Journal of Algebra and Related Topics
Vol. 12, No 02, (2024), pp 47-51

TWO NOTES ON "ON THE MAXIMAL SPECTRUM OF A MODULE AND ZARISKI TOPOLOGY"

S.C. HAN* AND J.N. HO

Abstract. For unitary modules over commutative rings with nonzero identity, Question 3.28 in [Bull. Malays. Math. Sci. Soc. 38(1) (2015) 303] asks if the maximal spectrum of a Max-injective module is a Max-spectral topological space and Proposition 3.19 in it contains an assertion that the maximal spectrum of a Max-injective module is a T_2-space. In this paper, we give a negative answer to the above question and show that the above assertion is false by two examples of multiplication modules.

1. Introduction

Throughout this paper, unless otherwise stated, all rings are commutative with nonzero identity and all modules are unitary with maximal submodule. For the concepts and notations used below, we refer to [2].

For unitary modules with maximal submodule over commutative rings with nonzero identity, Ovlyaee-Sarmazdeh and Maleki-Roudposhti in [6] studied further properties of a Max-injective module and established some conditions for the maximal spectrum of the module to be a Max-spectral topological space. Ansari-Toroghy and Keyvani in [2] investigated the interplay between the algebraic properties of a module and topological properties of its maximal spectrum, and asked a question if the maximal spectrum of a Max-injective module is a Max-spectral topological space (see [2, Question 3.28]). They also stated an

MSC(2020): Primary: 13C13; Secondary: 54G20
Keywords: maximal submodule, max-injective module, max-spectral space, multiplication module, Zariski topology.
Received: 10 March 2023, Accepted: 29 January 2024.
∗Corresponding author.

assertion with "straightforward" proof that the maximal spectrum of a Max-injective module is a T_2-space (see [2, Proposition 3.19]).

In this paper, we give a negative answer to the above question and show that the above assertion is false by two examples of multiplication modules.

2. Preliminaries

In this paper, R is a commutative ring with nonzero identity and M is a unitary R-module with maximal submodule. Also $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{Z}$ the ring of integers. For every ideal I of R containing $\mathrm{Ann}_R(M)$, $\overline{I}$ denotes the ideal $I/\mathrm{Ann}_R(M)$ of $\overline{R} = R/\mathrm{Ann}_R(M)$.

For a submodule N of M, let $(N : M) = \{r \in R | \ rM \subseteq N\}$. Then $(N : M)$ is an ideal of R. A proper submodule P of M is said to be *prime* in M if $rm \in P$ with $r \in R$ and $m \in M$ implies that $m \in P$ or $r \in (P : M)$. The set of all prime submodules of M is denoted by $\mathrm{Spec}_R(M)$. If Q is a maximal submodule of M, then Q is a prime submodule of M and $(Q : M)$ is a maximal ideal of R. The set of all maximal submodules of M is denoted by $\mathrm{Max}_R(M)$. The mapping $\phi : \mathrm{Max}_R(M) \to \mathrm{Max}(\overline{R})$ such that $\phi(Q) = \overline{(Q : M)}$ for all $Q \in \mathrm{Max}_R(M)$ is called the *natural map* of $\mathrm{Max}_R(M)$. M is said to be *Max-injective* if the natural map of $\mathrm{Max}_R(M)$ is injective [2].

For a submodule N of M, let

$$V_M(N) = \{P \in \mathrm{Spec}_R(M) | \ (N : M) \subseteq (P : M)\}.$$

The *Zariski topology* on $\mathrm{Spec}_R(M)$ is the topology τ_M taking the set of all such $V_M(N)$ that N is a submodule of M as the set of closed subsets of $\mathrm{Spec}_R(M)$. The topological space $(\mathrm{Spec}_R(M), \ \tau_M)$ is called the *prime spectrum* of M.

The subspace $(\mathrm{Max}_R(M), \ \tau_M^m)$ of the prime spectrum of M is called the *maximal spectrum* of M, where τ_M^m is the relative topology induced by τ_M on $\mathrm{Max}_R(M)$ having the set of all

$$V_M^m(N) = \{Q \in \mathrm{Max}_R(M) | \ (N : M) \subseteq (Q : M)\}$$

where N is a submodule of M as the set of closed subsets of $\mathrm{Max}_R(M)$.

In particular, $\mathrm{Spec}_R(R) = \mathrm{Spec}(R)$, $\mathrm{Max}_R(R) = \mathrm{Max}(R)$ and for an ideal I of R, $V_R(I) = \{p \in \mathrm{Spec}(R) | \ I \subseteq p\}$ and $V_R^m(I) = \{q \in \mathrm{Max}(R) | \ I \subseteq q\}$.

A topological space W is said to be *Max-spectral* if W is homeomorphic with the maximal ideal space $\mathrm{Max}(S)$ of some ring S [2]. A topological space W is Max-spectral iff W is a compact T_1-space [5].

If $\mathrm{Max}_R(M)$ is a Max-spectral topological space, then M is a Max-injective R-module [2].

3. Main results

Lemma 3.1. [3] *Let M be a nonzero multiplication R-module.*

(1) *Every proper submodule of M is contained in a maximal submodule of M.*

(2) *K is a maximal submodule of M iff there exists a maximal ideal p of R such that $K = pM \neq M$.*

Lemma 3.2. [1, 4] *Let M be a nonzero multiplication R-module. M is finitely generated iff $\mathrm{Spec}_R(M)$ is a compact space.*

Lemma 3.3. [2] *If M is a nonzero multiplication R-module, then M is a Max-injective R-module.*

[2, Question 3.28] is as follows: Let M be a Max-injective R-module. Is $\mathrm{Max}_R(M)$ a Max-spectral topological space?

The following theorem gives a negative answer to this question.

Theorem 3.4. *Let $R = \prod_{i\in\mathbb{N}} \mathbb{Z}_2$ and $M = \oplus_{i\in\mathbb{N}}\mathbb{Z}_2$. Then the following hold.*

(1) *M is a multiplication R-module.*

(2) *M is a Max-injective R-module.*

(3) $\mathrm{Spec}(R) = \mathrm{Max}(R)$*.*

(4) $\mathrm{Spec}_R(M) = \mathrm{Max}_R(M)$*.*

(5) $\mathrm{Max}_R(M)$ *is not a Max-spectral topological space.*

Proof. R is a commutative ring with nonzero identity and M is a nonzero unitary R-module. Note that every element in R is multiplicatively idempotent.

(1) Let N be an R-submodule of M. Then N is an ideal of R and for every element $a \in N$, $a = a^2 \in NM$, which implies $N = NM$.

(2) Follows from Lemma 3.3.

(3) Suppose that $p \in \mathrm{Spec}(R)$. Then R/p is an integral domain and for every $r \in R$, $(r + p)(r + p) = r^2 + p = r + p$, so either $r + p = 0 + p$ or $r + p = 1 + p$. Hence R/p is a field, thus $p \in \mathrm{Max}(R)$.

(4) Suppose that $P \in \mathrm{Spec}_R(M)$. Then $(P : M) \in \mathrm{Spec}(R)$ and $(P : M) \in \mathrm{Max}(R)$ by part (3). Since $(P : M)M = P \neq M$, $P \in \mathrm{Max}_R(M)$ by Lemma 3.1(2).

(5) Obviously, the multiplication R-module M is not finitely generated. By Lemma 3.2, $\mathrm{Spec}_R(M)$ is not compact, neither is $\mathrm{Max}_R(M)$ by part (4). □

[2, Proposition 3.19] is as follows: Let M be an R-module. Then the following are equivalent.

(1) M is Max-injective.

(2) $\mathrm{Max}_R(M)$ is a T_0-space.

(3) $\mathrm{Max}_R(M)$ is a T_1-space.

(4) $\mathrm{Max}_R(M)$ is a T_2-space.

Here it is easily verified that (1)⇒(3)⇒(2)⇒(1). The following theorem shows that (1)⇒(4) is false.

Theorem 3.5. *$\mathbb{Z}$ is a Max-injective $\mathbb{Z}$-module, but $\mathrm{Max}_{\mathbb{Z}}(\mathbb{Z})$ is not a T_2-space.*

Proof. $\mathbb{Z}$ is a nonzero multiplication $\mathbb{Z}$-module, so by Lemma 3.3, $\mathbb{Z}$ is a Max-injective $\mathbb{Z}$-module. Meanwhile, let Ω be the set of all prime numbers. Then $\mathrm{Max}_{\mathbb{Z}}(\mathbb{Z}) = \mathrm{Max}(\mathbb{Z}) = \{p\mathbb{Z}|\ p \in \Omega\}$ is an infinite set and every nonempty open subset of $\mathrm{Max}_{\mathbb{Z}}(\mathbb{Z})$ is the complement of a finite subset in $\mathrm{Max}_{\mathbb{Z}}(\mathbb{Z})$, so the intersection of arbitrary two nonempty open subsets of $\mathrm{Max}_{\mathbb{Z}}(\mathbb{Z})$ is not empty. Hence $\mathrm{Max}_{\mathbb{Z}}(\mathbb{Z})$ is not a T_2-space. □

Acknowledgments

The authors would like to thank the referee for careful reading.

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Song-Chol Han

Faculty of Mathematics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea

`Email: sc.han@ryongnamsan.edu.kp`

Jin-Nam Ho
Faculty of Mathematics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea
Email: cioc12@ryongnamsan.edu.kp