semiring $\mathbb{Z}_0^-$. Throughout this section, R denotes a commutative ternary semiring with identity element. A proper ideal I of R is called α-prime, if $ab(c + c) \in I, a, b, c \in R$, then either $a+a \in I$ or $b+b \in I$ or $c+c \in I$. Since R is commutative, equivalently, we have a proper ideal I of R is α-prime, if $(a + a)bc \in I, a, b, c \in R$, then either $a + a \in I$ or $b + b \in I$ or $c + c \in I$. Clearly, every prime ideal of a ternary semiring R is an α-prime ideal. Now we prove some elementary results.

Lemma 2. *Let I be a proper ideal of a ternary semiring R with identity element e such that $e + e \in I$. Then I is an α-prime ideal.*

Proof. Let I be a proper ideal of a ternary semiring R and $e + e \in I$. Therefore, $a + a = ae(e + e) \in I$ for all $a \in R$. Hence I is an α-prime ideal. □

Lemma 3. *Let I be a proper ideal of the ternary semiring $\mathbb{Z}_0^-$. If $-2 \in I$, then I is an α-prime ideal.*

Proof. It follows from Lemma 2. □

Lemma 4. *Let I be a proper ideal of the ternary semiring $\mathbb{Z}_0^-$. Then I is an α-prime ideal and $-2 \in I$ if and only if $I = \langle -2\rangle$ or $I = \langle -2, n\rangle$, where n is an odd number ≤ -3.*

Proof. Let I be an α-prime ideal of $\mathbb{Z}_0^-$ and $-2 \in I$. Suppose that $I \neq \langle -2\rangle$. Choose largest odd number $n \leq -3$ such that $n \in I$. Then clearly $\langle -2, n\rangle \subseteq I$. Let $a \in I$. If a is an even number, then $a \in \langle -2, n\rangle$. If a is an odd number, then $a = n + (-2)(-1)r$, for some $r \in \mathbb{Z}_0^-$. Hence $a \in \langle -2, n\rangle$. Thus $I \subseteq \langle -2, n\rangle$. Now $I = \langle -2, n\rangle$. Converse follows by Lemma 3. □

Here, $I = \langle -2, -5\rangle$ is an α-prime ideal in $\mathbb{Z}_0^-$ but by Theorem 2, I is not a prime ideal.

Lemma 5. *Let I be an ideal of the ternary semiring $\mathbb{Z}_0^-$, $-2 \notin I$ and $-4, -6 \in I$. Then I is an α-prime ideal.*

Proof. Let I be an ideal of the ternary semiring $\mathbb{Z}_0^-$ such that $-2 \notin I$ and $-4, -6 \in I$. Then clearly $a + a \in I$ for all $a \in \mathbb{Z}_0^- \setminus \{-1\}$. Hence I is an α-prime ideal. $\square$

Lemma 6. *Let I be an ideal of the ternary semiring $\mathbb{Z}_0^-$, $-2 \notin I$ and $-4, -6 \in I$. Then I is an α-prime ideal if and only if $I = \langle -4, -6 \rangle$ or $I = \langle -4, -6, n \rangle$ or $I = \langle -4, -6, n, n-2 \rangle$, where n is an odd number ≤ -3.*

Proof. Let I be an α-prime ideal of $\mathbb{Z}_0^-$ such that $-2 \notin I$ and $-4, -6 \in I$. Suppose that $I \neq \langle -4, -6 \rangle$ and $I \neq \langle -4, -6, n, n-2 \rangle$, where n is an odd number ≤ -3. Choose the largest odd number $n \leq -3$ such that $n \in I$. Then, clearly $\langle -4, -6, n \rangle \subseteq I$. Let $b \in I$. If b is an even number, then $b \in \langle -4, -6, n \rangle$. If b is an odd number, then $b = n + (-2)(-1)r$, for some $r \in \mathbb{Z}_0^- \setminus \{-1\}$. Hence $b \in \langle -4, -6, n \rangle$. Thus $I \subseteq \langle -4, -6, n \rangle$. Now $I = \langle -4, -6, n \rangle$. The converse follows by using Lemma 5. $\square$

The following theorem gives a characterization of principal α-prime ideals in the ternary semiring $\mathbb{Z}_0^-$.

Theorem 3. *Let I be a proper ideal of $\mathbb{Z}_0^-$. Then I is a principal α-prime ideal if and only if $I = \langle 0 \rangle$ or $I = \langle -p \rangle$ or $I = \langle -2p \rangle$, where p is a prime number.*

Proof. Let I be a principal α-prime ideal. If $I = \langle 0 \rangle$, then we are through. Let $I \neq \langle 0 \rangle$. Suppose that $I = \langle m \rangle$, where $m \in \mathbb{Z}_0^-$ and $m \leq -2$. Let $m = (-p_1)^{r_1}(-p_2)^{r_2} \cdots (-p_k)^{r_k}(-1)^{(\sum_{i=1}^{k} r_i)+1}$, where $p_1, p_2, \ldots, p_k \in \mathbb{N}$ are pairwise distinct prime numbers and $r_i, k \in \mathbb{N}$.
Case (1): Assume that all $p_i's$ are odd prime numbers.
If $k \geq 2$, then $2m = [2(-p_1)^{r_1}(-1)^{r_1+1}][(-p_2)^{r_2}(-1)^{r_2+1}][(-p_3)^{r_3} \cdots (-p_k)^{r_k}(-1)^{(\sum_{i=3}^{k} r_i)+1}] \in I$ but $2(-p_1)^{r_1}(-1)^{r_1+1} \notin I$, $2(-p_2)^{r_2}(-1)^{r_2+1} \notin I$ and $2(-p_3)^{r_3} \cdots (-p_k)^{r_k}\ (-1)^{(\sum_{i=3}^{k} r_i)+1} \notin I$, which is not possible. Hence, assume that $I = \langle (-p_1)^{r_1}(-1)^{r_1+1} \rangle$, where p_1 is an odd prime number. If $r_1 \geq 2$, then $2(-p_1)^{r_1}(-1)^{r_1+1} = [2(-p_1)][(-p_1)^{r_1-1}\ (-1)^{r_1}](-1) \in I$ but $2(-p_1) \notin I$, $2(-p_1)^{r_1-1}(-1)^{r_1} \notin I$ and $2(-1) \notin I$, which is not possible. Hence $r_1 = 1$. Thus, $I = \langle -p \rangle$, where p is an odd prime number.
Case (2): Let $p_1 = 2$ and other $p_i's$ are odd prime numbers. If $k \geq 3$, then

$$
\begin{aligned}
m &= (-2)^{r_1}(-p_2)^{r_2} \cdots (-p_k)^{r_k}(-1)^{(\sum_{i=1}^{k} r_i)+1} \\
&= [2(-2)^{r_1-1}(-1)^{r_1}][(-p_2)^{r_2}(-1)^{r_2+1}][(-p_3)^{r_3} \cdots (-p_k)^{r_k}(-1)^{(\sum_{i=3}^{k} r_i)+1}] \in I.
\end{aligned}
$$

But $2(-2)^{r_1-1}(-1)^{r_1-1+1} \notin I, 2(-p_2)^{r_2}(-1)^{r_2+1} \notin I$ and $2(-p_3)^{r_3} \cdots (-p_k)^{r_k}\ (-1)^{(\sum_{i=3}^{k} r_i)+1} \notin I$, which is not possible. Hence $k = 1$ or $k = 2$. If $k = 2$, then $I = \langle (-2)^{r_1}(-p_2)^{r_2}(-1)^{r_1+r_2+1} \rangle$, where p_2 is an odd prime number. If $r_2 \geq 2$, then $(-2)^{r_1}(-p_2)^{r_2}(-1)^{r_1+r_2+1} = [2(-2)^{r_1-1}(-1)^{r_1}]$ $[(-p_2)^{r_2}\ (-1)^{r_2+1}](-1) \in I$ but $2(-2)^{r_1-1}(-1)^{r_1} \notin I, 2(-p_2)^{r_2}(-1)^{r_2+1} \notin I$ and $2(-1) \notin I$, which is not possible. Hence $r_2 = 1$. Therefore, $I = \langle (-2)^{r_1}(-p_2)(-1)^{r_1} \rangle$, where p_2 is an odd prime number. If $r_1 \geq 2$, then $(-2)^{r_1}(-p_2)(-1)^{r_1} = [2(-2)^{r_1-1}(-1)^{r_1}](-p_2)(-1) \in I$ but $2(-2)^{r_1-1}(-1)^{r_1} \notin I, 2(-p_2) \notin I$ and $2(-1) \notin I$, which is not possible. Hence $r_1 = 1$. Thus, $I = \langle -2p_2 \rangle$, where p_2 is an odd prime number. If $k = 1$, then $I = \langle (-2)^{r_1}(-1)^{r_1+1} \rangle$. If $r_1 \geq 3$, then $(-2)^{r_1}(-1)^{r_1+1} = [2(-2)][(-2)^{r_1-2}(-1)^{r_1-1}]\ (-1) \in I$ but $2(-2) \notin I, 2(-2)^{r_1-2}(-1)^{r_1-1} \notin I$ and $2(-1) \notin I$, which is not possible. Hence $r_1 = 1$ or 2. Thus, $I = \langle -2 \rangle$ or $I = \langle (-2)^2(-1) \rangle$

$= \langle -4 \rangle$.
Conversely, assume that $I = \langle 0 \rangle$ or $I = \langle -p \rangle$ or $I = \langle -2p \rangle$, where p is a prime number. By Theorem 2, $\langle 0 \rangle$ and $\langle -p \rangle$ are prime ideals. Hence they are α-prime ideals. Suppose that $I = \langle -2p \rangle$. Let $ab(c+c) \in I = \langle -2p \rangle$, where $a, b, c \in \mathbb{Z}_0^-$. Since p is a prime number, $(-p) \mid a$ or $(-p) \mid b$ or $(-p) \mid c$. Therefore, $a + a \in I$ or $b + b \in I$ or $c + c \in I$. Hence I is an α-prime ideal. □

For $n \in \mathbb{Z}^-$, we denote $I_n = \{r \in \mathbb{Z}^- : r \leq n\} \cup \{0\}$. Then clearly I_n is an α-prime ideal if and only if $n = -2$ or $n = -3$ or $n = -4$. Here, $I_{-2} = \langle -2, -3 \rangle$, $I_{-3} = \langle -3, -4, -5 \rangle$ and $I_{-4} = \langle -4, -5, -6, -7 \rangle$.

The following theorem gives a characterization of non-principal finitely generated α-prime ideals I in the ternary semiring $\mathbb{Z}_0^-$, where $I = \langle m_1, m_2, \ldots, m_k \rangle$ and $m_k < \cdots < m_2 < m_1 < -1$, $m_i \nmid m_j$ for all $i < j$ and $(m_1, m_2, \ldots, m_k) = 1$.

Theorem 4. *Let I be a proper, non-principal ideal in the ternary semiring $\mathbb{Z}_0^-$ and $I = \langle m_1, m_2, \ldots, m_k \rangle$, where $m_k < \cdots < m_2 < m_1 < -1$, $m_i \nmid m_j$ for all $i < j$ and $(m_1, m_2, \ldots, m_k) = 1$. Then I is an α-prime ideal if and only if I is one of the following types:*

1) $\langle -2, n \rangle$, where n is an odd number ≤ -3;

2) $\langle -3, -4 \rangle$;

3) $\langle -3, -4, -5 \rangle$;

4) $\langle -4, -5, -6 \rangle$;

5) $\langle -4, -5, -6, -7 \rangle$;

6) $\langle -4, -6, n \rangle$, where n is an odd number ≤ -7;

7) $\langle -4, -6, n, n-2 \rangle$, where n is an odd number ≤ -7.

Proof. Let I be an α-prime ideal. If $m_1 \leq -5$, then by Lemma 1, choose smallest $t \geq 3$ such that $((-2)^t(-1)^{t+1})(-1)(-1) \in I$ i.e. $(-2)^t(-1)^{t+1} \in I$. Now $(-2)^t(-1)^{t+1} = [2(-2)^{t-2}(-1)^{t-1}](-2)(-1) \in I$ but $2(-2)^{t-2}(-1)^{t-1} \notin I$, $2(-2) \notin I$ and $2(-1) \notin I$, which is not possible. Hence $m_1 = -4$ or -3 or -2.
Case (1): Let $m_1 = -4$. If $m_2 \leq -7$, then by Lemma 1, choose smallest $t \geq 2$ such that $(-3)^t(-1)^{t+1} \in I$. Therefore, $2(-3)^t(-1)^{t+1} \in I$ i.e. $[2(-3)][(-3)^{t-1}(-1)^t](-1) \in I$ but $2(-3) \notin I$, $2(-3)^{t-1}(-1)^t \notin I$ and $2(-1) \notin I$, which is not possible. Hence $m_2 = -6$ or $m_2 = -5$.
case (i): Let $m_2 = -6$. Since $(m_1, m_2, \ldots, m_k) = 1$, m_3 must exist and it must be an odd number. For any odd number $m_3 \leq -7$, if $I = \langle -4, -6, m_3 \rangle$, then we are through. If $I \neq \langle -4, -6, m_3 \rangle$, then by Lemma 6, $I = \langle -4, -6, m_3, m_3 - 2 \rangle$.
case (ii): Let $m_2 = -5$. If m_3 does not exist, then $I = \langle m_1, m_2 \rangle = \langle -4, -5 \rangle = \{-4, -5, -8, -9, -10\} \cup I_{-12}$. Now $[2(-3)](-3)(-1) = -18 = (-2)(-4)(-1) + (-2)(-5)(-1) \in I$ but $2(-3) \notin I$, $2(-1) \notin I$, a contradiction. Hence m_3 exists and $m_3 = -6$ or -7 or -11. If $I = \langle -4, -5, -6 \rangle = I_{-4} \setminus \{-7\}$, then we are through. If $I \neq \langle -4, -5, -6 \rangle$, then $m_4 = -7$.

Hence $I = \langle -4,-5,-6,-7\rangle = I_{-4}$. If $m_3 = -7$ or -11, then $[2(-3)](-3)(-1) = -18 = (-2)(-4)(-1)+(-2)(-5)(-1) \in I$ but $2(-3) \notin I$, $2(-1) \notin I$, a contradiction. Hence $m_3 \neq -7$ and $m_3 \neq -11$.

Case (2): Let $m_1 = -3$. If $m_2 \leq -5$, then by Lemma 1, choose smallest number $t \geq 3$ such that $((-2)^t(-1)^{t+1})(-1)(-1) \in I$. Then $[2(-2)^{t-2}(-1)^{t-1}](-2)\ (-1) \in I$ but $2(-2)^{t-2}(-1)^{t-1} \notin I$, $2(-2) \notin I$, $2(-1) \notin I$, which is not possible. Hence $m_2 = -4$. If $I = \langle -3,-4\rangle = I_{-3} \setminus \{-5\}$, then we are through. If $I \neq \langle -3,-4\rangle$, then $m_3 = -5$. Hence $I = I_{-3} = \langle -3,-4,-5\rangle$.

Case (3): Let $m_1 = -2$. Then, by Lemma 4, for any odd number $n \leq -3, I = \langle -2, n\rangle$.

Conversely, assume that I is one of the given types. If $I = \langle -2, n\rangle$, where n is an odd number, then by Lemma 4, I is an α-prime ideal. Clearly, $\langle -3,-4\rangle = \{-3,-4\} \cup I_{-5}$ is an α-prime ideal. Also, $I_{-3} = \langle -3,-4,-5\rangle$ is an α-prime ideal. By Lemma 6, $\langle -4,-5,-6\rangle$, $\langle -4,-5,-6,-7\rangle$, $\langle -4,-6,n\rangle$, $\langle -4,-6,n,n-2\rangle$ are α-prime ideals, where n is an odd number ≤ -7. □

Lemma 7. *Let $I = \langle -4, 2n\rangle$ be an ideal of $\mathbb{Z}_0^-$, where $n \in \mathbb{Z}^-$. Then I is an α-prime ideal if and only if $-4 \leq n \leq -1$.*

Proof. Let $I = \langle -4, 2n\rangle$ be an α-prime ideal of ternary semiring $\mathbb{Z}_0^-$, where $n \in \mathbb{Z}^-$. If $n \leq -5$, then by Lemma 1, choose smallest number $t \geq 2$ such that $2(-3)^t(-1)^t(-1) \in I$. Now $[2(-3)][(-3)^{t-1}(-1)^t](-1) \in I$ but $2(-3) \notin I$, $2(-3)^{t-1}(-1)^t \notin I$ and $2(-1) \notin I$, which is not possible. Hence $-4 \leq n \leq -1$.

Conversely, assume that $-4 \leq n \leq -1$. If $n = -1,-2,-4$, then $I = \langle -4,-2\rangle = \langle -2\rangle$, $I = \langle -4,-4\rangle = \langle -4\rangle$ and $I = \langle -4,-8\rangle = \langle -4\rangle$ respectively. Therefore, by Theorem 3, I is an α-prime ideal. If $n = -3$, then by Lemma 6, $\langle -4,-6\rangle$ is an α-prime ideal. □

The following theorem gives a characterization of non-principal finitely generated α-prime ideals in the ternary semiring $\mathbb{Z}_0^-$.

Theorem 5. *Let I be a proper, non-principal ideal of $\mathbb{Z}_0^-$. Suppose that $I = \langle m_1, m_2, \ldots, m_k\rangle$, where $m_k < \cdots < m_2 < m_1 < -1$ and $m_i \nmid m_j$ for all $i < j$ and $(m_1, m_2, \ldots, m_k) = d$, where $-1 > -d = (-p_1)^{r_1}(-p_2)^{r_2}\cdots(-p_s)^{r_s}(-1)^{(\sum_{i=1}^{s} r_i)+1} > m_1$, where $p_1, p_2, \ldots, p_k \in \mathbb{N}$ are pairwise distinct prime numbers and $r_i, s \in \mathbb{N}$. Then I is an α-prime ideal if and only if $I = \langle -4,-6\rangle$ or $I = \langle -2p, np\rangle$, where p is an odd prime number and n is an odd number ≤ -3.*

Proof. Let I be an α-prime ideal. If $r_i \geq 2$, for some i (say $r_1 \geq 2$), then by Lemma 1, choose the smallest number $t \geq 1$ such that $(-2)^t(-d)(-1)^t \in I$. Now $[2(-2)^{t-1}(-1)^t][(-p_1)^{r_1-1}(-p_2)^{r_2}\cdots(-p_s)^{r_s}(-1)^{(\sum_{i=1}^{s} r_i)}](-p_1) \in I$ but $2(-2)^{t-1}(-1)^t \notin I$, $2(-p_1)^{r_1-1}(-p_2)^{r_2}\cdots(-p_s)^{r_s}(-1)^{(\sum_{i=1}^{s} r_i)} \notin I$ and $2(-p_1) \notin I$, a contradiction. Hence $r_i = 1$, for all i. So, $-d = (-p_1)(-p_2)\cdots(-p_s)(-1)^{s+1}$. If $s \geq 2$, then by Lemma 1, choose the smallest $t \geq 1$ such that $((-2)^t(-1)^{t+1})(-d)(-1) \in I$ i.e. $[2(-2)^{t-1}(-1)^t][(-p_1)][(-p_2)\cdots(-p_s)(-1)^s] \in I$ but $2(-2)^{t-1}(-1)^t \notin I$, $2(-p_1) \notin I$ and $2[(-p_2)\cdots(-p_s)(-1)^s] \notin I$, a contradiction. Hence $s = 1$. So, $d = -p_1 = -p$ (say). Let $m_i = (-1)n_i(-p)$, where $n_k < \cdots < n_2 < n_1 < -1$. If $n_1 \leq -3$, then by Lemma 1, choose the smallest $t \geq 2$ such that $((-2)^t(-1)^{t+1})(-p)(-1) \in I$ i.e.$[2(-2)^{t-1}(-1)^t](-p)(-1) \in I$ but $2(-2)^{t-1}(-1)^t \notin I$, $2(-p) \notin I$ and $2(-1) \notin I$, a contradiction. Hence $n_1 = -2$. Since $m_1 \nmid m_2$ i.e. $(-2) \nmid n_2$, we have n_2 is an odd number ≤ -3. Denote $n_2 = n$. Now $\langle (-1)(-2)(-p), (-1)n(-p)\rangle = \{(-2)r(-p) : \frac{n+1}{2} \leq r \leq 0\} \cup \{(-1)r(-p) : r \leq n\}$ and $n = n_2 > n_3$ implies n_3 does not exist i.e. m_3 does not exist. Hence $\langle (-1)(-2)(-p), (-1)n(-p)\rangle = I$.

Case (1): If $p = 2$, then $I = \langle -4, (-1)n(-2)\rangle$. Then, by Lemma 7, $-4 \leq n < -1$. Since I is a non-principal ideal, we have $n \neq -1$, $n \neq -2$, $n \neq -4$. Hence $n = -3$. Therefore, $I = \langle -4, -6\rangle$.
Case (2): If p is an odd prime number, then $I = \langle(-1)(-2)(-p), (-1)n(-p)\rangle = \{(-2)r(-p) : \frac{n+1}{2} \leq r \leq 0\} \cup \{(-1)r(-p) : r \leq n\}$.
Conversely, assume that $I = \langle -4, -6\rangle$ or $I = \langle(-1)(-2)(-p), (-1)n(-p)\rangle$, where p is an odd prime number and n is an odd number ≤ -3. If $I = \langle -4, -6\rangle$, then by Lemma 7, I is an α-prime ideal. Suppose that $I = \langle(-1)(-2)(-p), (-1)n(-p)\rangle = \{(-2)r(-p) : \frac{n+1}{2} \leq r \leq 0\} \cup \{(-1)r(-p) : r \leq n\}$. Let $(2a)bc \in I$ and $2abc \neq 0$. Then $(-p)|2abc$ implies $(-p)|a$ or $(-p)|b$ or $(-p)|c$. We may assume that $(-p)|a$. Therefore, $a = (-1)t(-p)$. If $\frac{n+1}{2} \leq t \leq 0$, then $2a = (-2)t(-p) \in I$. If $t < \frac{n+1}{2}$, then $2t \leq n$. Hence $2a = (-2)t(-p) \in I$. Thus, I is an α-prime ideal. □

From Theorem 3, Theorem 4 and Theorem 5, we have the following corollary in which a characterization of α-prime ideals in the ternary semiring $\mathbb{Z}_0^-$ is proved.

Corollary 1. *A proper ideal I of the ternary semiring $\mathbb{Z}_0^-$ is an α-prime ideal if and only if I is one of the following types:*
1) $I = \langle 0\rangle$;
2) $I = \langle -p\rangle$, where p is a prime number;
3) $I = \langle -2p\rangle$, where p is a prime number;
4) $I = \langle -2p, np\rangle$, where p is an odd prime and n is an odd number ≤ -3;
5) $I = \langle -2, n\rangle$, where n is an odd number ≤ -3;
6) $I = \langle -3, -4\rangle$;
7) $I = \langle -3, -4, -5\rangle$;
8) $I = \langle -4, -5, -6\rangle$;
9) $I = \langle -4, -5, -6, -7\rangle$;
10) $I = \langle -4, -6\rangle$;
11) $I = \langle -4, -6, n\rangle$, where n is an odd number ≤ -7;
12) $I = \langle -4, -6, n, n-2\rangle$, where n is an odd number ≤ -7.

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References

[1] P. J. Allen and L. Dale, *Ideal theory in the semiring $\mathbb{Z}_0^+$*, Publ. Math. Debrecen, **22** (1975), 219–224.

[2] D. R. Bonde and J. N. Chaudhari, *On α-prime ideals in the semiring of non-negative integers*, Journal of the Indian Math. Soc., (3–4) **88** (2021), 250–257.

[3] D. R. Bonde and J. N. Chaudhari, *On α-prime and weakly α-prime ideals in semirings*, Asian-European Journal of Mathematics, **8** (2021), 2150128 (9 pages).

[4] J. N. Chaudhari and K. J. Ingale, *Ideals in the ternary semiring of non-positive integers*, Bull. Malaysian Math. Sci. Soc., (2)(4) **37** (2014), 1149–1156.

[5] J. N. Chaudhari and K. J. Ingale, 3-*absorbing principal T-ideals in the ternary semiring of non-positive integers*, Journal of the Indian Math. Soc., (1-2) **86** (2019), 38–45.

[6] T.K. Dutta and S. Kar, *On regular ternary semiring*, Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific (2003), 343–355.

[7] T.K. Dutta, S. Kar, *On prime ideals and prime radical of ternary semirings*, Bull. Calcutta Math. Soc., (5) **97** (2005), 445–454.

[8] T. K. Dutta and S. Kar, *On semiprime ideals and irreducible ideals of ternary semirings*, Bull. Calcutta Math. Soc., (5) **97** (2005), 467–476.

[9] J. S. Golan, *Semirings and their applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[10] S. Kar, *Ideal theory in the ternary semiring $\mathbb{Z}_0^-$*, Bull. Malays. Math. Sci. Soc., (1)(2) **34** (2011), 69–77.

[11] T. Khumprapussorn, *On α-prime and weakly α-prime submodules*, European J. Pure Appl. Math., **3** (2018), 730–739.

[12] W. G. Lister, *Ternary rings*, Trans. Amer. Math. Soc., **154** (1971), 37–55.