Journal of Algebra and Related Topics
Vol. 12, No 02, (2024), pp 53-73

NEUTROSOPHIC PRIMARY SUBMODULE, LOCALIZATION AND RESIDUAL QUOTIENTS

S. AZIMI, N. ZAMANI*, M.R. HOOSHMANDASL AND A. KHOJALI

Abstract. Let R be a commutative ring with identity, M be a unital R-module and let L be a complete Heyting algebra. In this paper, among results on colon structures of L-neutrosophic submodules and L-neutrosophic ideals, we introduce and study the notion of primary (and prime) L-neutrosophic submodules and give connections with primary (prime) behavior of its t, i and f components. Then, for a multiplicatively closed subset S of R, we define the notion of localization formation for an L-neutrosophic submodule λ of M and study its behavior. Some types of L-neutrosophic quotients will also be investigated.

1. Introduction

Fuzzy set theory originated with the 1965 publication of the paper ″Fuzzy sets″ by Lotfi Zadeh [19]. This seminal paper opened up new insights in a vast range of science and generalized into various basic mathematical concepts including algebra. The first attempt to the natural fuzzification of the main concepts of algebra was done by A. Rosenfeld. His well-cited paper [12] was the starting point and inspiration for most of the subsequent work in the field of fuzzy algebra. So that, fuzzy invariant subgroups, fuzzy ideals and submodules, fuzzy Galois theory and most of the fuzzy versions of the crisp abstract algebra came to exist. Most of the work done in the three decades leading up to 2000 on fuzzy (commutative) algebra is summarized in the book

MSC(2010): Primary: 06F35, 03G25; Secondary: 08A72, 13C99, 13B99
Keywords: Neutrosophic modules, Neutrosophic Primary submodules, localization, residual quotients, single valued neutrosophic subset.
Received: 31 May 2023, Accepted: 02 December 2024.
∗Corresponding author .

by J. Mordeson and D. S. Malik [9] and we refer to it for unexplained notation.
The fuzzy theory approach deals only the degree of membership but sometimes, in order to obtain the results in more realistic way, it may necessary to deal with non-membership function. To overcome the above fact, Atanassov [1] presented the intuitionistic fuzzy set in 1983 which is a generalization of the fuzzy set. The intuitionistic fuzzy set deals with the non-membership function of the element in the set and as compared to fuzzy set it is based on more intuition. Intuitionistic fuzzy structures began with the work by Biswas [2], where he applied the concept of intuitionistic fuzzy set to group theory and studied the intuitionistic subgroups of a group. In the last few years considerable works have been done on fuzzy and intuitionistic fuzzy structures in general and, fuzzy and intuitionistic fuzzy prime and primary ideals and sub-modules in particular [5, 14, 14].

In real life, we sometimes encounter the concept of degree of uncertainty that cannot be explained by fuzzy logic or intuitive logic. To overcome this problem, Smarandache introduced the concept of the neutrosophic set [3, 15, 17]. The theory of neutrosophic sets is a powerful tool to deal with incomplete, indeterminate and inconsistent information which exist in the real world [6, 10]. A type of neotrosophistic algebraic structures based on numerical components t =truth, i =indeterminacy and f = falsity (which is different from the neutrosophic algebraic structures based on neutrosophic numbers) was introduced and studied by F. Smarandache and Vasantha [16]. Then, based on this definition and structure V. Çetkin and H. Aygün introduced and studied some of the properties of neutrosophic submodules [4]. However, on the author's knowledge, there is not any other contribution on the structure of neutrosophic submodules and thus the results appeared may throw a light on the subject. In this paper we define the concepts of colons and primary (prime) neutrosophic submodules, and study their properties. Then, we introduce and study the idea of localization formation and types of quotients neutrosophic submodules. In section 2, we recall some preliminaries and pay attention to colon neutrosophic ideals and submodules. It contains some key lemmas on products and sums of neutrosophic submodules and a theorem (Theorem 2.4) saying that colons of neutrosophic submodules (ideals) are neutrosophic ideals (submodules). Section 3 contains the notion of primary neutrosophic submodules π from which we give a result for this kind of objects π in terms of its component functions t_π, i_π and f_π. Section 4 deals with the important notion of localization. For a multiplicatively closed subset S of a commutative ring R, and for an L-neutrosophic submodule λ of the

R-module M we define and give somehow lower ($[S]\lambda$) and upper ($\lambda[S]$) approximations for λ, such that $[S]\lambda \sqsubseteq \lambda \sqsubseteq \lambda[S]$. Then we formulate the localization object $S^{-1}\lambda$ and study its behaviour and the relations with $[S]\lambda$ and $\lambda[S]$. In the final section 5, we study quotient rings, quotient modules and L-single valued neutrosophic quotients. Throughout the paper, R is a commutative ring with identity element $1 \neq 0$ and M is a unital R-module with zero element θ. Let $(L, \vee, \wedge, \leq, 0, 1)$ be a complete Heyting algebra with minimal and maximal element 0 and 1, respectively.

2. L-neutrosophic residual quotients

Let U be an arbitrary non-empty set. Following [18], an L-single valued neutrosophic subset of U is defined as

$$\lambda = \{< x, t_\lambda(x), i_\lambda(x), f_\lambda(x) > | x \in U\},$$

where $t_\lambda, i_\lambda, f_\lambda : U \to L$ is (the so called) truth membership function, indeterminacy membership function and falsity membership function, respectively. The class of all L-single valued neutrosophic subsets of U will be denoted by $LSVN(U)$ and for each $\lambda \in LSVN(U)$ we may, for simplicity, denote λ by $(t_\lambda, i_\lambda, f_\lambda)$. For $x \in U$ the triple $\lambda(x) := (t_\lambda(x), i_\lambda(x), f_\lambda(x))$ is called an L-single valued neutrosophic element of λ, and for two L-single valued neutrosophic elements $\lambda(x)$ and $\mu(y)$ in λ and μ, put

$$\lambda(x) \,\bar{\wedge}\, \mu(y) = (t_\lambda(x) \wedge t_\mu(y), i_\lambda(x) \vee i_\mu(y), f_\lambda(x) \vee f_\mu(y)) \qquad (*)$$

and

$$\lambda(x) \bar{\vee} \mu(y) = (t_\lambda(x) \vee t_\lambda(y), i_\lambda(x) \wedge i_\mu(y), f_\lambda(x) \wedge f_\mu(y)), \qquad (\star)$$

for their meet and joint. For a subset $V \subseteq U$ and $t, i, f \in L$ we define $(t, i, f)_V \in LSVN(U)$ as

$$(t, i, f)_V(x) = \begin{cases} (t, i, f) & \text{if} \quad x \in V \\ (0, 1, 1) & \quad \text{otherwise.} \end{cases} \qquad (2.1)$$

When, $V = \{x\}$, then $(t, i, f)_x := (t, i, f)_{\{x\}}$ with

$$(t, i, f)_x(y) = \begin{cases} (t, i, f) & \text{if} \quad y = x \\ (0, 1, 1) & \quad \text{otherwise,} \end{cases} \qquad (2.2)$$

is called an L-single valued neutrosophic point (abbreviated neutrosophic point) (of value (t, i, f)) in U.

Following [4], for each $\lambda \in LSVN(U)$ and each $\ell \in L$, we consider the ℓ-level sets of λ as: $(t_\lambda)_\ell = \{x \in U | t_\lambda(x) \geq \ell\}$, $(i_\lambda)_\ell = \{x \in U | i_\lambda(x) \leq \ell\}$, $(f_\lambda)_\ell = \{x \in U | f_\lambda(x) \leq \ell\}$. Also, $\lambda_{\mathbf{1}} = \{x \in U | \lambda(x) = (1, 0, 0)\}$.

Definition 2.1. (Extension Principle) Let U, W be two non-empty sets and $H : U \to W$ be a surjective map. Let λ, ν be single valued neutrosophic subsets of U, W respectively. Then we can define $H(\lambda) : W \to L^3$ and $H^{-1}(\nu) : U \to L^3$ as

$$\begin{aligned} H(\lambda)(y) &:= (t_{H(\lambda)}(y), i_{H(\lambda)}(y), f_{H(\lambda)}(y)) \\ &= (\vee\{t_\lambda(x)|x \in U, H(x) = y\}, \wedge\{i_\lambda(x)|x \in U, H(x) = y\}, \\ &\qquad \wedge\{f_\lambda(x)|x \in U, H(x) = y\}), \end{aligned}$$

for all $y \in W$, and

$$H^{-1}(\nu)(x) = \nu(H(x)) = (t_\nu(H(x)), i_\nu(H(x)), f_\nu(H(x))),$$

for all $x \in U$.

Definition 2.2. ([8]) Let $\lambda = (t_\lambda, i_\lambda, f_\lambda), \mu = (t_\mu, i_\mu, f_\mu) \in LSVN(U)$.

(i) λ is L-neutrosophic subset of μ, denoted by $\lambda \sqsubseteq \mu$, if for all $x \in X$, $\lambda(x) \preceq \mu(x)$; i.e., $t_\lambda(x) \leq t_\mu(x), i_\lambda(x) \geq i_\mu(x)$ and $f_\lambda(x) \geq f_\mu(x)$. If $\lambda \sqsubseteq \mu$ and $\mu \sqsubseteq \lambda$, we set $\lambda = \mu$. For $t, i, f \in L$ and for $x \in U$, we put $(t, i, f)_x \in \lambda$ if $(t, i, f)_x \sqsubseteq \lambda$. So, $\lambda = \mu$ if and only if

$$\forall t, i, f \in L, \forall x \in U, (t, i, f)_x \in \lambda \Leftrightarrow (t, i, f)_x \in \mu.$$

(ii) The union of λ and μ, is defined as an L-neutrosophic subset $\alpha = (t_\alpha, i_\alpha, f_\alpha) \in LSVN(U)$, where for each $x \in U$, $t_\alpha(x) = t_\lambda(x) \vee t_\mu(x)$, $i_\alpha(x) = i_\lambda(x) \wedge t_\mu(x)$, and $f_\alpha(x) = f_\lambda(x) \wedge f_\mu(x)$. We simply write $\lambda \Cup \mu$ for α. For a family λ_j, $j \in J$, of L-neutrosophic subsets of X, we put

$$(\Cup_{j\in J}\lambda_j)(x) := (\vee_{j\in J} t_{\lambda_j}(x), \wedge_{j\in J} i_{\lambda_j}(x), \wedge_{j\in J} f_{\lambda_j}(x)).$$

(iii) The intersection of λ and μ is as $\beta = (t_\beta, i_\beta, f_\beta) \in LSVN(U)$, where for each $x \in U$, $t_\beta(x) = t_\lambda(x) \wedge t_\mu(x)$, $i_\beta(x) = i_\lambda(x) \vee t_\mu(x)$, and $f_\beta(x) = f_\lambda(x) \vee f_\mu(x)$.

(iv) The complement of $\lambda \in LSVN(U)$ is $\lambda^c = (f_\lambda, 1 - i_\lambda, t_\lambda) \in LSVN(U)$.

(v) If $\lambda \in LSVN(U)$, then we say that λ has the *supinf property*, if every subset of $\lambda(U)$ (i.e., the image of λ), has a supinf; i.e., for any subset V of U, there exists $v_0 \in V$ such that

$$(t_\lambda(v_0), i_\lambda(v_0), f_\lambda(v_0)) = (\vee_{v\in V} t_\lambda(v), \wedge_{v\in V} i_\lambda(v), \wedge_{v\in V} f_\lambda(v)).$$

For any unexplained facts and results about the neutrosophic single valued sets we refer to [8, 18].

Let X be a subset of the R-module M and Let I be an ideal of R. We recall that $< X >$ denotes the submodule of M generated by X, and for an element $x \in X$, $< x >$ is the cyclic submodule of M generated by $\{x\}$. The set $\mathrm{Rad}(I) := \{r \in R | r^n \in R, \text{for some} n \in \mathbb{N}\}$ denotes the radical of I. For a submodule N of M, $N : M = \{r \in R | rM \subseteq N\}$ is the residual quotient of M by N, which is an ideal of R. In this section we will study the residual quotients of L-single valued neutrosophic subsets of M and R. The next definition will be needed in the sequel.

Definition 2.3. Let $\lambda = (t_\lambda, i_\lambda, f_\lambda), \mu \in LSVN(M)$ and let $\alpha = (t_\alpha, i_\alpha, f_\alpha) \in LSVN(R)$. We define $\alpha \cdot \lambda$, and the neutrosophic residual quotients $\lambda : \mu$ and $\lambda : \alpha$ as follows:

(1) $(\alpha \cdot \lambda)(x) = \{< x, \vee(t_\alpha(r) \wedge t_\lambda(y)), \wedge(i_\alpha(r) \vee i_\lambda(y)), \wedge(f_\alpha(r) \vee f_\lambda(y)) > | r \in R, y \in M, ry = x\}$, for all $x \in M$.
(2) $(\lambda : \mu) = \Cup\{\alpha | \alpha \in LSVN(R), \alpha \cdot \mu \sqsubseteq \lambda\}$.
(3) $(\lambda : \alpha) = \Cup\{\nu | \nu \in LSVN(M), \alpha \cdot \nu \sqsubseteq \lambda\}$.

Here we should note that by Definition 2.2(ii),

$$\begin{aligned} &\Cup \{\alpha | \alpha \in LSVN(R), \alpha \cdot \mu \sqsubseteq \lambda\} \\ &= \big(\vee \{t_\alpha | \alpha \in LSVN(R), \alpha \cdot \mu \sqsubseteq \lambda\}, \wedge\{t_\alpha | \alpha \in LSVN(R), \alpha \cdot \mu \sqsubseteq \lambda\}, \\ &\qquad \wedge \{t_\alpha | \alpha \in LSVN(R), \alpha \cdot \mu \sqsubseteq \lambda\}\big), \end{aligned}$$

and a similar definition is considered for $\Cup\{\nu | \nu \in LSVN(M), \alpha\cdot\nu \sqsubseteq \lambda\}$. We also note that $\alpha \cdot \lambda, \lambda : \alpha \in LSVN(M), \lambda : \mu \in LSVN(R)$, and it is easy to see that $\alpha \cdot (\lambda \cup \mu) = (\alpha \cdot \lambda) \cup (\alpha \cdot \mu)$.

Definition 2.4. (cf.[4]) Let $\lambda \in LSVN(M)$. We will call λ an L-single valued neutrosophic submodule of M (L-neutrosophic submodule for short) if the following conditions hold.

(LM1) $\lambda(\theta) = (1, 0, 0)$,
(LM2) $\lambda(x + y) \succeq \lambda(x) \,\bar{\wedge}\, \lambda(y)$, for all $x, y \in M$,
(LM3) $\lambda(rx) \succeq \lambda(x)$ for all $r \in R$ and all $x \in M$.

Here note, for example in (LM2), that we have $t_\lambda(x + y) \geq t_\lambda(x) \wedge t_\lambda(y)$, $i_\lambda(x + y) \leq i_\lambda(x) \vee i_\lambda(y)$ and $f_\lambda(x + y) \leq f_\lambda(x) \vee f_\lambda(y)$.
The set of all L-neutrosophic submodules of M will be denoted by $LNS(M)$ and $\lambda \in LNS(M)$ is nonconstant if $\lambda \neq (1, 0, 0)_M$. Also, for $M = R, LNI(R)$ denotes the set of all L-single valued neutrosophic ideals of R and $\xi \in LNI(R)$ is nonconstant if $\xi(x) \neq (1, 0, 0)_R$. By the notation as in Definition 2.1, it is easy to see that when U and W are two R-modules, $\lambda \in LNS(U)$, $\nu \in LNS(W)$ and $H : U \to W$ is an epimorphism, then $H(\lambda) \in LNS(W)$ and $H^{-1}(\nu) \in LNS(U)$.
The next theorem shows that, when $\lambda \in LNS(M)$, in the parts (2) and (3) of Definition 2.3 we can restrict the union in a relatively small sets. To prove this we need the following auxiliary lemma. Note that for $r \in R$, $x \in M$ and $t, i, f \in L$, $< (t, i, f)_r > \in LNI(R)$ is defined

as $(t,i,f)_{<r>} \cup (1,0,0)_0$, and $<(t,i,f)_x> \in LNS(M)$ is defined as $(t,i,f)_{<x>} \cup (1,0,0)_\theta$.

Lemma 2.5. *Assume that $\lambda \in LNS(M)$, $\nu \in LSVN(M)$, $\alpha \in LSVN(R)$, $r \in R, x \in M$ and $t,i,f \in L$.*

(1) *If $(t,i,f)_r \cdot \nu \sqsubseteq \lambda$, then $(t,i,f)_{<r>} \cdot \nu \sqsubseteq \lambda$.*
(2) *If $\alpha \cdot (t,i,f)_x \sqsubseteq \lambda$, then $\alpha \cdot (t,i,f)_{<x>} \sqsubseteq \lambda$.*

Proof. We prove (1); and (2) can be proven by a similar argument. Let $y \in M$. Then, by Definition 2.3(1),

$$
\begin{aligned}
&((t,i,f)_{<r>} \cdot \nu)(y)\\
&= (\bigvee_{\substack{s\in R, z\in M\\ sz=y}} (t_{<r>}(s) \wedge t_\nu(z)), \bigwedge_{\substack{s\in R, z\in M\\ sz=y}} (i_{<r>}(s) \vee i_\nu(z)),\\
&\quad \bigwedge_{\substack{s\in R, z\in M\\ sz=y}} (f_{<r>}(s) \vee f_\nu(z))\\
&= (\bigvee_{\substack{s\in <r>, z\in M\\ sz=y}} (t_r \wedge t_\nu(rz)), \bigwedge_{\substack{s\in <r>, z\in M\\ sz=y}} (i_r \vee i_\nu(rz)), \bigwedge_{\substack{s\in <r>, z\in M\\ sz=y}} (f_r \vee f_\nu(rz))\\
&\preceq (\bigvee_{\substack{a\in R, z\in M\\ arz=y}} (t_r \wedge t_\nu(rz)), \bigwedge_{\substack{a\in R, z\in M\\ arz=y}} (i_r \vee i_\nu(rz)), \bigwedge_{\substack{a\in R, z\in M\\ arz=y}} (f_r \vee f_\nu(rz))\\
&\preceq (\bigvee_{\substack{a\in R, z\in M\\ a(rz)=y}} ((t,i,f)_r \cdot \nu)(rz)\\
&\preceq \bigvee_{\substack{a\in R, z\in M,\\ a(rz)=y}} \lambda(rz) \quad \text{(by assumption in (1))}\\
&\preceq \bigvee_{\substack{a\in R, z\in M\\ a(rz)=y}} \lambda(arz)\\
&= \lambda(y).
\end{aligned}
$$

Thus $(t,i,f)_{<r>} \cdot \nu \sqsubseteq \lambda$. □

Theorem 2.6. *Assume that $\mu \in LSVN(M), \lambda \in LNS(M)$ and $\alpha \in LSVN(R)$. Then*

(1) $\lambda : \alpha = \Cup\{\nu | \nu \in LNS(M), \alpha \cdot \nu \sqsubseteq \lambda\}$.
(2) $\lambda : \mu = \Cup\{\alpha | \alpha \in LNI(R), \alpha \cdot \mu \sqsubseteq \lambda\}$.

Proof. (1) First, we show that $\lambda : \alpha = \Cup\{(t,i,f)_x | (t,i,f) \in L^3, x \in M, \alpha \cdot (t,i,f)_x \sqsubseteq \lambda\}$. To prove the nontrivial direction $\sqsubseteq$, assume that $\nu \in LSVN(M)$ and that $\alpha \cdot \nu \sqsubseteq \lambda$. Fix $x \in M$ and put $\nu(x) = (t_\nu(x), i_\nu(x), f_\nu(x)) = (t,i,f)$. Then, $(t,i,f)_x$ is an L-neutrosophic point in M and by Definition 2.3(1),

$$
\begin{aligned}
&(\alpha\cdot(t,i,f)_x)(y)\\
&=(\bigvee_{\substack{r\in R,z\in M\\ rz=y}}(t_\alpha(r)\wedge t_x(z)),\bigwedge_{\substack{r\in R,z\in M\\ rz=y}}(i_\alpha(r)\vee i_x(z)),\bigwedge_{\substack{r\in R,z\in M\\ rz=y}}(f_\alpha(r)\vee f_x(z))\\
&=(\bigvee_{\substack{r\in R\\ rx=y}}(t_\alpha(r)\wedge t_x(x)),\bigwedge_{\substack{r\in R\\ rx=y}}(i_\alpha(r)\vee i_x(x)),\bigwedge_{\substack{r\in R\\ rx=y}}(f_\alpha(r)\vee f_x(x))\\
&\preceq(\alpha.\nu)(y)\preceq\lambda(y),
\end{aligned}
$$

for each $y\in M$. This means that $(t,i,f)_x\in\lambda:\alpha$ and thus

$$\lambda:\alpha\sqsubseteq\Cup\{(t,i,f)_x|(t,i,f)\in L^3,x\in M,\alpha\cdot(t,i,f)_x\sqsubseteq\lambda\}.$$

Now, we always have $\Cup\{\nu|\nu\in LNS(M),\alpha\cdot\nu\sqsubseteq\lambda\}\sqsubseteq\lambda:\alpha$. To prove the opposite inclusion, let $x\in M$ and let $(t,i,f)\in L^3$ be such that $\alpha\cdot(t,i,f)_x\sqsubseteq\lambda$. Put $\nu=<(t,i,f)_x>$. Then, we have

$$
\begin{aligned}
\alpha\cdot\nu\ &=(\alpha\cdot((1,0,0)_\theta\cup(t,i,f)_{<x>}))\\
&=(\alpha\cdot(1,0,0)_\theta)\cup(\alpha\cdot(t,i,f)_{<x>})\\
&\sqsubseteq[(1,0,0)_\theta\cup(\alpha\cdot(t,i,f)_{<x>})]\\
&\sqsubseteq[(1,0,0)_\theta\cup\lambda]=\lambda;
\end{aligned}
$$

where the first inclusion is by the definition of $\alpha\cdot(1,0,0)_\theta$ and the second inclusion is by Lemma 2.1(2). Therefore,

$$
\begin{aligned}
\lambda:\alpha\ &=\ \Cup\{(t,i,f)_x|(t,i,f)\in L^3,x\in M,\alpha\cdot(t,i,f)_x\sqsubseteq\lambda\}\\
&\sqsubseteq\ \Cup\{\nu|\nu\in LNS(M),\alpha\cdot\nu\sqsubseteq\lambda\}.
\end{aligned}
$$

The proof of part (1) now is complete.
Part (2) is proved in a similar method. Here as an auxiliary statement one could first show that

$$\lambda:\mu=\Cup\{(t,i,f)_r|(t,i,f)\in L^3,r\in R,(t,i,f)_r\cdot\mu\sqsubseteq\lambda\}.$$

□

We omit the straightforward proof of the following lemma.

Lemma 2.7. *Let $\mu,\nu\in LSVN(M)$. For each $x\in M$, put*

$$
\begin{aligned}
&(\mu+\nu)(x)\\
&\quad=(\bigvee_{\substack{y,z\in M\\ y+z=x}}(t_\mu(y)\wedge t_\nu(z)),\bigwedge_{\substack{y,z\in M\\ y+z=x}}(i_\mu(y)\vee i_\nu(z)),\bigwedge_{\substack{y,z\in M\\ y+z=x}}(f_\mu(y)\vee f_\nu(z))).
\end{aligned}
$$

Then,

(1) $\mu\sqsubseteq\mu+\nu=\nu+\mu$,
(2) *if $\mu,\nu\in LNS(M)$, then $\mu+\nu\in LNS(M)$.*

Theorem 2.8. *Assume that $\lambda\in LNS(M),\mu\in LSVN(M)$ and $\alpha\in LSVN(R)$. Then*

(1) $\lambda:\alpha\in LNS(M)$,

(2) $\lambda : \mu \in LNI(R)$.

Proof. We only prove (1), and (2) can be proven by a similar argument. Using Definition 2.3(1), it is clear that $\alpha \cdot (1,0,0)_\theta \sqsubseteq (1,0,0)_\theta \sqsubseteq \lambda$. So, by Definition 2.3(3), $(1,0,0)_\theta \sqsubseteq (\lambda : \alpha)$ and thus $(\lambda : \alpha)(\theta) = (1,0,0)$. This means that (LM1) of Definition 2.4 holds.
To prove (LM2), let $x, y \in M$. Then,

$$\begin{aligned}
&(\lambda : \alpha)(x+y)\\
&= (\Cup\{\nu | \nu \in LNS(M), \alpha \cdot \nu \sqsubseteq \lambda\})(x+y) \quad \text{(by Theorem 2.6,)}\\
&= \big(\vee \{t_\nu(x+y) | \nu \in LNS(M), \alpha \cdot \nu \sqsubseteq \lambda\},\\
&\quad \wedge\{i_\nu(x+y) | \nu \in LNS(M), \alpha \cdot \nu \sqsubseteq \lambda\},\\
&\quad \wedge\{f_\nu(x+y) | \nu \in LNS(M), \alpha \cdot \nu \sqsubseteq \lambda\}\big)\\
&\succeq \big(\vee\{t_{(\nu_1+\nu_2)}(x+y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot (\nu_1+\nu_2) \sqsubseteq \alpha \cdot \nu_1 + \alpha \cdot \nu_2 \sqsubseteq \lambda\},\\
&\quad \wedge\{i_{(\nu_1+\nu_2)}(x+y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot (\nu_1+\nu_2) \sqsubseteq \alpha \cdot \nu_1 + \alpha \cdot \nu_2 \sqsubseteq \lambda\},\\
&\quad \wedge\{f_{(\nu_1+\nu_2)}(x+y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot (\nu_1+\nu_2) \sqsubseteq \alpha \cdot \nu_1 + \alpha \cdot \nu_2 \sqsubseteq \lambda\}\big)\\
&\succeq \big(\vee\{t_{(\nu_1+\nu_2)}(x) \wedge t_{(\nu_1+\nu_2)}(y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot \nu_1 \sqsubseteq \lambda, \alpha \cdot \nu_2 \sqsubseteq \lambda\},\\
&\quad \wedge\{i_{(\nu_1+\nu_2)}(x) \vee i_{(\nu_1+\nu_2)}(y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot \nu_1 \sqsubseteq \lambda, \alpha \cdot \nu_2 \sqsubseteq \lambda\},\\
&\quad \wedge\{f_{(\nu_1+\nu_2)}(x) \vee f_{(\nu_1+\nu_2)}(y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot \nu_1 \sqsubseteq \lambda, \alpha \cdot \nu_2 \sqsubseteq \lambda\}\big)\\
&\succeq (\vee\{t_{\nu_1}(x) \wedge t_{\nu_2}(y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot \nu_1 \sqsubseteq \lambda, \alpha \cdot \nu_2 \sqsubseteq \lambda\},\\
&\quad \wedge\{i_{\nu_1}(x) \vee i_{\nu_2}(y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot \nu_1 \sqsubseteq \lambda, \alpha \cdot \nu_2 \sqsubseteq \lambda\},\\
&\quad \wedge\{f_{\nu_1}(x) \vee f_{\nu_2}(y) | \nu_1, \nu_2 \in LNS(M), \alpha \cdot \nu_1 \sqsubseteq \lambda, \alpha \cdot \nu_2 \sqsubseteq \lambda\})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma (2.7)(2))}\\
&= (\vee\{\nu_1(x) | \nu_1 \in LNS(M), \alpha \cdot \nu_1 \sqsubseteq \lambda\})\\
&\quad \wedge(\vee\{\nu_2(y) | \nu_2 \in LNS(M), \alpha \cdot \nu_2 \sqsubseteq \lambda\})\\
&= (\lambda : \alpha)(x) \wedge (\lambda : \alpha)(y).
\end{aligned}$$

For (LM3), let $r \in R, x \in M$. Then,

$$\begin{aligned}
(\lambda : \alpha)(rx) &= \big(\vee \{\nu(rx) | \nu \in LNS(M), \alpha \cdot \nu \sqsubseteq \lambda\},\\
&\succeq \vee\{\nu(x) | \nu \in LNS(M), \alpha \cdot \nu \sqsubseteq \lambda\}\\
&= (\lambda : \alpha)(x).
\end{aligned}$$

□

3. Primary L-neutrosophic submodules

In this section we study some properties of primary L-neutrosophic submodules of M. Recall from [7, Definition 3.1] that a submodule $P \neq M$ of M is called primary provided that for each $r \in R$ and $x \in M$; $rx \in P$ gives that $x \in P$ or $r^n \in (M : P)$ for some $n \in \mathbb{N}$. Inspired by this concept, for $\xi = (t_\xi, i_\xi, f_\xi) \in LSVN(R)$, we define the L-neutrosophic nil radical $\mathfrak{N}(\xi) \in LSVN(R)$ of ξ as

$$\mathfrak{N}(\xi)(r) = (\vee_{n\in\mathbb{N}} t_\xi(r^n), \wedge_{n\in\mathbb{N}} i_\xi(r^n), \wedge_{n\in\mathbb{N}} f_\xi(r^n)), \forall r \in R.$$

By adopting an argument similar to the one in the proof of [9, Theorem 3.8.3], one can see that if $\xi \in LNI(R)$, then $\mathfrak{N}(\xi) \in LNI(R)$, and that $\mathfrak{N}(\mathfrak{N}(\xi)) = \mathfrak{N}(\xi)$.

Definition 3.1. Let $\pi \in LNS(M)$ be nonconstant. We say that π is a primary (resp. prime) L-single valued neutrosophic submodule of M, if for each $\alpha \in LNI(R)$ and each $\nu \in LNS(M)$, $\alpha \cdot \nu \sqsubseteq \pi$, gives that either $\alpha \sqsubseteq \mathfrak{N}(\pi : (1,0,0)_M)$ (resp. $\alpha \sqsubseteq (\pi : (1,0,0)_M)$) or $\nu \sqsubseteq \pi$. A primary (prime) L-single valued neutrosophic submodule of R is called a primary (prime) L-neutrosophic ideal of R. In the following we use the expression ″primary (prime) L-neutrosophic submodule (ideal)″ instead of ″primary (prime) L-single valued neutrosophic submodule (ideal)″.

Theorem 3.2. *Let $\pi \in LSVN(M)$ be a primary L-neutrosophic submodule of M. Then, for each $\ell \in L$, $(t_\pi)_\ell, (i_\pi)_\ell$ and $(f_\pi)_\ell$ are all primary submodules of M and for each $y \in M \setminus (t_\pi)_1 \cup (i_\pi)_1 \cup (f_\pi)_1$, all $t_\pi(y), i_\pi(y)$ and $f_\pi(y)$ are prime elements of L.*

Proof. Suppose that π is a primary L-neutrosophic submodule of M and let $\ell \in L$. By [4, Proposition 3.13], with L instead of $[0,1]$, all $(t_\pi)_\ell, (i_\pi)_\ell, (f_\pi)_\ell$ are submodules of M. Thus it remain only to prove that these level sets are primary sub-mudules of M.
To do this, we first consider $(t_\pi)_\ell$. Let $r \in R$, $x \in M$ such that $rx \in (t_\pi)_\ell$, and let $x \notin (t_\pi)_\ell$. We show that $r^n \in ((t_\pi)_\ell : M)$ for some $n \in \mathbb{N}$. As $x \notin (t_\pi)_\ell$, we have $(t_\pi)(x) < \ell$. Let $\nu =< (\ell,0,0)_x >$ and $\alpha =< (1,0,0)_R > \cdot$ Then, for each $y \in M$,

$$\begin{aligned}&(\alpha \cdot \nu)(y)\\ &= (\bigvee_{\substack{s\in R, z\in M\\ sz=y}} (t_\alpha(s) \wedge t_\nu(z)), \bigwedge_{\substack{s\in R, z\in M\\ sz=y}} (i_\alpha(s) \vee i_\nu(z)), \bigwedge_{\substack{s\in R, z\in M\\ sz=y}} (f_\alpha(s) \vee f_\nu(z)))\\ &= (1,0,0)_M(y)\\ &\preceq \pi(y).\end{aligned}$$

This means that $\alpha \cdot \nu \sqsubseteq \pi$. As $t_\pi(x) < \ell$, $\nu \not\sqsubseteq \pi$, and π is a primary neutrosophic submodule of M, we must have $\alpha \sqsubseteq \mathfrak{N}(\pi : (1,0,0)_M)$. This in turn implies

$$\begin{aligned}&< (\ell,0,0)_R > (r) \preceq \mathfrak{N}(\pi : (1,0,0)_M)(r)\\ &\qquad = (\vee_{n\in\mathbb{N}} t_{(\pi:(1,0,0)_M)}(r^n), \wedge_{n\in\mathbb{N}} i_{(\pi:(1,0,0)_M)}(r^n), \wedge_{n\in\mathbb{N}} f_{(\pi:(1,0,0)_M)}(r^n)).\end{aligned}$$

In particular, $\ell \le t_{(\pi:(1,0,0)_M)}(r^n)$ for some $n \in \mathbb{N}$, meaning that $\ell \le \vee\{t_\xi(r^n) | \xi \in LNI(R), \xi \cdot (1,0,0)_M \sqsubseteq \pi\}$. So, there exists $\xi \in LNI(R)$ such that $\ell \le t_\xi(r^n)$ and $\xi \cdot (1,0,0)_M \sqsubseteq \pi$. This gives that for each

$z \in M$,

$$(\bigvee_{\substack{s\in R, y\in M\\ sy=r^n z}}(t_\xi(s)\wedge 1_M(y)), \bigwedge_{\substack{s\in R, y\in M\\ sy=r^n z}}(i_\xi(s)\vee 0_M(y)), \bigwedge_{\substack{s\in R, y\in M\\ sy=r^n z}}(f_\xi(s)\vee 0_M(y)))$$
$$= (\xi\cdot(1,0,0)_M)(r^n z)\preceq \pi(r^n z).$$

Thus, in particular, $t_\xi(r^n) = t_\xi(r^n)\wedge 1_M(z)\leq t_\pi(r^n z)$. So $\ell\leq t_\pi(r^n z)$ and $r^n z\in (t_\pi)_\ell$. As z is arbitrary, this means that $r^n\in ((t_\pi)_\ell : M)$ and the proof of this part is complete.

Now, we settle $(i_\pi)_\ell$ and show that this is also a primary submodule of M. Let $r\in R$, $x\in M$ such that $rx\in (i_\pi)_\ell$, and let $x\notin (i_\pi)_\ell$. As $x\notin (i_\pi)_\ell$, we have $\ell < (i_\pi)(x)$. Let $\mu =< (0,\ell,1)_x >$ and $\beta =< (0,1,1)_R >$. Then, for each $y\in M$, we have

$$\begin{aligned}&(\beta\cdot\mu)(y)\\ &= (\bigvee_{\substack{s\in R, z\in M\\ sz=y}}(t_\beta(s)\wedge t_\mu(z)), \bigwedge_{\substack{s\in R, z\in M\\ sz=y}}(i_\beta(s)\vee i_\mu(z)), \bigwedge_{\substack{s\in R, z\in M\\ sz=y}}(f_\beta(s)\vee f_\mu(z)))\\ &= (0,1,1)_M(y)\preceq\pi(y).\end{aligned}$$

This gives that $\beta\cdot\mu\sqsubseteq\pi$. As $\mu\not\sqsubseteq\pi$ (note that $\ell < (i_\pi)(x)$) and π is a primary L-neutrosophic submodule of M, we must have $\beta\sqsubseteq\mathfrak{N}(\pi : (1,0,0)_M)$. This in turn gives that

$$\begin{aligned}&< (0,\ell,1)_R > (r)\\ &\qquad\preceq (\bigvee_{n\in\mathbb{N}} t_{(\pi:(0,1,1)_M)}(r^n), \bigwedge_{n\in\mathbb{N}} i_{(\pi:(0,1,1)_M)}(r^n), \bigwedge_{n\in\mathbb{N}} f_{(\pi:(0,1,1)_M)}(r^n)).\end{aligned}$$

In particular $\ell\geq i_{(\pi:(0,1,1)_M)}(r^n) = \wedge\{i_\zeta(r^n) | \zeta\in LNI(R), \zeta\cdot(1,0,0)_M\sqsubseteq\pi\}$ for some $n\in\mathbb{N}$. Hence, $\ell\geq i_\zeta(r^n)$ and $\zeta\cdot(1,0,0)_M\sqsubseteq\pi$ for some $n\in\mathbb{N}$ and some $\zeta\in LNI(R)$. Therefore, for each $z\in M$,

$$(\bigvee_{\substack{s\in R, y\in M\\ sy=r^n z}}(t_\zeta(s)\wedge 1_M(y)), \bigwedge_{\substack{s\in R, y\in M\\ sy=r^n z}}(i_\zeta(s))\vee 0_M(y)), \bigwedge_{\substack{s\in R, y\in M\\ sy=r^n z}}(f_\zeta(s)\vee 0_M(y))$$
$$= (\zeta\cdot(1,0,0)_M)(r^n z)\preceq\pi(r^n z).$$

This, in turn gives that $i_\zeta(r^n) = i_\zeta(r^n)\vee 0_M(z)\geq\wedge_{s\in R, y\in M, sy=r^n z}(i_\zeta(s)\vee 0_M(y))\geq i_\pi(r^n z)$. So, $\ell\geq i_\pi(r^n z)$ and we have $r^n z\in (i_\pi)_\ell$ by the paragraph preceding definition 3.1. Thus, $r^n\in ((i_\pi)_\ell : M)$ as desired. A similar argument as for $(i_\pi)_\ell$ shows that $(f_\pi)_\ell$ is a also primary submodule of M.

Now, let $y\in M\setminus (t_\pi)_1\cup (i_\pi)_1\cup (f_\pi)_1$. We show that all $t_\pi(y), i_\pi(y)$ and $f_\pi(y)$ are prime elements of L. First, assume that $1\neq (t_\pi)(y) := c$ is not a prime element in L. Then, there exist $a, b\in L$ such that $a\wedge b\leq c$, $a\not\leq c$ and $b\not\leq c$. This gives that $(1,0,0)_\theta\cup (b,0,0)_M\not\sqsubseteq\pi$ and $(1,0,0)_0\cup (a,0,0)_R\not\sqsubseteq\mathfrak{N}(\pi : (1,0,0)_M)$. On the other hand, we see that

$$\big((1,0,0)_0\cup(a,0,0)_R\big)\cdot\big((1,0,0)_\theta\cup(b,0,0)_M\big)\sqsubseteq (1,0,0)_\theta\cup(c,0,0)_M\sqsubseteq\pi.$$

As π is a primary L-neutrosophic submodule of M, this is a contradiction, and $c = (t_\pi)(y)$ must be a prime element of L.
Second, we consider $s = i_\pi(y)$ and assume that s is not a prime element of L. Then, there exist $u, v \in L$ such that $u \wedge v \leq s, u \nleq s, v \nleq s$. So, $(0,1,1)_\theta \cup (0,v,1)_M \not\sqsubseteq \pi$ and $(0,1,1)_0 \cup (0,u,1)_R \not\sqsubseteq \mathfrak{N}(\pi : (1,0,0)_M)$. But, we see that

$$\big((0,1,1)_0 \cup (0,u,1)_R\big) \cdot \big((0,1,1)_\theta \cup (0,v,1)_M\big) \sqsubseteq (1,0,0)_\theta \cup (0,s,1)_M \sqsubseteq \pi.$$

As π is a primary L-neutrosophic submodule of M, this is a contradiction, and $s = i_\pi(y)$ must be a prime element of L.
A similar argument shows that $f_\pi(y)$ must be a prime element of L. □

The following remark will be used in the sequel. Its proof is straightforward.

Remark 3.3. Let $\lambda \in LNS(M)$ such that $\lambda = (1,0,0)_{\lambda_\mathbf{1}} \cup (t,i,f)_M$. Then,

$$\lambda : (1,0,0)_M = (1,0,0)_{(\lambda_\mathbf{1}:M)} \bar{\vee} (t,i,f)_R$$

and that

$$\mathfrak{N}(\lambda : (1,0,0)_M) = (1,0,0)_{\mathrm{Rad}(\lambda_\mathbf{1}:M)} \cup (t,i,f)_R.$$

Lemma 3.4. *Let $\lambda \in LNS(M)$ be a primary L-neutrosophic submodule of M. Then, $\lambda = 1_{\lambda_\mathbf{1}} \cup (t,i,f)_M$, where $\lambda_\mathbf{1}$ is a primary submodule of M and t, i, f are prime elements in L.*

Proof. Assume that λ is a primary L-neotrosophic submodule of M. We note that by [4, Proposition 3.13], $\lambda_\mathbf{1} = (t_\lambda)_1 \cap (i_\lambda)_0 \cap (f_\lambda)_0$ is a submodule of M, and by definition of a primary L-neotrosophic submodule, $|\lambda(M)| \geq 2$. Let $x, y \notin \lambda_\mathbf{1}$ and let $\lambda(x) = (t,i,f)$. As $(t,i,f) = \lambda(x) \leq \lambda(rx)$, for all $r \in R$, we have $(t,i,f)_{\langle x \rangle} \sqsubseteq \lambda$. Now,

$$\begin{aligned}((1,0,0)_0 \cup (t,i,f)_R) \cdot \langle (1,0,0)_x \rangle \\ &= ((1,0,0)_0 \cdot \langle (1,0,0)_x \rangle) \cup \big((t,i,f)_R \cdot \langle (1,0,0)_x \rangle \\ &= (1,0,0)_\theta \cup \langle (t,i,f)_x \rangle \\ &= \langle (t,i,f)_x \rangle \sqsubseteq \lambda.\end{aligned}$$

and $\langle (1,0,0)_x \rangle \nsubseteq \lambda$. Since λ is a primary L-neutrosophic submodule of M, we should have

$$(1,0,0)_0 \cup (t,i,f)_R \sqsubseteq \mathfrak{N}\big(\lambda : (1,0,0)_M\big).$$

Now, Put $\Xi = \{\xi \in LNI(R) | \xi \cdot (1,0,0)_M \sqsubseteq \lambda\}$. Then,

$$\begin{aligned}
&\lambda(x)\\
&= (t,i,f) = \big((1,0,0)_0 \cup (t,i,f)_R\big)(1)\\
&\leq \mathfrak{N}\big(\lambda : (1,0,0)_M\big)(1)\\
&= \big(\vee_{n\in\mathbb{N}}\{t_{(\lambda:(1,0,0)_M)}(1^n), \wedge_{n\in\mathbb{N}}\{i_{(\lambda:(1,0,0)_M)}(1^n), \wedge_{n\in\mathbb{N}}\{f_{(\lambda:(1,0,0)_M)}(1^n)\big)\\
&= (\vee_{n\in\mathbb{N}} \vee_{\xi\in\Xi} t_\xi(1^n), \wedge_{n\in\mathbb{N}} \wedge_{\xi\in\Xi} i_\xi(1^n), \wedge_{n\in\mathbb{N}} \wedge_{\xi\in\Xi} f_\xi(1^n))\\
&= (\vee_{\xi\in\Xi} \vee_{n\in\mathbb{N}} t_\xi(1^n), \wedge_{\xi\in\Xi} \wedge_{n\in\mathbb{N}} i_\xi(1^n), \wedge_{\xi\in\Xi} \wedge_{n\in\mathbb{N}} f_\xi(1^n))\\
&= \big(\vee\{t_{\mathfrak{N}(\xi)}(1) | \xi \in \Xi\}, \wedge\{i_{\mathfrak{N}(\xi)}(1) | \xi \in \Xi\}, \wedge\{f_{\mathfrak{N}(\xi)}(1) | \xi \in \Xi\}\big)\\
&= (\vee\{t_\xi(1) | \xi \in \Xi\}, \wedge\{i_\xi(1) | \xi \in \Xi\}, \wedge\{f_\xi(1) | \xi \in \Xi\})\\
&= (\bigvee_{\xi\in\Xi}(t_\xi(e) \wedge (1,0,0)_M(y)), \bigwedge_{\xi\in\Xi}(i_\xi(1) \vee (0,1,1)_M(y)),\\
&\quad \bigwedge_{\xi\in\Xi}(f_\xi(1) \vee (0,1,1)_M(y)))\\
&\preceq \Cup\{\xi \cdot (1,0,0)_M(y) | \xi \in \Xi\} \preceq \lambda(y).
\end{aligned}$$

As x, y are arbitrary, we see that $\lambda(x) = \lambda(y)$ and,

$$\lambda(M) = \{x \in M | \lambda(x) = (t,i,f)\} \cup \{x \in M | \lambda(x) = (1,0,0)\}.$$

Thus $\lambda = (1,0,0)_{\lambda_\mathbf{1}} \cup (t,i,f)_M$. To prove $\lambda_\mathbf{1}$ is a primary submodule of M, let $rx \in \lambda_\mathbf{1}$ for some $r \in R$ and $x \in M$. This gives that $\langle(1,0,0)_x\rangle \cdot \langle(1,0,0)_r\rangle \sqsubseteq \lambda$, and thus either $\langle(1,0,0)_x\rangle \sqsubseteq \lambda$ or $\langle(1,0,0)_r\rangle \sqsubseteq \mathfrak{N}(\lambda : (1,0,0)_M)$. This, in turn, gives that $x \in \lambda_\mathbf{1}$ or $r^m M \subseteq \lambda_\mathbf{1}$ for some $m \in \mathbb{N}$. Hence the claim is true.

Next assume that t is not a prime element of L. Then, there exists $u, v \in L$ such that $u \nleq t$, $v \nleq t$, and $u \wedge v \leq t$. We see that

$$\begin{aligned}
&\big((1,0,0)_0 \cup (u,0,0)_R\big) \cdot \big((1,0,0)_\theta \cup (v,0,0)_M\big)\\
&\quad = [(1,0,0)_0 \cdot (1,0,0)_\theta] \cup [(1,0,0)_0 \cdot (v,0,0)_M]\\
&\qquad \cup [(u,0,0)_R \cdot (1,0,0)_\theta] \cup [(u,0,0)_R \cdot (v,0,0)_M]\\
&\quad \sqsubseteq (1,0,0)_\theta \cup (t,i,f)_M \sqsubseteq \lambda,
\end{aligned}$$

but neither $(1,0,0)_\theta \cup (v,0,0)_M \sqsubseteq \lambda$ nor and $(1,0,0)_0 \cup (u,0,0)_R \sqsubseteq \mathfrak{N}(\lambda : (1,0,0)_M)$. This contradicts with the assumption that λ is a primary L-neutrosophic submodule of M.

Now, we show that i is also a prime element of L. If it is not, then there exists $j, k \in L$, such that $j \nleq i, k \nleq i$, but $j \wedge k \leq i$. Then, we see that

$$\begin{aligned}
&((0,1,1)_0 \cup (0,j,1)_R) \cdot ((0,1,1)_\theta \cup (0,k,1)_M)\\
&\quad = (0,1,1)_0 \cdot (0,1,1)_\theta \cup (0,1,1)_0 \cdot (0,k,1)_M\\
&\qquad \cup (0,j,1)_R \cdot (0,1,1)_\theta \cup (0,j,1)_R (0,k,1)_M\\
&\quad \sqsubseteq (0,1,1)_\theta \cup (t,i,f)_M \sqsubseteq \lambda,
\end{aligned}$$

but neither $(0,1,1)_\theta \cup (0,k,1)_M \sqsubseteq \lambda$ nor $(0,1,1)_0 \cup (0,j,1)_R \sqsubseteq \mathfrak{N}(\lambda : (1,0,0)_M)$. With this contradiction i must be a prime element of L. A similar argument shows that f is also a prime element of L. □

Theorem 3.5. *Assume that* $\lambda \in LNS(M)$ *is a primary* L*-neutrosophic submodule of* M*. Then,* $\lambda_{\mathbf{1}}$ *is a primary submodule of* M *and* $(\lambda : (1,0,0)_M)(1)$ *is a prime triple of* L*.*

Proof. By Lemma 3.2, $\lambda = 1_{\lambda_{\mathbf{1}}} \cup (t,i,f)_M$, where $\lambda_{\mathbf{1}}$ is a primary submodule of M and t,i,f are prime elements in L. Also, according the remark preceding the same lemma, and Theorem 2.2(2) we have

$$\big(\lambda : (1,0,0)_M\big)(1) = \big((1,0,0)_{(\lambda_{\mathbf{1}}:M)} \Cup (t,i,f)_R\big)(1) = (t,i,f),$$

a prime triple, and the claim follows. □

By a minor modification in the proofs of the above results one can deduce the following results concerning the prime L-neutrosophic submodules of M.

Theorem 3.6. *Let* $\pi \in LSVN(M)$ *be a prime* L*-neutrosophic submodule of* M*. Then, for each* $\ell \in L$*,* $(t_\pi)_\ell, (i_\pi)_\ell, (f_\pi)_\ell$ *are all prime submodules of* M *and for each* $y \in M \setminus (t_\pi)_1 \cup (i_\pi)_1 \cup (f_\pi)_1$*, all* $t_\pi(y), i_\pi(y)$ *and* $f_\pi(y)$ *are prime elements of* L*.*

Lemma 3.7. *Let* $\lambda \in LNS(M)$ *be a prime* L*-neutrosophic submodule of* M*. Then,* $\lambda = 1_{\lambda_{\mathbf{1}}} \cup (t,i,f)_M$*, where* $\lambda_{\mathbf{1}}$ *is a prime submodule of* M *and* t,i,f *are prime elements in* L*.*

Theorem 3.8. *Assume that* $\lambda \in LNS(M)$ *is a prime* L*-neutrosophic submodule of* M*. Then,* $\lambda_{\mathbf{1}}$ *is a prime submodule of* M *and* $(\lambda : (1,0,0)_M)(1)$ *is a prime triple of* L*.*

We were not able to prove the converses of the above theorems and lemmas 3.1-3.6. Therefore, to give a proof for the inverse direction of the above results, or finding conditions under which the inverse of these results also holds true, could be the goal of further studies.

4. Localization in L-single valued neutrosophic submodules

Recall that for a multiplicative closed subset S of R, $S^{-1}R$ (resp. $S^{-1}M$) denotes the ring of fractions of R (resp. the module of fractions of M) with respect to S. We note that, the set $\mathfrak{o}(S) = \{r \in R | rs = 0, \text{for some}\, s \in S\} = \cup_{s\in S}(0 :_R s)$ is an ideal of R and the map $u : R \to S^{-1}R$, with $u(r) = r/1$ is a ring homomorphism with its kernel $\mathfrak{o}(S)$. If I is an ideal of R, then the ideal generated by $\{u(r)|r \in I\}$ is called the extended ideal of I in $S^{-1}R$ and it is denoted by I^e. If J is an ideal of $S^{-1}R$, then $J^c = \{r \in R|r/1 \in J\}$ is an ideal of R and it is called the contracted ideal of J.
In a similar way, the set $\mathfrak{O}(S) = \{x \in M|sx = 0, \text{for some}\, s \in S\} =$

$\cup_{s\in S}(\theta :_M s)$ is a submodule of M and the map $M/\mathfrak{O}(S) \hookrightarrow S^{-1}M$ is an R-monomorphism, so that one can consider $\bar{M} := M/\mathfrak{O}(S)$ as a submodule of $S^{-1}M$. Let $\Pi : M \to \bar{M}$, be the natural homomorphism $x \to \bar{x} = x + \mathfrak{O}(S)$, for all $x \in M$. By [14, 9.11 (v)], each submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M. Moreover, by [14, Theorem 3.4], there is a one to one correspondence between the set of all prime (resp. primary) submodules P of M with $(P : M) \cap S = \emptyset$ and the set of all prime (resp. primary) submodules of $S^{-1}M$ given by $P \to S^{-1}P$. In what follows we will mainly deal with the L-single valued neutrosophic analogues of this concepts.
For $\lambda \in LNS(M)$, we assign two L-single valued neutrosophic subsets $[S]\lambda$ and $\lambda[S]$ of M and an L-single valued neutrosophic subset $S^{-1}\lambda$ of $S^{-1}M$ defined by

$$[S]\lambda = \cap_{s\in S}\big((1,0,0)_s \cdot \lambda\big), \qquad (\flat)$$
$$\lambda[S](x) = (\vee_{s\in S}(t_\lambda(sx), \wedge_{s\in S} i_\lambda(sx), \wedge_{s\in S} f_\lambda(sx)) \qquad (\dagger)$$

and

$$(S^{-1}\lambda)(x/s) = \{< x/s, t_{(S^{-1}\lambda)}(x/s), i_{(S^{-1}\lambda)}(x/s), f_{(S^{-1}\lambda)}(x/s) | x/s \in S^{-1}M\}, \quad (\natural)$$

where for each $x \in M$ and $s \in S$

$$t_{(S^{-1}\lambda)}(x/s) = \vee\{\ell \in L | \bar{x}/s \in S^{-1}((t_{\Pi(\lambda)})_\ell)\},$$
$$i_{(S^{-1})\lambda)}(x/s) = \wedge\{\ell \in L | \bar{x}/s \in S^{-1}((i_{\Pi(\lambda)})_\ell)\},$$
$$f_{(S^{-1}\lambda)}(x/s) = \wedge\{\ell \in L | \bar{x}/s \in S^{-1}((f_{\Pi(\lambda)})_\ell)\}.$$

We call these sets the lower approximation, the upper approximation and the localization of λ with respect to S. It is easy to see that $(S^{-1}\lambda)(x/1) = (S^{-1}\lambda)(x/s) = (S^{-1}\lambda)(ux/1)$, for $x \in M$ and for $s, u \in S$.
Using the Extension Principle (and our abbreviated notation), one sees that

$$(S^{-1}\lambda)(x/s) = ((t_{(S^{-1}\lambda)}(x/s), i_{(S^{-1}\lambda)}(x/s), f_{(S^{-1}\lambda)}(x/s)),$$

where

$$t_{(S^{-1}\lambda)}(x/s) =$$
$$\vee\{\ell \in L | \exists r, u \in S, y \in M, rux - rsy \in \mathfrak{O}(S), \vee_{e\in\mathfrak{O}(S)} t_\lambda(y+e) \geq \ell\},$$
$$i_{(S^{-1}\lambda)}(x/s) =$$
$$\wedge\{\ell \in L | \exists r, u \in S, y \in M, rux - rsy \in \mathfrak{O}(S), \wedge_{e\in\mathfrak{O}(S)} i_\lambda(y+e) \leq \ell\},$$
$$f_{(S^{-1}\lambda)}(x/s) =$$
$$\wedge\{\ell \in L | \exists r, u \in S, y \in M, rux - rsy \in \mathfrak{O}(S), \wedge_{e\in\mathfrak{O}(S)} f_\lambda(y+e) \leq \ell\}.$$

Theorem 4.1. *Let $\lambda \in LNS(M)$. Then,*

(i) $[S]\lambda, \lambda[S] \in LNS(M)$ *and we have* $[S]\lambda \sqsubseteq \lambda \sqsubseteq \lambda[S]$.

(ii) $S^{-1}\lambda \in LNS(S^{-1}M)$.
(iii) *The elements of* $LNS(S^{-1}M)$ *are extended, i.e., for each* $\Lambda \in LNS(S^{-1}M)$*, there exists* $\lambda \in LNS(M)$ *such that* $\Lambda = S^{-1}\lambda$.

Proof. Parts (i) and (ii) are straightforward and we only prove (iii). Let $H : M \to S^{-1}M$ be the homomorphism $H(x) = x/1$. We show that $\Lambda = S^{-1}(H^{-1}(\Lambda))$. Note that by the paragraph immediately after definition 2.2, $H^{-1}(\Lambda) \in LNS(M)$, and by definition 2.2(LM3) it is concluded that $\Lambda(x/s) = \Lambda(x/1) = \Lambda(ux/1)$ for each $x \in M$ and each $s, u \in S$. So, let $t, i, f \in L$, $x \in M$ such that $(t, i, f)_{x/1} \in S^{-1}(H^{-1}(\Lambda))$. Then, $(S^{-1}(H^{-1}(\Lambda)))(x/1) \geq (t, i, f)$. But,

$$\begin{aligned}
&(S^{-1}(H^{-1}(\Lambda)))(x/1) \geq (t, i, f)\\
&\Leftrightarrow \vee\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, \vee_{e\in\mathfrak{D}(S)} t_{H^{-1}(\Lambda)}(y + e) \geq \ell\} \geq t,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, \wedge_{e\in\mathfrak{D}(S)} i_{H^{-1}(\Lambda)}(y + e) \leq \ell\} \leq i,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, \wedge_{e\in\mathfrak{D}(S)} f_{H^{-1}(\Lambda)}(y + e) \leq \ell\} \leq f\\
&\Leftrightarrow \vee\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, \vee_{e\in\mathfrak{D}(S)} t_{\Lambda}(H(y + e)) \geq \ell\} \geq t,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, \wedge_{e\in\mathfrak{D}(S)} i_{\Lambda}(H(y + e)) \leq \ell\} \leq i,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, \wedge_{e\in\mathfrak{D}(S)} f_{\Lambda}(H(y + e)) \leq \ell\} \leq f\\
&\Leftrightarrow \vee\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, t_{\Lambda}(H(y)) \geq \ell\} \geq t,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, i_{\Lambda}(H(y)) \leq \ell\} \leq i,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, f_{\Lambda}(H(y)) \leq \ell\} \leq f\\
&\Leftrightarrow \vee\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, t_{\Lambda}(y/1) \geq \ell\} \geq t,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, i_{\Lambda}(y/1) \leq \ell\} \leq i,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, f_{\Lambda}(y/1) \leq \ell\} \leq f\\
&\Leftrightarrow \vee\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, t_{\Lambda}(x/1) \geq \ell\} \geq t,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, i_{\Lambda}(x/1) \leq \ell\} \leq i,\\
&\wedge\{\ell \in L|\exists \bar{y} \in \bar{M}, s \in S, \bar{x}/1 = \bar{y}/s, f_{\Lambda}(x/1) \leq \ell\} \leq f\\
&\Leftrightarrow t_{\Lambda}(x/1) \geq t, i_{\Lambda}(x/1) \leq i, f_{\Lambda}(x/1) \leq f\\
&\Leftrightarrow \Lambda(x/1) \geq (t, i, f)\\
&(t, i, f)_{x/1} \in \Lambda.
\end{aligned}$$

This proves the claim. □

Theorem 4.2. *Let* $\alpha \in LNI(R)$, $\alpha_{\mathbf{1}} = \{x \in R|\alpha(x) = (1, 0, 0)\}$ *and let* $\lambda \in LNS(M)$ *has the supremum property.*

(i) *If* $S \cap (\lambda : (1, 0, 0)_M)_{\mathbf{1}} = \emptyset$*, then* $[S]\lambda = \lambda$.
(ii) *If* $S \cap (\lambda : (1, 0, 0)_M)_{\mathbf{1}} \neq \emptyset$*, then* $S^{-1}\lambda = (1, 0, 0)_{S^{-1}M}$.
(iii) *If* $S \cap \alpha_{\mathbf{1}} \neq \emptyset$,*, then* $(\alpha \cdot \lambda)[S] = \lambda[S]$.
(iv) $S^{-1}(\lambda[S]) = S^{-1}\lambda$.

Proof. (i) Let $s \in S$. Then, $s \notin (\lambda : (1, 0, 0)_M))_{\mathbf{1}}$. It is easy to see that $(1, 0, 0)_s \cdot \lambda = \lambda$ for all $s \in S$. Thus, $[S]\lambda = \lambda$.
(ii) We only need to show that $(1, 0, 0)_{S^{-1}M} \sqsubseteq S^{-1}\lambda$. To do this, it

suffices to show that for each $x \in M$, $(S^{-1}\lambda)(x/1) = (1,0,0)$. By our assumption, there exists $s \in S \cap (\lambda : (1,0,0)_M)_{\mathbf{1}}$. This, in particular, gives that

$$(1,0,0)_s(s) \sqsubseteq \cup\{\alpha(s) | \alpha \in LNI(R), \alpha \cdot (1,0,0)_M \sqsubseteq \lambda\} = (1,0,0)$$

and hence $(1,0,0)_{sM} = (1,0,0)_s \cdot (1,0,0)_M \sqsubseteq \lambda$. Now, assume that $(S^{-1}\lambda)(x/1) = (t,i,f)$. Then

$$\begin{array}{l} \vee\{\ell \in L | \bar{x}/1 \in S^{-1}((t_{\Pi(\lambda)})_\ell)\} = t \\ \wedge\{\ell \in L | \bar{x}/1 \in S^{-1}((i_{\Pi(\lambda)})_\ell)\} = i \\ \wedge\{\ell \in L | \bar{x}/1 \in S^{-1}((f_{\Pi(\lambda)})_\ell)\} = f. \end{array}$$

This, by the above observation, gives that

$$\begin{array}{l} \vee\{\ell \in L | \bar{x}/1 \in S^{-1}((t_{\Pi((1,0,0)_{sM})})_\ell)\} \le t \\ \wedge\{\ell \in L | \bar{x}/1 \in S^{-1}((i_{\Pi((1,0,0)_{sM}))})_\ell)\} \ge i \\ \wedge\{\ell \in L | \bar{x}/1 \in S^{-1}((f_{\Pi((1,0,0)_{sM}))})_\ell)\} \ge f. \end{array}$$

Therefore,

$$\begin{array}{l} \vee\{\ell \in L | \exists u, v \in S, y \in M, uv\bar{x} = u\bar{y}, \vee_{e\in\mathfrak{O}(S)} 1_{sM}(y+e) \ge \ell\} \le t \\ \wedge\{\ell \in L | \exists u, v \in S, y \in M, uv\bar{x} = u\bar{y}, \wedge_{e\in\mathfrak{O}(S)} 0_{sM}(y+e) \le \ell\} \ge i \\ \wedge\{\ell \in L | \exists u, v \in S, y \in M, uv\bar{x} = u\bar{y}, \wedge_{e\in\mathfrak{O}(S)} 0_{sM}(y+e) \le \ell\} \ge f. \end{array}$$

Hence,

$$\begin{array}{l} 1 = \vee\{\ell \in L | \vee_{e\in\mathfrak{O}(S)} 1_{sM}(sx+e) \ge \ell\} \le t \\ 0 = \wedge\{\ell \in L | \wedge_{e\in\mathfrak{O}(S)} 0_{sM}(sx+e) \le \ell\} \ge i \\ 0 = \wedge\{\ell \in L | \wedge_{e\in\mathfrak{O}(S)} 0_{sM}(sx+e) \le \ell\} \ge f. \end{array}$$

Thus $1_{S^{-1}M}(x/1) = (1,0,0) \preceq (t,i,f)$. As we always have $(t,i,f) \preceq (1,0,0)$, the result follows.
(iii) We note that $(\alpha\cdot\lambda) \sqsubseteq \lambda$ by Definition 2.1(1) and so $(\alpha\cdot\lambda)[S] \sqsubseteq \lambda[S]$ by definition (†).
For the reverse inclusion, fix $u \in S \cap \alpha_{\mathbf{1}}$, and let $x \in M$, $(t,i,f) \in L^3$ be such that $(\lambda[S])(x) \succeq (t,i,f)$. This means that

$$(\vee_{s\in S}(t_\lambda(sx), \wedge_{s\in S} i_\lambda(sx), \wedge_{s\in S} f_\lambda(sx)) \succeq (t,i,f)$$

and thus by the supinf property of λ there exist $s \in S$, such that $(t_\lambda(sx), i_\lambda(sx), f_\lambda(sx)) \ge (t,i,f)$. Now,

$$\begin{array}{rcl} (\alpha\cdot\lambda)[S](x) & = & (\vee_{v\in S} t_{(\alpha\cdot\lambda)}(vx), \wedge_{v\in S} i_{(\alpha\cdot\lambda)}(vx), \wedge_{v\in S} i_{(\alpha\cdot\lambda)}(vx)) \\ & \succeq & (t_{(\alpha\cdot\lambda)}(sux), i_{(\alpha\cdot\lambda)}(sux), f_{(\alpha\cdot\lambda)}(sux)) \\ & \succeq & (t_\alpha(u) \wedge t_\lambda(sx), i_\alpha(u) \vee i_\lambda(sx), f_\alpha(u) \vee f_\lambda(sx)) \\ & \succeq & (t_\lambda(sx), i_\lambda(sx), f_\lambda(sx) \ge (t,i,f). \end{array}$$

This completes this part.
(iv) As $\lambda \sqsubseteq \lambda[S]$, the inclusion $S^{-1}\lambda \sqsubseteq S^{-1}(\lambda[S])$ follows immediately from definition (♮), and we only prove the nontrivial direction

$S^{-1}(\lambda[S]) \sqsubseteq S^{-1}\lambda$. Note that for each $e \in \mathfrak{O}[S]$, there exists $u_e \in S$ such that $u_e e = \theta$. Let $x \in M$. Then,

$$\begin{aligned}
&S^{-1}(\lambda[S])(x/1)\\
&= \big(\vee\{\ell \in L|\bar{x}/1 \in S^{-1}((t_{\Pi(\lambda[S])})_\ell)\}, \wedge\{\ell \in L|\bar{x}/1 \in S^{-1}((i_{\Pi(\lambda[S])})_\ell)\},\\
&\ \wedge\{\ell \in L|\bar{x}/1 \in S^{-1}((f_{\Pi(\lambda[S])})_\ell)\}\big)\\
&= (\vee\{\ell \in L|\exists s', u \in S, z \in M, s'u\bar{x} = u\bar{z}, \vee_{e\in\mathfrak{O}(S)} t_{\lambda[S]}(z+e) \geq_\ell)\},\\
&\wedge\{\ell \in L|\exists s', u \in S, z \in M, s'u\bar{x} = u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} i_{\lambda[S]}(z+e) \leq_\ell)\},\\
&\wedge\{\ell \in L|\exists s', u \in S, z \in M, s'u\bar{x} = u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} f_{\lambda[S]}(z+e) \leq_\ell)\})\\
&= (\vee\{\ell\in L|\exists s', u \in S, z \in M, s'u\bar{x} = u\bar{z}, \vee_{e\in\mathfrak{O}(S)} \vee_{s\in S} t_\lambda(ssz+e) \geq_\ell)\},\\
&\wedge\{\ell\in L|\exists s', u \in S, z \in M, s'u\bar{x} = u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} \wedge_{s\in S} i_\lambda(sz+se) \leq_\ell)\},\\
&\wedge\{\ell\in L|\exists s', u \in S, z \in M, s'u\bar{x} = u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} \wedge_{s\in S} f_\lambda(sz+se) \leq_\ell)\}) \preceq\\
&(\vee\{\ell \in L|\exists s', u\in S, z\in M, s'u\bar{x}=u\bar{z}, \vee_{e\in\mathfrak{O}(S)} \vee_{s\in S} \vee_{u_e\in S} t_\lambda(su_ez + su_ee) \geq_\ell)\},\\
&\wedge\{\ell\in L|\exists s', u \in S, z \in M, s'u\bar{x}=u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} \wedge_{s\in S} \wedge_{u_e\in S} i_\lambda(su_ez + su_ee) \leq_\ell)\},\\
&\wedge\{\ell\in L|\exists s', u \in S, z \in M, s'u\bar{x}=u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} \wedge_{s\in S} \wedge_{u_e\in S} f_\lambda(su_ez+su_ee) \leq_\ell)\})\\
&\preceq (\vee\{\ell\in L|\exists s', u \in S, z \in M, s'u\bar{x}=u\bar{z}, \vee_{e\in\mathfrak{O}(S)} \vee_{s\in S} \vee_{u_e\in S} t_\lambda(su_ez) \geq_\ell)\},\\
&\wedge\{\ell\in L|\exists s', u \in S, z \in M, s'u\bar{x}=u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} \wedge_{s\in S} \wedge_{u_e\in S} i_\lambda(su_ez) \leq_\ell)\},\\
&\wedge\{\ell\in L|\exists s', u \in S, z \in M, s'u\bar{x}=u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} \wedge_{s\in S} \wedge_{u_e\in S} f_\lambda(su_ez) \leq_\ell)\}).
\end{aligned}$$

Now, as λ has the supinf property, this last expression equals

$$\begin{aligned}
&(\vee\{\ell \in L|\exists s', u, w \in S, z \in M, s'u\bar{x} = u\bar{z}, t_\lambda(wz) \geq_\ell)\},\\
&\ \wedge\{\ell \in L|\exists s', u, w \in S, z \in M, s'u\bar{x} = u\bar{z}, i_\lambda(wz) \leq_\ell)\},\\
&\ \wedge\{\ell \in L|\exists s', u, w \in S, z \in M, s'u\bar{x} = u\bar{z}, f_\lambda(wz) \leq_\ell)\})\\
&\preceq (\vee\{\ell \in L|\exists s', u, w \in S, z \in M, s' \quad u\bar{x} = u\bar{z}, \vee_{e\in\mathfrak{O}(S)} t_\lambda(wz+e) \geq_\ell)\},\\
&\qquad \wedge\{\ell \in L|\exists s', u, w \in S, z \in M, s'u\bar{x} = u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} i_\lambda(wz+e) \leq_\ell)\},\\
&\qquad \wedge\{\ell \in L|\exists s', u, w \in S, z \in M, s'u\bar{x} = u\bar{z}, \wedge_{e\in\mathfrak{O}(S)} f_\lambda(wz+e) \leq_\ell)\})\\
&\preceq (\vee\{\ell \in L|\exists s', u, w \in S, z \in M, s'u\bar{x} = u\bar{z}, t_{\Pi(\lambda)}(w\bar{z}) \geq_\ell)\},\\
&\qquad \wedge\{\ell \in L|\exists s', u, w \in S, z \in M, s'u\bar{x} = u\bar{z}, i_{\Pi(\lambda)}(w\bar{z}) \leq_\ell)\},\\
&\qquad \wedge\{\ell \in L|\exists s', u, w \in S, z \in M, s'u\bar{x} = u\bar{z}, f_{\Pi(\lambda)}(w\bar{z}) \leq_\ell)\})\\
&= \big(\vee\{\ell \in L|\exists v \in S, \bar{x}/1 = w\bar{z}/v, w\bar{z} \in (t_{\Pi(\lambda)})_\ell\},\\
&\qquad \wedge\{\ell \in L|\exists v \in S, \bar{x}/1 = w\bar{z}/v, w\bar{z} \in (i_{\Pi(\lambda)})_\ell\},\\
&\qquad \wedge\{\ell \in L|\exists v \in S, \bar{x}/1 = w\bar{z}/v, w\bar{z} \in (f_{\Pi(\lambda)})_\ell\}\big)\\
&= \big(\vee\{\ell \in L|\bar{x}/1 \in S^{-1}((t_{\Pi(\lambda)})_\ell)\}, \wedge\{\ell \in L|\bar{x}/1 \in S^{-1}((i_{\Pi(\lambda)})_\ell)\},\\
&\qquad \wedge\{\ell \in L|\bar{x}/1 \in S^{-1}((f_{\Pi(\lambda)})_\ell)\}\big) = S^{-1}(\lambda)(x/1).
\end{aligned}$$

Thus for each $s \in S$ and each $x \in M$,

$$S^{-1}(\lambda[S])(x/s) = S^{-1}(\lambda[S])(x/1) \leq S^{-1}(\lambda)(x/1) = S^{-1}(\lambda)(x/s),$$

and the result follows. □

Now the proof of the following theorem is straightforward.

Theorem 4.3. *Let $\lambda \in LNS(M)$. If λ is an L-neutrosophic single valued primary (prime) submodule of M such that $S \cap (\lambda : (1,0,0)_M)_{\mathbf{1}} = \emptyset$, then $S^{-1}\lambda$ is an L-neutrosphic single valued primary (prime) submodule of $S^{-1}M$. The converse is also true when λ has the supinf property.*

5. Quotients and L-single valued neutrosophic quotients

In this section, the aim is to construct and study the concepts of quotient ring by an L-single valued neutrosophic ideal, quotient module by an L-single valued neutrosophi submodule, and L-single valued neutrosophic submodule of a quotient module. To do so, let $\xi \in LNI(R)$, $\lambda \in LNS(M)$ and let N be an R-submodule of M.
For each $r \in R$, $(1,0,0)_r + \xi \in LNS(R)$ defined by

$$((1,0,0)_r+\xi)(a) := \xi(r-a) = (t_\xi(r-a), i_\xi(r-a), f_\xi(r-a)), \text{for all } a \in R$$

will be called a coset of ξ by r and it will be denoted by $r+\xi$ for brief. Similarly, for each $x \in M$, $(1,0,0)_x + \lambda \in LSNS(M)$ defined by

$$((1,0,0)_x + \lambda)(y) = \lambda(x-y), \qquad \text{for all } y \in M$$

will be considered a coset of λ by x, and will be denoted by $x+\lambda$. It is easy to see that for each $x, y \in M$, $x+\lambda = y+\lambda$ if and only if $\lambda(x-y) = (1,0,0)$, and that the set $R/\lambda = \{x+\lambda | x \in M\}$, with operations as $(x+\lambda)+(y+\lambda) = (x+y)+\lambda$ and $r(x+\lambda) = rx+\lambda$ is an R-module called the quotient module of M by λ. It is clear that $\theta+\lambda = \lambda$ is the zero element of M/λ, and for each $x \in M$, $-(x+\lambda) = -x+\lambda$. Similar statements holds true for R, and we can construct the quotient ring $R/\xi = \{r+\xi | r \in R\}$ of R by ξ. The proof of the following theorem is straightforward now.

Theorem 5.1. *With the above notation $R/\xi_1 \cong R/\xi$ as two rings and $M/\lambda_1 \cong M/\lambda$ as two R-modules.*

Let M, N be two R-modules and $H : M \to N$ be a homomorphism of R-modules. Let $\lambda \in LNS(M)$. We say that λ is *H-invariant* if for all $x, y \in M$, $H(x) = H(y)$ imply that $\lambda(x) = \lambda(y)$.
In the following theorem, we collect the behavior of the L-single valued neutrosophic submodules under a module homomorphism. As the proofs are easy, we will leave most parts of it.

Theorem 5.2. *Let M, N be two R-modules and $H : M \to N$ be an epimorphism of R-modules. Assume that $\lambda \in LNS(M)$, $\nu \in LNS(N)$ and that λ is constant on* $\mathrm{Ker}(H)$. *Then,*

(i) $H(\lambda) \in LNS(N)$, *and* $H(\lambda_{\mathbf{1}}) = H(\lambda)_{\mathbf{1}}$.
(ii) $H^{-1}(\nu) \in LNS(M)$ *and it is constant on* $\mathrm{Ker}(H)$ *(here the assumption that λ is constant on* $\mathrm{Ker}(H)$ *is not needed).*
(iii) $H^{-1}(\nu_{\mathbf{1}}) = (H^{-1}(\nu))_{\mathbf{1}}$.
(iv) $(H^{-1}oH)(\lambda) = \lambda$.
(v) $(HoH^{-1})(\nu) = \nu$.

(vi) *if* λ *is an* H*-invariant primary* L*-neutrosophic submodule of* M*, then* $H(\lambda)$ *is a primary* L*-neutrosophic submodule of* N*.*
(vii) *if* ν *is a primary* L*-neutrosophic submodule of* N*, then* $H^{-1}(\nu)$ *is a primary* L*-neutrosophic submodule of* M*.*

Proof. We only prove part (i). To show that $H(\lambda) \in LNS(N)$, we prove the condition (LM2), and two others conditions can be proved by an statment. So, let $x, y \in M$ such that $H(x), H(y) \in N$. Then,

$$\begin{aligned}
&H(\lambda)(H(x)+H(y))\\
&= (t_{H(\lambda)}(H(x+y)), i_{H(\lambda)}(H(x+y)), f_{H(\lambda)}(H(x+y)))\\
&= (\vee\{t_\lambda(z)|z \in M, H(z) = H(x+y)\},\\
&\quad \wedge\{i_\lambda(z)|z \in M, H(z) = H(x+y)\},\\
&\quad \wedge\{f_\lambda(z)|z \in M, H(z) = H(x+y)\})\\
&\succeq (\vee\{t_\lambda(u+v)|u, v \in M, H(u) = H(x), H(v) = H(y)\},\\
&\quad \wedge\{i_\lambda(u+v)|u, v \in M, H(u) = H(x), H(v) = H(y)\},\\
&\quad \wedge\{f_\lambda(u+v)|u, v \in M, H(u) = H(x), H(v) = H(y)\})\\
&\succeq (\vee\{t_\lambda(u) \wedge t_\lambda(v)|u, v \in M, H(u) = H(x), H(v) = H(y)\},\\
&\quad \wedge\{i_\lambda(u) \vee i_\lambda(v)|u, v \in M, H(u) = H(x), H(v) = H(y)\},\\
&\quad \wedge\{f_\lambda(u) \vee f_\lambda(v)|u, v \in M, H(u) = H(x), H(v) = H(y)\})\\
&= ((\vee(t_\lambda(u)|u \in M, H(u) = H(x)) \wedge (\vee(t_\lambda(v)|v \in M, H(v) = H(y)),\\
&(\wedge(i_\lambda(u)|u \in M, H(u) = H(x)) \vee (\wedge(i_\lambda(v)|v \in M, H(v) = H(y))),\\
&(\wedge(i_\lambda(u)|u \in M, H(u) = H(x))) \vee (\wedge(i_\lambda(v)|v \in M, H(v) = H(y))))\\
&= H(\lambda)(H(x)) \wedge H(\lambda)(H(y)).
\end{aligned}$$

To prove $H(\lambda_{\mathbf{1}}) = H(\lambda)_{\mathbf{1}}$, let $x \in M$ and $H(x) \in H(\lambda_{\mathbf{1}})$. Then,

$$\begin{aligned}
H(\lambda)(H(x)) &= (t_{H(\lambda)}(H(x)), i_{H(\lambda)}(H(x)), f_{H(\lambda)}(H(x)))\\
&= (\vee\{t_\lambda(y)|y \in M, H(y) = H(x)\}, \wedge\{i_\lambda(y)|y \in M, H(y) = H(x)\},\\
&\qquad \wedge\{f_\lambda(y)|y \in M, H(y) = H(x)\})\\
&= (1, 0, 0),
\end{aligned}$$

and thus $H(\lambda_{\mathbf{1}}) \subseteq H(\lambda)_{\mathbf{1}}$.
For the opposite direction, assume that $x \in M$ and $H(x) \in N$, such that

$$\begin{aligned}
(1, 0, 0) &= H(\lambda)(H(x)\\
&= (\vee\{t_\lambda(y)|y \in M, H(y) = H(x)\}, \wedge\{i_\lambda(y)|y \in M, H(y) = H(x)\},\\
&\qquad \wedge\{f_\lambda(x)|y \in M, H(y) = H(x)\}).
\end{aligned}$$

So, by the supinf property of λ, there exist some $y \in M$ such that $(1, 0, 0) = (t_\lambda(y), i_\lambda(y), f_\lambda(y))$ and $H(x) = H(y)$. This means that $H(x) \in H(\lambda_{\mathbf{1}})$. □

Using some parts of the above theorem we can formulate the following theorem we state without proof.

Theorem 5.3. *Let M, N be two R-modules and let $H : M \to N$ be an epimorphism of R-modules. Then, there is a one-to-one order preserving correspondence between the elements of $LNS(M)$ that are constant on* $\mathrm{Ker}(H)$ *and the elements of $LNS(N)$. Moreover, this correspondence maps H-invariant primary L-neutrosophic submodules of M to the primary neutrosophic submodules of N.*

Now, let $\lambda \in LNS(M)$, N be a submodule of M and put $\bar{M} = M/N$. For each $\bar{x} = x + N \in \bar{M}$, set

$$(\lambda/N)(\bar{x}) = \big(\vee \{t_\lambda(y)|y \in \bar{x}\}, \wedge\{i_\lambda(y)|y \in \bar{x}\}, \wedge\{f_\lambda(y)|y \in \bar{x}\}\big).$$

Obviously, $(\lambda/N) \in LSVN(M/N)$. In the following theorem we prove that $(\lambda/N) \in LNS(M/N)$. We call it quotient L-(single valued) neutrosophic submodule of λ with respect to N.

Theorem 5.4. *With the above notation, $(\lambda/N) \in LNS(M/N)$.*

Proof. First note that

$$\begin{aligned}(\lambda/N)(\bar{\theta}) &= \big(\vee \{t_\lambda(y)|y \in N\}, \wedge\{i_\lambda(y)|y \in N\}, \wedge\{f_\lambda(y)|y \in N\}\big)\\ &= (1, 0, 0),\end{aligned}$$

and (LM1) holds. Now, for $x, y \in M$,

$$\begin{aligned}
&(\lambda/N)(\bar{x} + \bar{y})\\
&\quad = \big(\vee \{t_\lambda(z)|z \in \bar{x} + \bar{y}\}, \wedge\{i_\lambda(z)|z \in \bar{x} + \bar{y}\}, \wedge\{f_\lambda(z)|z \in \bar{x} + \bar{y}\}\big)\\
&= (\vee\{t_\lambda(x + y + u)|u \in N\}, \wedge\{i_\lambda(x + y + u)|u \in N\},\\
&\quad \wedge\{f_\lambda(x + y + u)|u \in N\})\\
&\succeq (\vee\{t_\lambda(x + y + u + v)|u, v \in N\}, \wedge\{i_\lambda(x + y + u + v)|u, v \in N\},\\
&\quad \wedge\{f_\lambda(x + y + u + v)|u, v \in N\})\\
&\succeq (\vee\{t_\lambda(x + u) \wedge t_\lambda(y + v)|u, v \in N\}, \wedge\{i_\lambda(x + u) \vee i_\lambda(y + v)|u, v \in N\},\\
&\quad \wedge\{f_\lambda(x + u) \vee f_\lambda(y + v)|u, v \in N\}) \qquad (\text{as } \lambda \in LNS(M),)\\
&= (\vee\{t_\lambda(x + u)|u \in N\}, \wedge\{i_\lambda(x + u)|u \in N\}, \wedge\{f_\lambda(x + u)|u \in N\})\, \bar{\wedge}\\
&\quad (\vee\{t_\lambda(y + v)|v \in N\}, \wedge\{i_\lambda(y + v)|v \in N\}, \wedge\{f_\lambda(y + v)|v \in N\})\\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{by assumption on } L \text{ and } (*))\\
&= (\vee\{t_\lambda(u)|u \in \bar{x}\}, \wedge\{i_\lambda(u)|u\bar{x}\}, \wedge\{f_\lambda(u)|u \in \bar{x}\})\\
&\quad \bar{\wedge}\, (\vee\{t_\lambda(v)|v \in \bar{y}\}, \wedge\{i_\lambda(v)|v\bar{y}\}, \wedge\{f_\lambda(v)|v \in \bar{y}\})\\
&= (\lambda/N)(\bar{x})\bar{\wedge}(\lambda/N)(\bar{y}).
\end{aligned}$$

This proves (LM2). Part (LM3) is proved easily. □

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S. Azimi
Department of Mathematics, Faculty of Science, University of Mohaghegh Ardabili, Ardabil, Iran
Email: azimi_y2012@yahoo.com
N. Zamani
Department of Mathematics, Faculty of Science, University of Mohaghegh Ardabili, Ardabil, Iran
Email: naserzaka@yahoo.com
M. R. Hooshmandasl
Department of Computer Science, Faculty of Science, University of Mohaghegh Ardabili, Ardabil, Iran
Email: hooshmandasl@uma.ac.ir
A. Khojali
Department of Mathematics, Faculty of Science, University of Mohaghegh Ardabili, Ardabil, Iran
Email: khojali@uma.ac.ir