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ON THE SEMITOTAL DOMINATING SETS OF GRAPHS

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Abstract. A set D of vertices in an isolate-free graph G is a semitotal dominating set of G, if D is a dominating set of G and every vertex in D is within distance 2 from another vertex of D. The semitotal domination number of G is the minimum cardinality of a semitotal dominating set of G and is denoted by $\gamma_{t2}(G)$. In this paper, after computation of semitotal domination number of specific graphs, we count the number of this kind of dominating sets of arbitrary size in some graphs.

1. Introduction

A dominating set of a graph $G = (V, E)$ is any subset S of V such that every vertex not in S is adjacent to at least one member of S. The minimum cardinality of all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. This parameter has been extensively studied in the literature and there are hundreds of papers concerned with domination. For a detailed treatment of domination theory, the reader is referred to [8]. Also, the concept of domination and related invariants have been generalized in many ways. Among the best know generalizations are total, independent, and connected dominating, each of them with the corresponding domination number. Most of the papers published so far deal with structural aspects of domination, trying to determine exact expressions for $\gamma(G)$ or some upper and/or lower bounds for it. There were no paper concerned with

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the enumerative side of the problem by 2008. Regarding to enumerative side of dominating sets, Alikhani and Peng [5], have introduced the domination polynomial of a graph. The domination polynomial of graph G is the generating function for the number of dominating sets of G, i.e.,

$$D(G,x) = \sum_{i=1}^{|V(G)|} d(G,i)x^i$$

(see [1, 5]). This polynomial and its roots has been actively studied in recent years (see for example [4]).

It is natural to count the number of another kind of dominating sets ([2, 3]). Motivated by papers such as [2, 3], we consider another type of dominating set of a graph in this paper.

A total dominating set, abbreviated a TD-set, of a graph G with no isolated vertex is a set D of vertices of G, such that every vertex in $V(G)$ is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in a book ([9]). A set D of vertices in an isolate-free graph G, is a semitotal dominating set of G if D is a dominating set of G and every vertex in D is within distance 2 from another vertex of D. The semitotal domination was introduced by Goddard, Henning and McPillan [7], and studied further in [10, 11] and elsewhere.

The semitotal domination number of G is the minimum cardinality of a semitotal dominating set of G and is denoted by $\gamma_{t2}(G)$. By the definition, it is easy to see that for any graph G with no isolated vertices, $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. Straight from the definition, we see that $\gamma_{t2}(G) \geq 2$ but in this paper we consider $\gamma_{t2}(K_n) = 1$. Recently, Henning, Pal and Pradhan [12] studied the semitotal domination number in block graphs. They presented a linear time algorithm to compute a minimum semitotal dominating set in block graphs. Also they studied the complexity of the semitotal domination problem.

A domination-critical (domination-super critical, respectively) vertex in a graph G is a vertex whose removal decreases (increases, respectively) the domination number. Bauer et al. [6] introduced the concept of domination stability in graphs. The domination stability, or just γ-stability, of a graph G is the minimum number of vertices whose removal changes the domination number. Motivated by domination stability, we consider the semi-total stability of a graph.

In Section 2, we compute the semitotal domination number of specific graphs. In Section 3, we count the number of semitotal dominating sets of arbitrary size in some graphs. Finally in Section 4, we introduce semitotal domination stability of a graph and compute it for some graphs.

2. Semitotal domination number of specific graphs

In this section, we study the semitotal domination number of some specific graphs. Here, we recall some graph products. The *corona product* $G \circ H$ of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H. The Cartesian product of graphs G and H is a graph denoted $G\Box H$ whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. The *join* of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1)$ and $v \in V(G_2)\}$. We begin with computation of the semi-total domination number of specific graphs which is straightforward to compute.

Theorem 2.1. (i) *For every $n \geq 3$, $\gamma_{t2}(P_n) = \gamma_{t2}(C_n) = \lceil \frac{2n}{5} \rceil$.*
(ii) *If W_n is a wheel of order n, then $\gamma_{t2}(W_n) = \lceil \frac{n-1}{3} \rceil$.*
(iii) *If F_n is a friendship graph (join of K_1 and nK_2), then $\gamma_{t2}(F_n) = n$.*
(iv) *If B_n is a book graph (the Cartesian product $K_{1,n}\Box P_2$), then $\gamma_{t2}(B_n) = n + 1$.*
(v) $\gamma_{t2}(K_{m,n}) = \begin{cases} min\{m, n\} & 2 \leq n, m \leq 4 \\ 4 & m, n \geq 5. \end{cases}$

The following theorem gives the difference between $\gamma(G)$ and $\gamma_{t_2}(G)$ for certain graphs, which are easy to prove.

Theorem 2.2. (i) *For the Petersen graph P, $\gamma_{t2}(P) = \gamma(P)$.*
(ii) $\gamma_{t2}(B_n) = \gamma(B_n) + n - 1$.
(iii) $\gamma_{t2}(F_n) - \gamma(F_n) = n - 1$.
(iv) *For the star graph $S_n = K_{1,n}$, $\gamma_{t2}(S_n) - \gamma(S_n) = n - 1$.*
(v) $\gamma_{t2}(W_n) - \gamma(W_n) = \lceil \frac{n-1}{3} \rceil - 1$.
(vi) *For the complete bipartite graph $K_{m,n}$ with $m \leq n$,*

$$\gamma_{t2}(K_{m,n}) - \gamma(K_{m,n}) = \begin{cases} m - 2 & 2 \leq m \leq 4 \\ 2 & m \geq 5. \end{cases}$$

The following theorem is about the semitotal domination number of corona and join products of two graphs.

Theorem 2.3. (i) *If G_1 and G_2 are two graphs, then*

$$\gamma_{t2}(G_1 \circ G_2) \leq \gamma_{t2}(G_1) + \gamma_{t2}(G_2) \times (|V(G_1)| - \gamma_{t2}(G_1)).$$

Moreover, this inequality is sharp, when G_2 is a complete graph.

(ii) *For two graphs G and H (which are not complete graphs) of order at least three,*

$$\gamma_{t2}(G \vee H) = min\{\gamma_{t2}(G), \gamma_{t2}(H), 4\}$$

Proof. (i) By the construction of $G_1 \circ G_2$, the vertices in the semitotal dominating set of G_1 covers all the verticies of copies of G_2 which adjacent to them. Suppose that D is a semitotal dominating set of G_1. Every vertex in $V(G_1) \setminus D$ adjacent to one copy of G_2, and therefore, these vertices cover by the semitotal dominating set of G_2. So $\gamma_{t2}(G_1 \circ G_2) \leq \gamma_{t2}(G_1) + \gamma_{t2}(G_2) \times (|V(G_1)| - \gamma_{t2}(G_1))$.

(ii) By the construction of $G \vee H$, all of the vertices of G are adjacent to all of the vertices of H, and therefore any semitotal dominating set of G is a semitotal dominating set of $G \vee H$ and also any semitotal dominating set of H is a semitotal dominating set of $G \vee H$. If $\gamma_{t2}(G) > 4$, $\gamma_{t2}(H) > 4$, then we can cover all vertices of $G \vee H$ by four vertices. So we have the result.

Theorem 2.4. *If G is a complete graph and H is an arbitrary graph (which is not complete graph), then*

$$\gamma_{t2}(G \vee H) = \gamma_{t2}(H).$$

Proof. Since in $G \vee H$ all vertices of G are adjacent to all vertices of H, so the semitotal dominating set of H is a semitotal dominating set of $G \vee H$. On the other hand, in G the distance between two vertices is equal one and so we have $\gamma_{t2}(G \vee H) = \gamma_{t2}(H)$. □

Theorem 2.5. $\gamma_{t2}(P_n \square P_m) = \lceil \frac{2n}{5} \rceil \times \lceil \frac{m}{3} \rceil + \lfloor \frac{m}{3} \rfloor \times (n - \lceil \frac{2n}{5} \rceil)$.

Proof. By the construction of $P_n \square P_m$ we have m copies of P_n. The vertices of the second copy that adjacent to the semitotal dominating set of the first copy cover by these vertices and we can cover other vertices of the second copy by the complement of the semitotal dominating set of the third copy. By continuing this method for other copies, the number of at least vertices that we can cover all vertices of $P_n \square P_m$ by them, is $\lceil \frac{2n}{5} \rceil \times \lceil \frac{m}{3} \rceil + \lfloor \frac{m}{3} \rfloor \times (n - \lceil \frac{2n}{5} \rceil)$. □

3. The number of semitotal dominating sets

Counting the number of some kind of dominating sets has been the focus of researchers in recent years. In this section, we consider the problem of the number of the semitotal dominating sets of any size in a graph G. Let $\mathcal{D}_{t2}(G,i)$ be the family of semitotal dominating sets of a graph G with cardinality i and let $d_{t2}(G,i) = |\mathcal{D}_{t2}(G,i)|$. We denote the generating function for the number of semitotal dominating sets of G by $D_{t2}(G,x)$ and is the polynomial

$$D_{t2}(G,x) = \sum_{i=1}^{|V(G)|} d_{t2}(G,i)x^i,$$

and we call it semitotal domination polynomial of G. Here, we try to count the number of this kind of dominating sets and study the semitotal domination polynomial for certain graphs.

Theorem 3.1. *(i) For every $i \neq n$, $d_{t2}(K_{1,n}, i) = 0$ and $d_{t2}(K_{1,n}, n) = 1$.*
(ii) *For every $n \geq 3$, $D_{t2}(K_{1,n}, x) = x^n$.*

Proof. (i) Since $\gamma_{t2}(K_{1,n}) = n$ so for $i < n, d_{t2}(K_{1,n}, i) = 0$ and since the distance between central vertex with other vertices is equal one, so this vertex is not in the semitotal dominating set of $K_{1,n}$, and so $d_{t2}(K_{1,n}, n) = 1$.
(ii) Follows from Part (i) and the definition of the semitotal domination polynomial.

□

Theorem 3.2. *For a bipartite graph $K_{m,n}$ with $3 \geq m$ and $m \leq n$, we have*

$$d_{t2}(K_{m,n}, i) = \begin{cases} 0 & i \leq m-1 \\ \binom{m+n}{m} - \binom{n}{m} - m\binom{n}{m-1} & i = m \\ \binom{m+n}{i} - \binom{n}{i} - m\binom{n}{i-1} - n\binom{m}{i-1} & i > m, i \neq n \\ \binom{m+n}{n} - mn - n\binom{m}{n-1} & i = n \end{cases}$$

Proof. If $m \leq 3, \gamma_{t2}(K_{m,n}) = m$ and so $d_{t2}(K_{m,n}, i) = 0$ for $i \leq m-1$. If $i \geq m$, the number of sets with i vertices is $\binom{m+n}{i}$, but since the distance between one vertex of the first section and one vertex of the second section is one, so some of these sets are not semitotal dominating set. The number of i-sets which cannot be semitotal dominating sets, are $\binom{m}{1} \times \binom{n}{i-1}$, $\binom{n}{1} \times \binom{m}{i-1}$ and also for $i \neq n$, $\binom{n}{i}$. So we have the result. □

Similarly, we have the following theorem:

Theorem 3.3. *For a bipartite graph $K_{m,n}$ with $4 \leq m \leq n$, we have*

$$d_{t2}(K_{m,n}, i) = \begin{cases} 0 & i \leq 3 \\ \binom{m+n}{m} - \binom{n}{m} - m\binom{n}{m-1} & i = m \\ \binom{m+n}{i} - \binom{n}{i} - m\binom{n}{i-1} - n\binom{m}{i-1} & i > 3, i \neq n \\ \binom{m+n}{n} - mn - n\binom{m}{n-1} & i = n \end{cases}$$

Theorem 3.4. (i) *For every $i \geq n \geq 2$, $d_{t2}(F_n, i) = 2^n\binom{n}{i-n}$.*
(ii) *For every $n \geq 2$, $D_{t2}(F_n, x) = 2^n x^n (1+x)^n$.*

Proof. (i) Since $\gamma_{t2}(F_n) = n$ so if $i < n$, $d_{t2}(F_n, i) = 0$. If $i \geq n$, then first we should select n vertex from sides of n triangles by 2^n methods and then select $i-n$ vertex with $\binom{n}{i-n}$ ways. Therefore $d_{t2}(F_n, i) = 2^n\binom{n}{i-n}$.
(ii) It follows by Part (i) and definition of semitotal domination polynomial.

□

We need the following lemma to obtain more results:

Lemma 3.5. [7] *If G is a connected graph on $n \geq 4$ vertices, then $\gamma_{t_2}(G) \leq \frac{n}{2}$.*

Goddard, Henning, and McPillan in [7] characterized the trees with semitotal domination number exactly one-half order. They defined a family $\mathcal{T}$ of trees as follows. Let H be a nontrivial tree and for each vertex v of H, add either a P_2 or a P_4 and identify v with one end vertex of the path. They proved the following theorem:

Theorem 3.6. *Let T be a tree of order $n \geq 4$. Then $\gamma_{t_2}(T) = \frac{n}{2}$ if and only if $T \in \mathcal{T}$ or $T = K_{1,3}$.*

Now, we state and prove the following result:

Theorem 3.7. *For every tree $T \in \mathcal{T}$, $D_{t_2}(T, x) = D(T, x)$.*

Proof. We should prove that for every $i \geq \gamma_{t_2}(T)$, $d_{t_2}(T, i) = d(T, i)$. Suppose that $i \geq \gamma_{t_2}(T)$, by Lemma 3.6, every dominating set of T with cardinality $i \geq \gamma_{t_2}(T)$ is a semitotal dominating set of T. Therefore, we have the result. □

Goddard, Henning, and McPillan in [7] extended Theorem 3.6 from trees to all graphs. For given graphs G and H and every vertex in G, form a copy of H and identify one vertex in the copy of H with the corresponding vertex in G. Let to denote this as $G \diamond H$ (See $P_5 \diamond C_4$ in Figure 1). The following theorem characterize the graphs with with minimum degree at least 2 whose semitotal domination number is exactly one-half order.

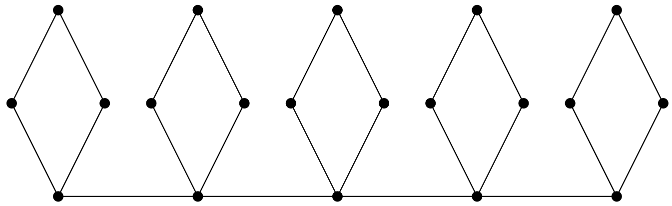

FIGURE 1. The graph $P_5 \diamond C_4$.

Theorem 3.8. [7] *Let G be a connected graph of order $n \geq 4$ with minimum degree at least 2. Then $\gamma_{t_2}(T) = \frac{n}{2}$ if and only if G is C_6, C_8, a spanning subgraph of K_4 or $H \diamond C_4$ for some graph H.*

Now, we have the following result:

Theorem 3.9. $D_{t_2}(H \diamond C_4, x) = D(H \diamond C_4, x)$.

Proof. We should prove that for every $i \geq \gamma_{t_2}(H \diamond C_4)$, $d_{t_2}(H \diamond C_4, i) = d(H \diamond C_4, i)$. Suppose that $i \geq \gamma_{t_2}(H \diamond C_4)$, by Lemma 3.8, every dominating set of $H \diamond C_4$ with cardinality $i \geq \gamma_{t_2}(H \diamond C_4)$ is a semitotal dominating set of $H \diamond C_4$. Therefore, we have the result. □

A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. Figure 2 shows a split graph partitioned into a clique (induced graph by $\{1, 2, 3\}$) and an independent set induced graph by $\{4, 5\}$.

Theorem 3.10. *If G is a connected split graph G with no dominating vertex (a vertex adjacent to all the other vertice), then*

$$D_t(G, x) = D_{t_2}(G, x) = D(G, x).$$

Proof. First we show that $\gamma(G) = \gamma_{t_2} = \gamma_t(G)$. It is suffices to prove that $\gamma_t(G) \leq \gamma(G)$. Suppose that $V(G) = C \cup I$ is a partition of the vertices of G into a clique C and an independent set I. Consider a minimum dominating set of G contained in C such as D. If D contains $v \in I$, then since no neighbour of v such as u is in D, $(D \setminus \{v\}) \cup \{u\}$ is a minimum dominating set containing less vertices of I. Since G has no dominating vertex, every dominating set contained in C is a total dominating set and so $\gamma_t(G) \leq \gamma(G)$. So every semitotal dominating set of cardinality i of G is a total dominating set of cardinality i and is a dominating set of G with cardinality i. Therefore $d(G, i) = d_t(G, i) = d_{t2}(G, i)$ and so we have the result. □

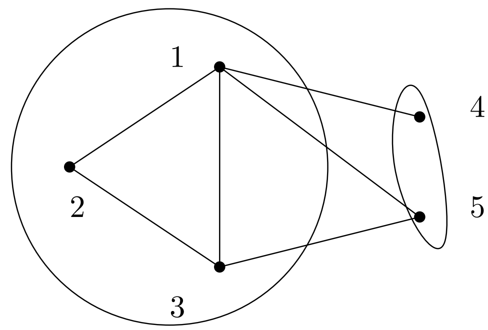

Figure 2. Example of split graph.

4. Stability of semitoal domination number

In this section, we introduce the semitotal domination stability of a graph and compute this parameter for some specific graphs.

Definition 4.1. Let G be a graph of order $n \geq 2$. The stabilizing on the semitotal domination number, or just semitotal, $st_{\gamma_{t2}}(G)$ of graph G is the minimum number of vertices whose removal changes the semitotal domination number.

Theorem 4.2. *If $m \leq n$, then*

$$st_{\gamma_{t2}}(K_{m,n}) = \begin{cases} 0, & m = 2 \\ 1, & 3 \leq m \leq 4 \\ m-3, & m > 4. \end{cases}$$

Proof. Suppose that $3 \leq m \leq 4$ and $m \leq n$. In this case $\gamma_{t2}(K_{m,n}) = \min\{m,n\} = m$ and so removing one vertex changes $\gamma_{t2}(K_{m,n})$. Therefore in this case, $st_{\gamma_{t2}}(K_{m,n}) = 1$. Suppose that $4 < m \leq n$, in this case $\gamma_{t2}(K_{m,n}) = 4$, so the number of minimum vertex whose removal changes the $\gamma_{t2}(K_{m,n})$ is $m-3$.

Theorem 4.3.

$$st_{\gamma_{t2}}(P_n) = st_{\gamma_{t2}}(C_n) = \begin{cases} 1, & n = 5k+1, \quad n = 5k+3 \\ 2, & n = 5k+2, \quad n = 5k-1 \\ 3, & n = 5k. \end{cases}$$

Proof. We know that $\gamma_{t2}(P_n) = \lceil \frac{2n}{5} \rceil$ (Theorem 2.1). For $n = 5k+1$ and $n = 5k+3$, $\gamma_{t2}(P_{n-1}) = \gamma_{t2}(P_n) - 1$. So in this case $st_{\gamma_{t2}}(P_n) = 1$. With similar arguments we have the results for another cases.

Theorem 4.4.

$$st_{\gamma_{t2}}(W_n) = \begin{cases} 1, & n = 3k+2 \\ 2, & n = 3k \\ 3, & n = 3k+1. \end{cases}$$

Proof. We know that $\gamma_{t2}(W_n) = \lceil \frac{n-1}{3} \rceil$ (Theorem 2.1). Since $\lceil \frac{3k+2-1}{3} \rceil = \lceil \frac{3k+1-1}{3} \rceil + 1$ so for the case $n = 3k+2$, $st_{\gamma_{t2}}(W_n) = 1$. With similar arguments we have the results for another cases.

Theorem 4.5. *(i) If* $5 \leq n \leq 10$ *and* $m > n$ *then*

$$st_{\gamma_{t2}}(P_n \vee P_m) = \begin{cases} 1 & n = 5k+1, \quad n = 5k+3 \\ 2 & n = 5k+2, \quad n = 5k-1 \\ 3 & n = 5k. \end{cases}$$

(ii) *If* $n > 10$ *and* $n \leq m$ *then* $st_{\gamma_{t2}}(P_n \vee P_m) = n - 7$.

Proof. (i) Suppose that $5 \leq n \leq 10$ and $m > n$. Since $\gamma_{t2}(P_n \vee P_m) = \gamma_{t2}(P_n)$ so in this case $st_{\gamma_{t2}}(P_n \vee P_m) = st_{\gamma_{t2}}(P_n)$.
(ii) If $n > 10$ by Theorem 2.2, $\gamma_{t2}(P_n \vee P_m) = 4$ and by attention to $\gamma_{t2}(P_7) = \lceil \dfrac{2 \times 7}{5} \rceil = 3$ we conclude $st_{\gamma_{t2}}(P_n \vee P_m) = n - 7$.

□

Theorem 4.6. $st_{\gamma_{t2}}(P_n \square P_m) = \lceil \frac{2n}{5} \rceil$.

Theorem 4.7. *(i) If* F_n *is a friendship graph, then* $st_{\gamma_{t2}}(F_n) = 2$.
(ii) *If* B_n *is a book graph (the Cartesian product* $K_{1,n} \square P_2$*), then* $st_{\gamma_{t2}}(B_n) = 1$.
(iii) *If* S_n *is a star graph then* $st_{\gamma_{t2}}(S_n) = 1$

Proof. (i) Since $\gamma_{t2}(F_n) = n$, so $\gamma_{t2}(F_{n-1}) = n-1$ and obviously to reach from F_n to F_{n-1}, we need to remove two vertices. So $st_{\gamma_{t2}}(F_n) = 2$.
(ii) Since $\gamma_{t2}(B_n) = n+1$, so $\gamma_{t2}(B_{n-1}) = n$ and obviously to reach from B_n to B_{n-1}, we need to remove two vertices. So $st_{\gamma_{t2}}(B_n) = 2$.
(iii) Since $\gamma_{t2}(S_n) = n$ and center vertex is not in semitotal dominating set, so by removing one vertex of S_n we reach to S_{n-1}. Therefore $st_{\gamma_{t2}}(S_n) = 1$.

□

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