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Solving a class of nonlinear two-dimensional Volterra integral equations by using two-dimensional triangular orthogonal functions

Farshid Mirzaee[*]

Department of Mathematics, Faculty of Science, Malayer University, Malayer, Iran
Email: f.mirzaee@malayeru.ac.ir

Abstract. In this paper, the two-dimensional triangular orthogonal functions (2D-TFs) are applied for solving a class of nonlinear two-dimensional Volterra integral equations. 2D-TFs method transforms these integral equations into a system of linear algebraic equations. The high accuracy of this method is verified through a numerical example and comparison of the results with the other numerical methods.

Keywords: Nonlinear two-dimensional Volterra integral equations, triangular orthogonal functions, Two-dimensional triangular orthogonal functions, orthogonal functions.
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1 Introduction

Many problems in applied mathematics and physics give rise to nonlinear two-dimensional Volterra integral equations of the second kind [10, 12]

$$u(x,y) = f(x,y) + \int_0^x \int_0^y k(x,y,t,s)[u(t,s)]^n dtds, \quad (x,y) \in D, \qquad (1)$$

[*]Corresponding author.
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where $u(x, y)$ is the unknown function on $D = [0, 1) \times [0, 1)$ and $f(x, y)$, $k(x, y, t, s)$ are given continuous functions defined on D and $E = D \times D$, respectively, and $n \in N$. While several numerical methods for approximating the solution of one-dimensional Volterra integral equations are known, only a few of them have been applied in two-dimensional ones. It seems that the numerical solution of these equations has been considered first by Beltyukov and Kuznechikhina in [1] where they proposed an explicit Rung-Kutta type method of order 3 without any convergence analysis. A bivariate cubic spline functions method of full continuity was obtained by Singh in [15]. Brunner et al. in [4] introduced collocation and iterated collocation methods for two-dimensional linear Volterra integral equations. An asymptotic error expansion of the iterated collocation solution for two-dimensional linear and nonlinear Volterra integral equations was obtained by Han and Zbang in [9] and Guoqiang et al. in [7], respectively. More recently, Hadizadeh et al. in [8] have investigated a differential transformation approach for nonlinear two-dimensional Volterra integral equations. Maleknejad et al. in [11] introduced two-dimensional block-pulse functions (2D-BFs) for two-dimensional nonlinear Volterra integral equations. In this paper, we are concerned with 2D-TFs method for nonlinear two-dimensional Volterra integral equations of the form (1).

2 Brief review of 2D-TFs

An $m.m = m^2$-set of 2D-BFs $\Phi_{i,j}(x, y)$ for each $i, j = 0, 1, 2, \ldots, m-1$ is defined in $(x, y) \in D$ as follows

$$\Phi_{i,j}(x, y) = \begin{cases} 1, & ih_1 \le x < (i+1)h_1 \text{ and } jh_2 \le y < (j+1)h_2, \\ 0\ , & \text{otherwise,} \end{cases}$$

where m is an arbitrary positive integer, and $h = h_1 = h_2 = \frac{1}{m}$ [11]. Now, we demonstrate the construction of 2D-TFs according to [13, 14]

$$\Phi_{i,j}(x, y) = T1_{i,j}(x, y) + T2_{i,j}(x, y), \quad (x, y) \in D,$$

where

$$T1_{i,j}(x, y) = \begin{cases} 1 - \frac{y-jh}{h}, & ih \le x < (i+1)h \text{ and } jh \le y < (j+1)h, \\ 0\ , & \text{otherwise,} \end{cases}$$

$$T2_{i,j}(x, y) = \begin{cases} \frac{y-jh}{h}, & ih \le x < (i+1)h \text{ and } jh \le y < (j+1)h, \\ 0, & \text{otherwise,} \end{cases}$$

We can generate two vectors of 2D-TFs, namely $T1(x,y)$ and $T2(x,y)$, such that

$$\Phi(x,y) = T1(x,y) + T2(x,y), \quad (x,y) \in D.$$

It could be said that these two vectors are complementary to each other as far as 2D-BFs are considered. We call $T1(x,y)$ and $T2(x,y)$ the left-handed 2D-TFs (LH2D-TFs) and the right-handed 2D-TFs (RH2D-TFs), respectively.

Now, if we divide the interval D into $m.m = m^2$ equal parts, we have

$$\begin{aligned} T1(x,y) &= [T1_{0,0}(x,y,)\dots, T1_{0,m-1}(x,y), \dots, T1_{m-1,0}(x,y), \\ &\qquad \dots, T1_{m-1,m-1}(x,y)]^T, \\ T2(x,y) &= [T2_{0,0}(x,y,)\dots, T2_{0,m-1}(x,y), \dots, T2_{m-1,0}(x,y), \\ &\qquad \dots, T2_{m-1,m-1}(x,y)]^T, \end{aligned} \tag{2}$$

for every $(x,y) \in D$. Let $T(x,y)$ be a $(2m^2)$-vector defined as

$$T(x,y) = \begin{pmatrix} T1(x,y) \\ T2(x,y) \end{pmatrix}, \quad (x,y) \in D,$$

where $T1(x,y)$ and $T2(x,y)$ have been defined in Eq. (2).

The orthogonality of LH2D-TFs set (similarly RH2D-TFs set) resulted from mutual disjointness of LH2D-TFs (and RH2D-TFs), i.e., for i_1, i_2, j_1, $j_2 = 0, 1, \dots, m-1$, we have

$$\int_0^1 \int_0^1 T1_{i_1,j_1}(x,y) T2_{i_2,j_2}(x,y) dx dy = \begin{cases} \frac{h^2}{3}, & i_1 = i_2 \text{ and } j_1 = j_2, \\ 0, & \text{otherwise.} \end{cases}$$

The following properties of the product of two 2D-TFs vectors will be used

$$T1(x,y)T1^T(x,y) = \begin{pmatrix} T1_{0,0}(x,y) & 0 & \dots & 0 \\ \ddots & \vdots & \vdots & \vdots \\ 0 & T1_{0,m-1}(x,y) & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & T1_{m-1,0}(x,y) & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T1_{m-1,m-1}(x,y) \end{pmatrix}, \tag{3}$$

$$T2(x,y)T2^T(x,y) = \begin{pmatrix} T2_{0,0}(x,y) & 0 & \dots & 0 \\ \ddots & \vdots & \vdots & \vdots \\ 0 & T2_{0,m-1}(x,y) & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & T2_{m-1,0}(x,y) & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T2_{m-1,m-1}(x,y) \end{pmatrix}, \tag{4}$$

where both of them are m^2-by-m^2 matrices, and

$$\begin{array}{l} T1(x,y)T2^T(x,y) = 0, \\ T2(x,y)T1^T(x,y) = 0, \end{array} \tag{5}$$

where 0 is the zero $m^2 \times m^2$ matrix [13].

We can approximate the function $f(x,y) \in L^2(D)$ by 2D-TFs as follows

$$f(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{i,j} T1_{i,j}(x,y) + \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{i,j+1} T2_{i,j}(x,y) \ , \tag{6}$$

where

$$\begin{cases} c_{i,j} & = & f(ih, jh), \\ c_{i,j+1} & = & f(ih, (j+1)h). \end{cases} \tag{7}$$

Hence, the expansion of $f(x,y)$ with respect to 2D-TFs can be written as

$$f(x,y) = F1^T T1(x,y) + F2^T T2(x,y) = F^T T(x,y) \ ,$$

where $F1$ and $F2$ are 2D-TFs coefficients with $F1_{i,j} = f(ih, jh)$ and $F2_{i,j} = f(ih, (j+1)h)$, for $i, j = 0, 1, 2, \ldots, m-1$. Also, $2m^2$-vector F is defined as

$$F = \begin{pmatrix} F1 \\ F2 \end{pmatrix} \ .$$

Approximating function $k(x,y,t,s) \in L^2(D \times D)$ by 2D-TFs, as described in Eq. (6), yields

$$\begin{aligned} k(x,y,t,s) \ &= \ T1^T(x,y)K11T1(t,s) + T1^T(x,y)K12T2(t,s) \\ &\quad +T2^T(x,y)K21T1(t,s) + T2^T(x,y)K22T2(t,s), \end{aligned} \tag{8}$$

where $K11$, $K12$, $K21$, $K22$ in above-stated equation, were previously defined in [13, 14]. It is apparent from Eqs. (7) and (8) that the representation by 2D-TFs does not need any integration to evaluate the coefficients, therefore a lot of computational efforts is reduced.

3 Main results

Let X be a $(2m^2)$-vector which can be written as $X^T = (X1^T \ X2^T)$ such that $X1$ and $X2$ are m^2-vectors. Now, it can be concluded from Eqs.

(3)-(8) that

$$
\begin{aligned}
T(x,y)T^T(x,y)X &= \begin{pmatrix} T1(x,y) \\ \\ T2(x,y) \end{pmatrix} \left(T1^T(x,y) \;\; T2^T(x,y) \right) \begin{pmatrix} X1 \\ \\ X2 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{diag}(T1(x,y)) & 0_{m^2\times m^2} \\ \\ 0_{m^2\times m^2} & \mathrm{diag}(T2(x,y)) \end{pmatrix}_{2m^2\times 2m^2} \begin{pmatrix} X1 \\ \\ X2 \end{pmatrix} \\
&= \mathrm{diag}(T(x,y))X = \mathrm{diag}(X)T(x,y).
\end{aligned}
$$

Therefore,

$$
T(x,y)T^T(x,y)X = \hat{X}T(x,y), \tag{9}
$$

where $\hat{X} = \mathrm{diag}(X)$ is a $2m^2 \times 2m^2$ diagonal matrix. Now, let B be a $2m^2 \times 2m^2$ matrix as

$$
B = \begin{pmatrix} B11_{m^2\times m^2} & B12_{m^2\times m^2} \\ B21_{m^2\times m^2} & B22_{m^2\times m^2} \end{pmatrix} .
$$

So, it can be similarly concluded from Eqs. (3)-(5) that

$$
\begin{aligned}
T(x,y)BT^T(x,y) &= \left(T1^T(x,y) \;\; T2^T(x,y) \right) \begin{pmatrix} B11 & B12 \\ B21 & B22 \end{pmatrix} \begin{pmatrix} T1(x,y) \\ T2(x,y) \end{pmatrix} \\
&= T1^T(x,y)B11T1(x,y) + T2^T(x,y)B22T2(x,y) \qquad (10) \\
&= \hat{B}11^T T1(x,y) + \hat{B}22^T T2(x,y),
\end{aligned}
$$

in which $\hat{B}11$ and $\hat{B}22$ are m^2-vectors with elements equal to the diagonal entries of matrices $B11$ and $B22$, respectively [2]. Therefore,

$$
T(x,y)BT^T(x,y) = \hat{B}^T T(x,y) \ ,
$$

where $\hat{B}$ is a $2m^2$-vector with elements equal to the diagonal entries of matrix B.

3.1 Operational matrix of integration

The integration of the vectors $T1(\tau_1,\tau_2)$ and $T2(\tau_1,\tau_2)$ defined in Eq. (2) can be approximately obtained as

$$
\begin{aligned}
\int_0^x \int_0^y T1(\tau_1,\tau_2)d\tau_1 d\tau_2 &= P1T1(x,y) + P2T2(x,y) \\
&= (E1 \otimes E)T1(x,y) + (E2 \otimes E)T2(x,y), \qquad (11)
\end{aligned}
$$

for $(x, y) \in D$ and

$$\begin{aligned}\int_0^x \int_0^y T2(\tau_1, \tau_2) d\tau_1 d\tau_2 &= P1T1(x, y) + P2T2(x, y) \\ &= (E1 \otimes E)T1(x, y) + (E2 \otimes E)T2(x, y), \qquad (12)\end{aligned}$$

for $(x, y) \in D$ where $P1$ and $P2$ are the operational matrix of integration for 2D-TFs and E, $E1$ and $E2$ are the operational matrix of one-dimensional triangular orthogonal functions defined over $[0, 1)$ with $h = \frac{1}{m}$ as follows

$$E = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m \times m},$$

$$E1 = \frac{h}{2} \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times m},$$

$$E2 = \frac{h}{2} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m \times m}.$$

In Eqs. (11) and (12), $\otimes$ denotes the Kronecker product defined which is defined as

$$A \otimes B = (a_{ij} B) .$$

3.2 Operational matrix

Expressing $\int_0^x \int_0^y T(\tau_1, \tau_2) d\tau_1 d\tau_2$ in terms of $T(x, y)$, and from Eqs. (11), (12), we can write

$$\begin{aligned}\int_0^x \int_0^y T(\tau_1, \tau_2) d\tau_1 d\tau_2 &= \int_0^x \int_0^y \begin{pmatrix} T1(\tau_1, \tau_2) \\ T2(\tau_1, \tau_2) \end{pmatrix} d\tau_1 d\tau_2 \\ &= \begin{pmatrix} P1T1(x, y) + P2T2(x, y) \\ P1T1(x, y) + P2T2(x, y) \end{pmatrix} \\ &= \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix} \begin{pmatrix} T1(x, y) \\ T2(x, y) \end{pmatrix} ,\end{aligned}$$

and hence

$$\int_0^x \int_0^y T(\tau_1, \tau_2) = PT(x, y), \qquad (13)$$

where $P_{2m^2 \times 2m^2}$, operational matrix of $T(x, y)$, is

$$P = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix}, \qquad (14)$$

where $P1$ and $P2$ are given by Eq. (10). So, for every function $f(\tau_1, \tau_2)$ we have

$$\int_0^x \int_0^y f(\tau_1, \tau_2) d\tau_1 d\tau_2 = \int_0^x \int_0^y F^T T(\tau_1, \tau_2) d\tau_1 d\tau_2 = F^T PT(x, y) \ .$$

4 Nonlinear two-dimensional Volterra integral equations

In this section, we present a 2D-TFs method for solving Eq. (1). Let us expand $f(x, y)$ and $u(x, y)$ by 2D-TFs (LH2D-TFs and RH2D-TFs) as follows

$$\begin{aligned} f(x, y) &= F^T T(x, y) = T^T(x, y) F \ , \\ u(x, y) &= U^T T(x, y) = T^T(x, y) U \ , \end{aligned} \qquad (15)$$

where $2m^2$-vectors F and U are 2D-TFs coefficients of $f(x, y)$ and $u(x, y)$, respectively.

As described in Section 2, we can expand $k(x, y, t, s) \in L^2(D \times D)$ by 2D-TFs. Suppose that this approximation is as follows

$$\begin{aligned} k(x, y, t, s) = \ & T1^T(x, y) K11 T1(t, s) + T1^T(x, y) K12 T2(t, s) \\ & + T2^T(x, y) K21 T1(t, s) + T2^T(x, y) K22 T2(t, s). \end{aligned}$$

Therefore

$$K(x, y, t, s) = T^T(x, y) K T(t, s), \qquad (16)$$

where K is a $2m^2 \times 2m^2$ 2D-TFs coefficient matrix and can be written as

$$K = \begin{pmatrix} K11 & K12 \\ K13 & K14 \end{pmatrix}.$$

Now, we require the following lemma.

Lemma 1. *Let $2m^2$ vectors U and U_n be 2D-TFs coefficients of $u(x,y)$ and $[u(x,y)]^n$, respectively. If*

$$\begin{aligned} U = (U1^T \quad U_2^T) \;&=\; (U1_{0,0}, \ldots, U1_{0,m-1}, \ldots, U1_{m-1,0}, \ldots, U1_{m-1,m-1}, \\ &\quad\; U2_{0,0}, \ldots, U2_{0,m-1}, \ldots, U2_{m-1,0}, \ldots, U2_{m-1,m-1})^T, \end{aligned}$$

then

$$\begin{aligned} U_n \;=\; & (U1^n_{0,0}, \ldots, U1^n_{0,m-1}, \ldots, U1^n_{m-1,0}, \ldots, U1^n_{m-1,m-1}, \\ & U2^n_{0,0}, \ldots, U2^n_{0,m-1}, \ldots, U2^n_{m-1,0}, \ldots, U2^n_{m-1,m-1})^T, \end{aligned}$$

where $n \geq 1$, is a positive integer.

Proof. See [14]. □

For solving Eq. (1), we substitute Eqs. (15) and (16) into equation (1). Therefore,

$$\begin{aligned} U^T T(x,y) \;&=\; F^T T(x,y) + \int_0^x \int_0^y T^T(x,y) K T(t,s) T^T(t,s) U_n dtds \\ &=\; F^T T(x,y) + T^T(x,y) K \int_0^x \int_0^y T(t,s) T^T(t,s) U_n dtds\ . \end{aligned}$$

By using Eq. (9) it follows that

$$\begin{aligned} U^T T(x,y) \;&=\; F^T T(x,y) + T^T(x,y) K \int_0^x \int_0^y \tilde{U}_n T(t,s) dtds \\ &=\; F^T T(x,y) + T^T(x,y) K \tilde{U}_n \int_0^x \int_0^y T(t,s) dtds, \end{aligned}$$

and using Eq. (13) gives

$$U^T T(x,y) = F^T T(x,y) + T^T(x,y) K \tilde{U}_n P T(x,y), \tag{17}$$

in which $K\tilde{U}_n P$ is a $2m^2 \times 2m^2$ matrix. By using Eq. (10) we have

$$T^T(x,y) K \tilde{U}_n P T(x,y) = \hat{V}_n T(x,y), \tag{18}$$

where $\hat{V}_n$ is a $2m^2$-vector with components equal to the diagonal entries of the matrix. Combining Eqs. (17) and (18) gives

$$U^T T(x,y) = F^T T(x,y) + \hat{V}_n^T T(x,y). \tag{19}$$

Eq. (19) is a nonlinear system of $2m^2$ algebraic equations for the $2m^2$ unknowns

$$U1_{0,0},\ldots,U1_{0,m-1},\ldots,U1_{m-1,0},\ldots,U1_{m-1,m-1},\ldots,$$
$$U2_{0,0},\ldots,U2_{0,m-1},\ldots,U2_{m-1,0},\ldots,U2_{m-1,m-1}\ .$$

The Newton's iteration method is used to solve the nonlinear system. Hence, an approximate solution $u(x,y)=U^T T(x,y)$, or

$$u(x,y)=U1^T T1(x,y)+U2^T T2(x,y),$$

can be computed for Eq. (1) without using any projection method.

5 Convergence analysis

Let $(C[J],\|.\|)$ be the Banach space of all continuous functions on $J=D$ with norm $\|f(x,y)\|=\max\limits_{(x,y)\in J}|f(x,y)|$. Let $\forall x,y,t,s\in[0,1)$, $|k(x,y,t,s)|\leq M$. Suppose the nonlinear term $[u(x,y)]^n$ satisfies the Lipschitz condition

$$|[u(x,y)]^n-[v(x,y)]^n|\leq L|u(x,y)-v(x,y)|.$$

We denote the error 2D-TFs by $e_{2D-TFs}=||u_m(x,y)-u(x,y)||$, where $u_m(x,y)$ and $u(x,y)$ show the approximate and exact solutions of the two-dimensional nonlinear Volterra integral equation, respectively. Note that the coefficients $c_{i,j}$'s and $c_{i,j+1}$'s in Eq. (7) are not optimal. By using the optimal coefficients, the representational errors e_{2D-TFs} can be reduced.

Theorem 1. *The solution of the two-dimensional nonlinear Volterra integral equation by using 2D-TFs approximation converges if* $0<\alpha<1$ *where* $\alpha=ML$.

Proof. Let

$$\begin{aligned}
&||u_m(x,y)-u(x,y)||\\
&\quad=\max_{(x,y)\in J}|u_m(x,y)-u(x,y)|\\
&\quad=\max|f(x,y)+\int_0^x\int_0^y k(x,y,t,s)[u_m(t,s)]^n dtds-f(x,y)\\
&\qquad-\int_0^x\int_0^y k(x,y,t,s)[u(t,s)]^n dtds|\\
&\quad\leq\max\int_0^x\int_0^y|k(x,y,t,s)||[u_m(t,s)]^n-[u(t,s)]^n|dtds\\
&\quad\leq ML\int_0^x\int_0^y\max|u_m(t,s)-u(t,s)|dsdt=ML||u_m-u||.
\end{aligned}$$

Therefore,

$$||u_m(x,y) - u(x,y)|| \leq \alpha||u_m(x,y) - u(x,y)||,$$

where $\alpha = ML$. We get $(1-\alpha)||u_m - u|| \leq 0$ and choose $0 < \alpha < 1$, by increasing m, it implies $||u_m - u|| \to 0$ as $m \to \infty$ and this completes the proof. □

6 Numerical experiments

In this section, the theoretical results of the previous sections are used for one numerical example. The numerical experiments are carried out for the selected grid points that are proposed as $2^{-l},\ l = 1,2,3,4,5,6$ and m terms of the 2D-TFs and 2D-BFs series.

Example 1. Consider the two-dimensional nonlinear Volterra integral equation [7, 8, 11]

$$u(x,y) = f(x,y) + \int_0^x \int_0^y u^2(t,s)dtds, \tag{20}$$

where $(x,y) \in D$ and

$$f(x,y) = x^2 + y^2 - \frac{1}{45}xy(9x^4 + 10x^2y^2 + 9y^4),$$

that the exact solution is $u(x,y) = x^2 + y^2$.

Table 1 shows the numerical results for $m = 16$ and $m = 32$ and comparison with the exact solution. Table 2 gives the comparison of the results of the error functions obtained by the present method and those of the 2D-TFs method for $m = 8,\ 16,\ 32$.

7 Conclusion

In this paper, we have worked out a computational method for the approximate solution of a class of nonlinear Volterra integral equation of the second kind, based on the expansion of the solution as series of 2D-TFs. Some advantage of the considered method (compared with methods based on basis set of different kinds) are:

1) Using 2D-TFs does not need any integration to evaluate the coefficients, therefore a lot of computational efforts have been reduced;

2) The matrices $P1$ and $P2$ introduced in Eq. (14) contain a large percentage of zero entries, which keeps computational effort within reasonable

Table 1: Numerical results for Example 1 by 2D-TFs method.

(x, y)	Presented method $m = 16$	Presented method $m = 32$	Error $m = 16$	Error $m = 32$	Exact solution
(0.0,0.0)	0	0	0	0	0
(0.1,0.1)	1.4844e-02	1.8945e-02	5.1560e-03	1.0550e-03	2.0000e-02
(0.2,0.2)	7.5792e-02	7.5393e-02	4.2080e-03	4.6070e-03	8.0000e-02
(0.3,0.3)	1.5315e-01	1.6935e-01	2.6850e-02	1.0650e-02	1.8000e-01
(0.4,0.4)	3.0173e-01	3.0083e-01	1.8270e-02	1.9170e-02	3.2000e-01
(0.5,0.5)	5.0060e-01	5.0019e-01	6.0000e-04	1.9000e-04	5.0000e-01
(0.6,0.6)	6.7821e-01	7.1310e-01	4.1790e-02	6.9000e-03	7.2000e-01
(0.7,0.7)	9.6562e-01	9.6368e-01	1.4380e-02	1.6320e-02	9.8000e-01
(0.8,0.8)	1.2061e-00	1.2520e-00	7.3900e-02	2.8000e-02	1.2800e-00
(0.9,0.9)	1.5183e-00	1.5782e-00	1.0170e-01	4.1800e-02	1.6200e-00

Table 2: Numerical results of the error functions of Eq. (20)

$(x, y) =$ $(2^{-l}, 2^{-l})$	Presented method $m = 8$	Presented method $m = 16$	Presented method $m = 32$	Method of [11] $m = 8$	Method of [11] $m = 16$	Method of [11] $m = 32$
$l = 1$	6.0e-02	2.7e-02	6.5e-03	1.3e-01	6.3e-02	3.1e-02
$l = 2$	1.2e-02	1.5e-02	6.7e-03	6.5e-02	3.2e-02	1.6e-02
$1 = 3$	4.6e-03	3.0e-03	3.8e-03	3.4e-02	1.6e-02	8.0e-03
$1 = 4$	1.6e-03	1.1e-03	7.4e-04	1.0e-03	8.5e-03	4.1e-03
$1 = 5$	1.4e-03	4.0e-04	7.0e-04	4.8e-03	2.5e-04	2.1e-03
$1 = 6$	1.2e-03	4.5e-04	1.5e-05	6.3e-03	1.2e-03	6.4e-05

limits.
An example with satisfactory results are used to demonstrate the application of this method.

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