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FUZZY GE-FILTERS OF GE-ALGEBRAS

R. BANDARU*, T. G. ALEMAYEHU AND Y. B. JUN

ABSTRACT. In this paper, notions of $\in_t$-set and Q_t-set of a fuzzy set f in a GE-algebra X are introduced and defined fuzzy GE-algebra in terms of $\in_t$-set. We provided conditions for the $\in_t$-set and Q_t-set of a fuzzy set f to be GE-subalgebras of X. We provided conditions for a fuzzy set in a GE-algebra to be a fuzzy GE-algebra. The concept of fuzzy GE-filter of a GE-algebra is introduced and investigated its properties. We explored the conditions under which the $\in_t$-set and Q_t-set can be GE-filters. Some characterizations of fuzzy GE-filters of GE-algebras are given.

1. INTRODUCTION

In mathematics, Hilbert algebras occur in the theory of von Neumann algebras in: Commutation theorem and Tomita-Takesaki theory. The concept of Hilbert algebra was introduced in early nineteen fifties by L. Henkin and T. Skolem for some investigations of implication in intuitionistic and other nonclassical logics. In 60- ties, these algebras were studied especially by A. Horn and A. Diego from algebraic point of view. Hilbert algebras are an important tool for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication ($\rightarrow$) and the constant 1 which is considered as the logical value "true". Many researchers studied various things about Hilbert algebras [2, 3, 4, 5, 6, 7, 8, 9, 10]. As a generalization of Hilbert algebras, R.K. Bandaru et al. [1] introduced the notion of GE-algebras. They have

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*Corresponding author .

studied the various properties and introduced different substructures of GE-algebras.

The fundamental concept of a fuzzy set, introduced by Zadeh in 1965 [12], provides a natural foundation for treating mathematically the fuzzy phenomena which exist pervasively in our real world and for building new branches of fuzzy mathematics. In [11], authors redefined a fuzzy point in such a way that it takes a crisp singleton, equivalently, an ordinary point, as a special case. As for the neighborhood structure of such a fuzzy point, in addition to the relation "$\in$" between fuzzy points and fuzzy sets and the corresponding neighborhood systems, they introduced another important relation "Q" between fuzzy points and fuzzy sets, called the Q-relation, and the corresponding neighborhood structure, called the Q-neighborhood system. In an ordinary topological space, as a special case of a fuzzy topological space, these concepts, neighborhood system and Q-neighborhood system, c-relation and Q-relation coincide respectively.

With this motivation, in this paper we introduce the notions of $\in_t$-set and Q_t-set of a fuzzy set f in a GE-algebra X and define fuzzy GE-algebra in terms of $\in_t$-set. We provide conditions for the $\in_t$-set and Q_t-set of a fuzzy set f to be GE-subalgebras of X. We provide conditions for a fuzzy set in a GE-algebra to be a fuzzy GE-algebra. We introduce the concept of fuzzy GE-filter of a GE-algebra and investigate its properties. We explore the conditions under which the $\in_t$-set and Q_t-set can be GE-filters. We give some characterizations of fuzzy GE-filters of GE-algebras.

2. Preliminaries

Definition 2.1 ([1]). By a *GE-algebra*, we mean a set X with a constant "1" and a binary operation " $*$ " satisfying the following axioms:

(GE1) $a * a = 1$,
(GE2) $1 * a = a$,
(GE3) $a * (b * c) = a * (b * (a * c))$

for all $a, b, c \in X$.

A binary relation " $\leq$ " in a GE-algebra X is defined by:

$$(\forall x, y \in X)(x \leq y \;\Leftrightarrow\; x * y = 1). \tag{2.1}$$

Proposition 2.2 ([1])**.** *Every GE-algebra X satisfies the following items.*

$$(\forall a \in X)\,(a * 1 = 1)\,. \tag{2.2}$$
$$(\forall a, b \in X)\,(a * (a * b) = a * b)\,. \tag{2.3}$$
$$(\forall a, b \in X)\,(a \leq b * a)\,. \tag{2.4}$$
$$(\forall a, b, c \in X)\,(a * (b * c) \leq b * (a * c))\,. \tag{2.5}$$
$$(\forall a \in X)\,(1 \leq a \ \Rightarrow \ a = 1)\,. \tag{2.6}$$
$$(\forall a, b \in X)\,(a \leq (a * b) * b)\,. \tag{2.7}$$

Definition 2.3 ([1])**.** A subset F of a GE-algebra X is called

- a *GE-subalgebra* of X if $a * b \in F$ for all $a, b \in F$.
- a *GE-filter* of X if it satisfies:

$$1 \in F, \tag{2.8}$$
$$(\forall a, b \in X)(a \in F,\, a * b \in F \ \Rightarrow \ b \in F). \tag{2.9}$$

A fuzzy set f in a set X of the form

$$f(b) := \begin{cases} t \in (0,1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $\frac{a}{t}$.

For a fuzzy set f in a set X and $t \in (0,1]$, we say that a fuzzy point $\frac{a}{t}$ is

(i) *contained* in f, denoted by $\frac{a}{t} \in f$, (see [11]) if $f(a) \geq t$.
(ii) *quasi-coincident* with f, denoted by $\frac{a}{t}\, q\, f$, (see [11]) if $f(a) + t > 1$.

If $\frac{a}{t}\, \alpha\, f$ is not established for $\alpha \in \{\in, q\}$, it is denoted by $\frac{a}{t}\, \overline{\alpha}\, f$.

3. Fuzzy GE-algebras

In what follows, let X be a GE-algebra unless otherwise specified. Given $t \in (0,1]$ and a fuzzy set f in X, consider the following sets

$$(f, t)_{\in} := \{x \in X \mid \tfrac{x}{t} \in f\} \text{ and } (f, t)_q := \{x \in X \mid \tfrac{x}{t}\, q\, f\}$$

which are called an *$\in_t$-set* and *Q_t-set* of f, respectively, in X.

Definition 3.1. A fuzzy set f in X is called a *fuzzy GE-algebra* of X if it satisfies:

$$x \in (f, t_a)_{\in},\ y \in (f, t_b)_{\in} \ \Rightarrow \ x * y \in (f, \min\{t_a, t_b\})_{\in} \tag{3.1}$$

for all $x, y \in X$ and $t_a, t_b \in (0,1]$.

Example 3.2. Let $X = \{1, 2, 3, 4, 5\}$ be a set with a binary operation "$*$" given by Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	3	5
3	1	2	1	1	5
4	1	2	1	1	5
5	1	2	4	4	1

Then it is routine to verify that $(X, *, 1)$ is a GE-algebra. Define a fuzzy set f in X as follows:

$$f : X \to [0, 1],\ x \mapsto \begin{cases} 0.5 & \text{if } x = 1, \\ 0.2 & \text{if } x = 2, \\ 0.3 & \text{if } x = 5, \\ 0.1 & \text{otherwise.} \end{cases}$$

It is routine to verify that f is a fuzzy GE-algebra of X.

Theorem 3.3. *A fuzzy set f in X is a fuzzy GE-algebra of X if and only if it satisfies:*

$$(\forall x, y \in X)(f(x * y) \geq \min\{f(x), f(y)\}). \tag{3.2}$$

Proof. Assume that f is a fuzzy GE-algebra of X. Note that $x \in (f, f(x))_\in$ and $y \in (f, f(y))_\in$ for all $x, y \in X$. Hence

$$x * y \in (f, \min\{f(x), f(y)\})_\in \text{ by (3.1)},$$

and so $f(x * y) \geq \min\{f(x), f(y)\}$ for all $x, y \in X$.

Conversely, suppose that f satisfies the condition (3.2). Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $x \in (f, t_a)_\in$ and $y \in (f, t_b)_\in$. Then $\frac{x}{t_a} \in f$ and $\frac{y}{t_b} \in f$, that is, $f(x) \geq t_a$ and $f(y) \geq t_b$. It follows from (3.2) that

$$f(x * y) \geq \min\{f(x), f(y)\} \geq \min\{t_a, t_b\}.$$

Hence $\frac{x*y}{\min\{t_a,t_b\}} \in f$, and so $x * y \in (f, \min\{t_a, t_b\})_\in$. Therefore f is a fuzzy GE-algebra of X. □

Corollary 3.4. *If f is a fuzzy GE-algebra of X, then $f(1) \geq f(x)$ for all $x \in X$.*

Proof. It is straightforward by the combination of (GE1) and (3.2). □

Corollary 3.5. *If a fuzzy GE-algebra f of X is order reversing, then it is constant.*

Proof. Note that $x * 1 = 1$, i.e., $x \leq 1$ for all $x \in X$. Since f is order reversing, it follows that $f(x) \geq f(1)$ for all $x \in X$. Hence $f(x) = f(1)$ for all $x \in X$. Therefore f is constant. $\square$

Theorem 3.6. *A fuzzy set f in X is a fuzzy GE-algebra of X if and only if the $\in_t$-set of f in X is a GE-subalgebra of X for all $t \in (0, 1]$.*

Proof. Assume that f is a fuzzy GE-algebra of X. Let $x, y \in (f, t)_\in$ for all $t \in (0, 1]$. Then $f(x) \geq t$ and $f(y) \geq t$. It follows from Theorem 3.3 that $f(x*y) \geq \min\{f(x), f(y)\} \geq t$, i.e., $\frac{x*y}{t} \in f$. Hence $x*y \in (f, t)_\in$, and therefore $(f, t)_\in$ is a GE-subalgebra of X.

Conversely, suppose that the $\in_t$-set of f in X is a GE-subalgebra of X for all $t \in (0, 1]$. If there are $a, b \in X$ such that $f(a * b) < \min\{f(a), f(b)\}$, then $\frac{a}{t} \in f$ and $\frac{b}{t} \in f$, that is, $a, b \in (f, t)_\in$ where $t = \min\{f(a), f(b)\}$. Hence $a * b \in (f, t)_\in$, and so $f(a * b) \geq t$. This is a contradiction, and thus $f(x * y) \geq \min\{f(x), f(y)\}$ for all $x, y \in X$. Therefore f is a fuzzy GE-algebra of X by Theorem 3.3. $\square$

Theorem 3.7. *If f is a fuzzy GE-algebra of X, then the set*

$$X_0 := \{x \in X \mid f(x) > 0\}$$

is a GE-subalgebra of X.

Proof. Let $x, y \in X_0$. Then $f(x) > 0$ and $f(y) > 0$. Since $x \in (f, f(x))_\in$ and $y \in (f, f(y))_\in$, it follows from (3.1) that

$$x * y \in (f, \min\{f(x), f(y)\})_\in.$$

Hence $f(x * y) \geq \min\{f(x), f(y)\} > 0$, and so $x * y \in X_0$. Therefore X_0 is a GE-subalgebra of X. $\square$

In the following example, we can observe the converse of Theorem 3.7 may not be true.

Example 3.8. Consider the GE-algebra $(X, *, 1)$ in Example 3.2. Define a mapping $f : X \to [0, 1]$ as follows:

$$f(x) = \begin{cases} 0.3 & \text{if } x = 1, \\ 0.2 & \text{if } x = 2, \\ 0.5 & \text{if } x = 5, \\ 0.1 & \text{otherwise.} \end{cases}$$

Then $X_0 = X$ is a GE-subalgebra of X. But f is not a fuzzy GE-algebra of X since $5 \in X$, $t_a = 0.5 \in (0, 1]$, $0.5 = \min\{t_a, t_a\}$ and $f(1) = 0.3$ but $5 * 5 = 1 \notin (f, \min\{t_a, t_a\})_\in$.

Theorem 3.9. *If a fuzzy set f in X satisfies:*

$$x \in (f, t_a)_\in,\ y \in (f, t_b)_\in \ \Rightarrow\ x * y \in (f, \max\{t_a, t_b\})_q \tag{3.3}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the set X_0 is a GE-subalgebra of X.

Proof. Assume that f satisfies the condition (3.3) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$. Let $x, y \in X_0$. Since $x \in (f, f(x))_\in$ and $y \in (f, f(y))_\in$, it follows from (3.3) that $x * y \in (f, \max\{f(x), f(y)\})_q$. If $x * y \notin X_0$, then $f(x * y) = 0$, and so

$$f(x * y) + \max\{f(x), f(y)\} = \max\{f(x), f(y)\} \leq 1.$$

Hence $\frac{x*y}{\max\{f(x),f(y)\}}\, \overline{q}\, f$, i.e., $x * y \notin (f, \max\{f(x), f(y)\})_q$. This is a contradiction, and so $x * y \in X_0$ which completes the proof. □

Theorem 3.10. *Let F be a GE-subalgebra of X and consider a fuzzy set f in X described as follows.*

$$f : X \to [0, 1],\ x \mapsto \begin{cases} t_0 \in (0, 1] & \text{if } x \in F, \\ t_1 & \text{otherwise} \end{cases}$$

where $t_0 > t_1$ in $(0, 1]$. Then f is a fuzzy GE-algebra of X.

Proof. Straightforward. □

Theorem 3.11. *If f is a fuzzy GE-algebra of X, then the set*

$$X_f := \{x \in X \mid f(x) = f(1)\}$$

is a GE-subalgebra of X.

Proof. Using Corollary 3.4, we know that

$$\begin{aligned} (f, f(1))_\in &= \{x \in X \mid \tfrac{x}{f(1)} \in f\} = \{x \in X \mid f(x) \geq f(1)\} \\ &= \{x \in X \mid f(x) = f(1)\} = X_f \end{aligned}$$

which is a GE-subalgebra of X by Theorem 3.6. □

We provide conditions for a fuzzy set to be a fuzzy GE-algebra.

Theorem 3.12. *If a fuzzy set f in X satisfies:*

$$b \in (f, t_b)_\in,\ c \in (f, t_c)_\in \ \Rightarrow\ a * b \in (f, \min\{t_b, t_c\})_\in \tag{3.4}$$

for all $t_b, t_c \in (0, 1]$ and $a, b, c \in X$ with $a \leq c$, then f is a fuzzy GE-algebra of X.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 1]$ be such that $x \in (f, t_a)_\in$ and $y \in (f, t_b)_\in$. Since $x \leq x$ for all $x \in X$, we have $x * y \in (f, \min\{t_a, t_b\})_\in$ by (3.4). Therefore f is a fuzzy GE-algebra of X. □

The following example shows that a fuzzy GE-algebra f of X does not satisfy the condition (3.4), that is, there exist $t_b, t_c \in (0,1]$ and $a, b, c \in X$ such that $a \leq c$ and

$$b \in (f, t_b)_\in,\ c \in (f, t_c)_\in \ \not\Rightarrow\ a * b \in (f, \min\{t_b, t_c\})_\in. \tag{3.5}$$

Example 3.13. Let $X = \{1, 2, 3, 4, 5\}$ be a set with a binary operation "$*$" given by Table 2. Then it is routine to verify that $(X, *, 1)$ is a

Table 2. Cayley table for the binary operation "$*$"

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	1	1
3	1	5	1	1	5
4	1	2	1	1	2
5	1	1	1	4	1

GE-algebra. Define a fuzzy set f in X as follows:

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = 1, \\ 0.8 & \text{if } x = 2, \\ 0.5 & \text{if } x = 3, \\ 0.7 & \text{otherwise.} \end{cases}$$

It is routine to verify that f is a fuzzy GE-algebra of X. Note that $3 \leq 1$, $2 \in (f, 0.8)_\in$ and $1 \in (f, 0.9)_\in$, but $3 * 2 = 5 \notin (f, 0.8)_\in = (f, \min\{0.8, 0.9\})_\in$. Hence f does not satisfy (3.5).

We provide conditions for the $\in_t$-set and Q_t-set of f to be GE-subalgebras of X.

Theorem 3.14. *If f is a fuzzy set in X that satisfies:*

$$(\forall x, y \in X)\,(\min\{f(x), f(y)\} \leq \max\{f(x * y), 0.5\})\,, \tag{3.6}$$

then the $\in_t$-set of f is a GE-subalgebra of X for all $t \in (0.5, 1]$.

Proof. Let $t \in (0.5, 1]$ and $x, y \in (f, t)_\in$. Then $\frac{x}{t} \in f$ and $\frac{y}{t} \in f$, that is, $f(x) \geq t$ and $f(y) \geq t$. It follows from (3.6) that

$$\max\{f(x * y), 0.5\} \geq \min\{f(x), f(y)\} \geq t > 0.5.$$

Hence $\frac{x*y}{t} \in f$, i.e., $x * y \in (f, t)_\in$. Therefore $(f, t)_\in$ is a GE-subalgebra of X. $\square$

Theorem 3.15. *The converse of Theorem 3.14 is also true, that is, if the $\in_t$-set of f is a GE-subalgebra of X for all $t \in (0.5, 1]$, then f satisfies the condition (3.6).*

Proof. Assume that the $\in_t$-set of f is a GE-subalgebra of X for all $t \in (0.5, 1]$. If the condition (3.6) is not established, then

$$s := \min\{f(a), f(b)\} > \max\{f(a * b), 0.5\}$$

for some $a, b \in X$. Then $s \in (0.5, 1]$, $\frac{a}{s} \in f$ and $\frac{b}{s} \in f$. Hence $a, b \in (f, s)_\in$ and thus $a * b \in (f, s)_\in$ by hypothesis. By the way, $\max\{f(a * b), 0.5\} < s$ induces $\frac{a*b}{s} \overline{\in} f$ and so $a * b \notin (f, s)_\in$. This is a contradiction, and therefore $\min\{f(x), f(y)\} \leq \max\{f(x * y), 0.5\}$ for all $x, y \in X$. □

Theorem 3.16. *If f is a fuzzy GE-algebra of X, then the Q_t-set of f is a GE-subalgebra of X for all $t \in (0, 1]$.*

Proof. Let $t \in (0, 1]$ and $x, y \in (f, t)_q$. Then $\frac{x}{t} \, q \, f$ and $\frac{y}{t} \, q \, f$, that is, $f(x) + t > 1$ and $f(y) + t > 1$. It follows from Theorem 3.3 that

$$f(x * y) + t \geq \min\{f(x), f(y)\} + t = \min\{f(x) + t, f(y) + t\} > 1.$$

Hence $\frac{x*y}{t} \, q \, f$, i.e., $x * y \in (f, t)_q$. Consequently, $(f, t)_q$ is a GE-subalgebra of X. □

Theorem 3.17. *If the Q_t-set of f is a GE-subalgebra of X, then f satisfies:*

$$x \in (f, t_a)_q, \; y \in (f, t_b)_q \;\Rightarrow\; x * y \in (f, \max\{t_a, t_b\})_\in \tag{3.7}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 0.5]$.

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 0.5]$ be such that $x \in (f, t_a)_q$ and $y \in (f, t_b)_q$. Then $x, y \in (f, \max\{t_a, t_b\})_q$ and so $x*y \in (f, \max\{t_a, t_b\})_q$ by hypothesis. Since $\max\{t_a, t_b\} \leq 0.5$, it follows that

$$f(x * y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\}$$

Hence $\frac{x*y}{\max\{t_a, t_b\}} \in f$, that is, $x * y \in (f, \max\{t_a, t_b\})_\in$. □

Theorem 3.18. *If a fuzzy set f in X satisfies the condition (3.7) for all $x, y \in X$ and $t_a, t_b \in (0, 1]$, then the set X_0 which is given in Theorem 3.7 is a GE-subalgebra of X.*

Proof. Let $x, y \in X_0$. Then $\frac{x}{1} \, q \, f$ and $\frac{y}{1} \, q \, f$. Thus

$$x * y \in (f, \max\{1, 1\})_\in = (f, 1)_\in$$

by (3.7). If $x * y \notin X_0$, then $f(x * y) = 0 < 1$ and so $x * y \notin (f, 1)_\in$, a contradiction. Hence $x * y \in X_0$ which completes the proof. □

4. Fuzzy GE-filters

Definition 4.1. A fuzzy set f in X is called a *fuzzy GE-filter* of X if it satisfies:

$$(\forall t \in (0,1])\,((f,t)_{\in} \neq \emptyset \;\Rightarrow\; 1 \in (f,t)_{\in}), \tag{4.1}$$

$$x * y \in (f,t_b)_{\in},\, x \in (f,t_a)_{\in} \;\Rightarrow\; y \in (f,\min\{t_a,t_b\})_{\in} \tag{4.2}$$

for all $x, y \in X$ and $t_a, t_b \in (0,1]$.

Example 4.2. Let $X = \{1,2,3,4,5\}$ be a set with a binary operation "$*$" given by Table 3.

Table 3. Cayley table for the binary operation "$*$"

$*$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	4	4
3	1	2	1	5	5
4	1	2	1	1	1
5	1	2	1	1	1

Then $(X, *, 1)$ is a GE-algebra (see [1]). Define a fuzzy set f in X as follows:

$$f : X \to [0,1]\ x \mapsto \begin{cases} 0.7 & \text{if } x = 1, \\ 0.4 & \text{if } x = 2, \\ 0.5 & \text{if } x = 5, \\ 0.3 & \text{otherwise.} \end{cases}$$

It is routine to verify that f is a fuzzy GE-filter of X.

Theorem 4.3. *A fuzzy set f in X is a fuzzy GE-filter of X if and only if it satisfies:*

$$(\forall x \in X)(f(1) \geq f(x)). \tag{4.3}$$

$$(\forall x, y \in X)(f(y) \geq \min\{f(x * y), f(x)\}). \tag{4.4}$$

Proof. Assume that f is a fuzzy GE-filter of X. Suppose there exists $a \in X$ such that $f(1) < f(a)$. Let $t_0 = \frac{1}{2}(f(1) + f(a))$. Then $f(1) < t_0$ and $0 < t_0 < f(a) \leq 1$. Hence $a \in (f,t_0)_{\in}$ and so $(f,t_0)_{\in} \neq \emptyset$. Thus $1 \in (f,t_0)_{\in}$, that is, $f(1) \geq t_0$, which is contradiction. Hence $f(1) \geq f(x)$ for all $x \in X$. Let $x, y \in X$ be such that $f(x) = t_1$ and $f(x * y) = t_2$. Then $x \in (f,t_1)_{\in}$ and $x * y \in (f,t_2)_{\in}$. Since f is a fuzzy GE-filter of X, we have $y \in (f,\min\{t_1,t_2\})_{\in}$. Hence $f(y) \geq \min\{t_1,t_2\} = \min(f(x), f(x * y))$.

Conversely, assume that f satisfies (4.3) and (4.4). Let $t \in (0,1]$ and $x \in (f,t)_\in$. Then $f(x) \geq t$ and hence $f(1) \geq f(x) \geq t$. Thus $1 \in (f,t)_\in$. Let $x, y \in X$ be such that $x \in (f,t_1)_\in$ and $x * y \in (f,t_2)_\in$. Then $f(x) \geq t_1$ and $f(x*y) \geq t_2$. Therefore $f(y) \geq \min\{f(x), f(x*y)\} \geq \min\{t_1,t_2\}$ by (4.4). Hence $y \in (f, \min\{t_1,t_2\})_\in$. Thus f is a fuzzy GE-filter of X. □

Proposition 4.4. *Every fuzzy GE-filter f of X satisfies:*

$$(\forall x, y \in X)(\forall t_a \in (0,1])\,(x \leq y,\, x \in (f,t_a)_\in \;\Rightarrow\; y \in (f,t_a)_\in)\,, \tag{4.5}$$

$$(\forall x, y, z \in X)(\forall t_a, t_b \in (0,1]) \left(\begin{array}{l} z \leq y * x,\, y \in (f,t_b)_\in,\, z \in (f,t_a)_\in \\ \Rightarrow\; x \in (f, \min\{t_a,t_b\})_\in \end{array} \right). \tag{4.6}$$

Proof. Let $x, y \in X$ and $t_a \in (0,1]$ be such that $x \leq y$ and $x \in (f,t_a)_\in$. Then $x*y = 1$, and so $f(y) \geq \min\{f(x*y), f(x)\} = \min\{f(1), f(x)\} = f(x) \geq t_a$ by Theorem 4.3. Hence $\frac{y}{t_a} \in f$, that is, $y \in (f,t_a)_\in$. Let $x, y, z \in X$ and $t_a, t_b \in (0,1]$ be such that $z \leq y * x$, $y \in (f,t_b)_\in$ and $z \in (f,t_a)_\in$. Then $z*(y*x) = 1$, $f(y) \geq t_b$ and $f(z) \geq t_a$. Hence

$$\begin{aligned} f(x) &\geq \min\{f(y*x), f(y)\} \\ &\geq \min\{\min\{f(z*(y*x)), f(z)\}, f(y)\} \\ &= \min\{\min\{f(1), f(z)\}, f(y)\} \\ &= \min\{f(y), f(z)\} \geq \min\{t_a, t_b\} \end{aligned}$$

and so $\frac{x}{\min\{t_a,t_b\}} \in f$, i.e., $x \in (f, \min\{t_a,t_b\})_\in$. □

The combination of (2.7) and (4.5) induces the following corollary.

Corollary 4.5. *Every fuzzy GE-filter f of X satisfies:*

$$(\forall x, a \in X)(\forall t \in (0,1])(a \in (f,t)_\in \;\Rightarrow\; (a*x)*x \in (f,t)_\in).$$

Proposition 4.6. *If f is a fuzzy GE-filter of X, then* (4.5) *and* (4.6) *are equivalent to the following facts, respectively.*

$$(\forall x, y \in X)(x \leq y \;\Rightarrow\; f(y) \geq f(x)), \tag{4.7}$$

$$(\forall x, y, z \in X)(z \leq y*x \;\Rightarrow\; f(x) \geq \min\{f(y), f(z)\}). \tag{4.8}$$

Proof. Suppose that (4.5) is valid and let $x, y \in X$ be such that $x \leq y$. Since $x \in (f, f(x))_\in$, we have $y \in (f, f(x))_\in$ by (4.5). Thus $f(y) \geq f(x)$. Suppose that (4.6) is valid and let $x, y, z \in X$ be such that $z \leq y*x$. Since $y \in (f, f(y))_\in$ and Since $z \in (f, f(z))_\in$, it follows from (4.6) that $x \in (f, \min\{f(y), f(z)\})_\in$. Hence $f(x) \geq \min\{f(y), f(z)\}$.

Conversely, suppose that (4.7) and (4.8) are valid. Let $x, y \in X$ and $t_a \in (0,1]$ be such that $x \leq y$ and $x \in (f,t_a)_\in$. Then $f(y) \geq$

$f(x) \geq t_a$ by (4.7), and so $y \in (f, t_a)_{\in}$. Now, let $x, y, z \in X$ and $t_a, t_b \in (0,1]$ be such that $z \leq y * x$, $y \in (f, t_b)_{\in}$ and $z \in (f, t_a)_{\in}$. Then $f(x) \geq \min\{f(y), f(z)\} \geq \min\{t_b, t_a\}$ by (4.8). Therefore $x \in (f, \min\{t_a, t_b\})_{\in}$. □

Proposition 4.7. *Every fuzzy GE-filter f of X satisfies:*

$$y \in (f, t_b)_{\in} \;\Rightarrow\; x * y \in (f, t_b)_{\in}, \tag{4.9}$$

$$y \in (f, t_b)_{\in},\ z \in (f, t_c)_{\in} \;\Rightarrow (y * (z * x)) * x \in (f, \min\{t_b, t_c\})_{\in} \tag{4.10}$$

for all $x, y, z \in X$ and $t_b, t_c \in (0,1]$.

Proof. Let $x, y \in X$ be such that $y \in (f, t_b)_{\in}$ for $t_b \in (0,1]$. Then

$$f(x * y) \geq \min\{f(y * (x * y)), f(y)\} = \min\{f(1), f(y)\} = f(y) \geq t_b$$

by (2.4) and Theorem 4.3. Hence $x * y \in (f, t_b)_{\in}$. Now let $x, y, z \in X$ be such that $y \in (f, t_b)_{\in}$ and $z \in (f, t_c)_{\in}$ for $t_b, t_c \in (0,1]$. Then $f(y) \geq t_b$ and $f(z) \geq t_c$. Using (GE3), (2.4), Theorem 4.3, and (4.4), we get

$$\begin{aligned} f(z * ((y * (z * x)) * x)) &= f(z * ((y * (z * x)) * (z * x))) \\ &\geq f((y * (z * x)) * (z * x)) \\ &\geq \min\{f(y), f(y * ((y * (z * x)) * (z * x)))\} \\ &= \min\{f(y), f(1)\} = f(y). \end{aligned}$$

It follows from (4.4) that

$$\begin{aligned} f((y * (z * x)) * x) &\geq \min\{f(z), f(z * ((y * (z * x)) * x))\} \\ &\geq \min\{f(y), f(z)\}. \end{aligned}$$

Hence $(y * (z * x)) * x \in (f, \min\{t_b, t_c\})_{\in}$. □

Let f be a fuzzy set in X that satisfies two conditions (4.9) and (4.10). Let $t \in (0,1]$ be such that $(f, t)_{\in} \neq \emptyset$. Then there exists $y \in (f, t)_{\in}$, and so $1 = y * y \in (f, t)_{\in}$ by (GE1) and (4.9). Let $x, y \in X$ be such that $x \in (f, t_a)_{\in}$ and $x * y \in (f, t_b)_{\in}$. Then

$$y = 1 * y = ((x * y) * (x * y)) * y \in (f, \min\{t_b, t_c\})_{\in}$$

by (GE2), (GE1) and (4.10). Therefore we have the following theorem.

Theorem 4.8. *If a fuzzy set f in X satisfies two conditions (4.9) and (4.10), then f is a fuzzy GE-filter of X.*

Theorem 4.9. *A fuzzy set f in X is a fuzzy GE-filter of X if and only if it satisfies (4.1) and*

$$(\forall x, y, z \in X)(\forall t_a, t_b \in (0,1]) \left(\begin{array}{c} y \in (f, t_b)_{\in},\ x * (y * z) \in (f, t_a)_{\in} \\ \Rightarrow\ x * z \in (f, \min\{t_a, t_b\})_{\in} \end{array} \right). \tag{4.11}$$

Proof. Assume that f is a fuzzy GE-filter of X. Let $x, y, z \in X$ and $t_a, t_b \in (0, 1]$ be such that $y \in (f, t_b)_\in$ and $x * (y * z) \in (f, t_a)_\in$. Then $y * (x * z) \in (f, t_a)_\in$ by (2.5) and (4.5). It follows from (4.2) that $x * z \in (f, \min\{t_a, t_b\})_\in$.

Conversely, suppose that f satisfies (4.1) and (4.11). Let $y, z \in X$ and $t_a, t_b \in (0, 1]$ be such that $y \in (f, t_b)_\in$ and $y * z \in (f, t_b)_\in$. Then $1 * (y * z) = y * z \in (f, t_b)_\in$ by (GE2), and so $z = 1 * z \in (f, t_b)_\in$ by (GE2) and (4.11). Hence f is a fuzzy GE-filter of X. □

Theorem 4.10. *If f and g are fuzzy GE-filters of X, then so is their intersection.*

Proof. Let f and g be fuzzy GE-filters of X. Then

$$(f \cap g)(1) = \min\{f(1), g(1)\} \geq \min\{f(x), g(x)\} = (f \cap g)(x)$$

and

$$\begin{aligned}(f \cap g)(y) &= \min\{f(y), g(y)\}\\ &\geq \min\{\min\{f(x), f(x * y)\}, \min\{g(x), g(x * y)\}\}\\ &= \min\{\min\{f(x), g(x)\}, \min\{f(x * y), g(x * y)\}\}\\ &= \min\{(f \cap g)(x)\,(f \cap g)(x * y)\}\end{aligned}$$

for all $x, y \in X$. It follows from Theorem 4.3 that $f \cap g$ is a fuzzy GE-filter of X. □

The following example shows that the union of fuzzy filters may not be a fuzzy filter.

Example 4.11. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be a set with a binary operation "$*$" given by Table 4.

Table 4. Cayley table for the binary operation "$*$"

$*$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	1	1	4	6	6	1	1
3	1	2	1	5	5	5	8	8
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1
7	1	2	1	6	6	6	1	1
8	1	2	5	5	5	5	3	1

Then $(X, *, 1)$ is a GE-algebra. Let f and g be fuzzy sets in X defined by

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = 1, \\ 0.8 & \text{if } x = 3, \\ 0.5 & \text{otherwise,} \end{cases}$$

and

$$g : X \to [0,1],\ x \mapsto \begin{cases} 0.95 & \text{if } x = 1, \\ 0.8 & \text{if } x = 8, \\ 0.4 & \text{otherwise,} \end{cases}$$

respectively. It is routine to verify that f and g are fuzzy GE-filters of X.
The union $f \cup g$ of f and g is given as follows.

$$f \cup g : X \to [0,1],\ x \mapsto \begin{cases} 0.95 & \text{if } x = 1, \\ 0.8 & \text{if } x \in \{3, 8\}, \\ 0.5 & \text{otherwise.} \end{cases}$$

But $f \cup g$ is not a fuzzy GE-filter of X since $8 \in (f \cup g, 0.63)_\in$ and $8 * 7 = 3 \in (f \cup g, 0.58)_\in$, but $7 \notin (f \cup g, 0.58)_\in = (f \cup g, \min\{0.63, 0.58\})_\in$.

We explore the conditions under which the $\in_t$-set and Q_t-set can be GE-filters.

Theorem 4.12. *Given a fuzzy set f in X, its $\in_t$-set $(f, t)_\in$ is a GE-filter of X for all $t \in (0.5, 1]$ if and only if f satisfies:*

$$(\forall x \in X)(f(x) \leq \max\{f(1), 0.5\}), \tag{4.12}$$

$$(\forall x, y \in X)(\min\{f(x), f(x * y)\} \leq \max\{f(y), 0.5\}). \tag{4.13}$$

Proof. Assume that the $\in_t$-set $(f, t)_\in$ of f is a GE-filter of X for all $t \in (0.5, 1]$. If there exists $a \in X$ such that $f(a) \not\leq \max\{f(1), 0.5\}$, then $t := f(a) \in (0.5, 1]$, $\frac{a}{t} \in f$ and $\frac{1}{t} \overline{\in} f$, that is, $a \in (f, t)_\in$ and $1 \notin (f, t)_\in$. This is a contradiction, and thus $f(x) \leq \max\{f(1), 0.5\}$ for all $x \in X$. If (4.13) is not valid, then

$$\min\{f(a), f(a * b)\} > \max\{f(b), 0.5\}$$

for some $a, b \in X$. If we take $t := \min\{f(a), f(a * b)\}$, then $t \in (0.5, 1]$, $\frac{a}{t} \in f$ and $\frac{a*b}{t} \in f$. Hence $a \in (f, t)_\in$ and $a * b \in (f, t)_\in$, which imply that $b \in (f, t)_\in$. Thus $\frac{b}{t} \in f$, and so $f(b) \geq t > 0.5$ which is a contradiction. Therefore $\min\{f(x), f(x * y)\} \leq \max\{f(y), 0.5\}$ for all $x, y \in X$.

Conversely, suppose that f satisfies (4.12) and (4.13). Let $(f, t)_\in \neq \emptyset$ for all $t \in (0.5, 1]$. Then there exists $a \in (f, t)_\in$ and thus $\frac{a}{t} \in f$, i.e., $f(a) \geq t$. It follows from (4.12) that $\max\{f(1), 0.5\} \geq f(a) \geq t > 0/5$.

Thus $\frac{1}{t} \in f$, i.e., $1 \in (f,t)_\in$. Let $t \in (0.5, 1]$ and $x, y \in X$ be such that $x \in (f,t)_\in$ and $x * y \in (f,t)_\in$. Then $\frac{x}{t} \in f$ and $\frac{x*y}{t} \in f$, that is, $f(x) \geq t$ and $f(x*y) \geq t$. Using (4.13), we get

$$\max\{f(y), 0.5\} \geq \min\{f(x), f(x*y)\} \geq t > 0.5$$

and so $\frac{y}{t} \in f$, i.e., $y \in (f,t)_\in$. Therefore $(f,t)_\in$ is a GE-filter of X for all $t \in (0.5, 1]$. □

Theorem 4.13. *A fuzzy set f in X is a fuzzy GE-filter of X if and only if the nonempty $\in_t$-set $(f,t)_\in$ of f in X is a GE-filter of X for all $t \in (0,1]$.*

Proof. Suppose that f is a fuzzy GE-filter of X. Let $t \in (0,1]$ be such that $(f,t)_\in \neq \emptyset$. Then there exists $a \in (f,t)_\in$, and so $\frac{a}{t} \in f$. It follows from (4.3) that $f(1) \geq f(a) \geq t$. Hence $1 \in (f,t)_\in$. Let $x, y \in X$ be such that $x \in (f,t)_\in$ and $x*y \in (f,t)_\in$. Then $f(x) \geq t$ and $f(x*y) \geq t$, which imply from (4.4) that

$$f(y) \geq \min\{f(x*y), f(x)\} \geq t.$$

Hence $y \in (f,t)_\in$. Consequently, $(f,t)_\in$ is a GE-filter of X for all $t \in (0,1]$.

Conversely, assume that the nonempty $\in_t$-set $(f,t)_\in$ is a GE-filter of X for all $t \in (0,1]$. If $f(1) < f(a)$ for some $a \in X$, then $a \in (f, f(a))_\in$ and $1 \notin (f, f(a))_\in$. This is a contradiction, and thus $f(1) \geq f(x)$ for all $x \in X$. If there exist $a, b \in X$ such that $f(b) < \min\{f(a), f(a*b)\}$, then $a \in (f,t)_\in$, $a*b \in (f,t)_\in$ but $b \notin (f,t)_\in$ for $t := \min\{f(a), f(a*b)\}$. This is a contradiction, and hence $f(y) \geq \min\{f(x), f(x*y)\}$ for all $x, y \in X$. Therefore f is a fuzzy GE-filter of X by Theorem 4.3. □

Theorem 4.14. *If f is a fuzzy GE-filter of X, then the nonempty Q_t-set $(f,t)_q$ of f is a GE-filter of X for all $t \in (0,1]$.*

Proof. Let f be a fuzzy GE-filter of X and assume that $(f,t)_q \neq \emptyset$ for all $t \in (0,1]$. Then there exists $a \in (f,t)_q$, and so $\frac{a}{t}\, q\, f$, i.e., $f(a)+t > 1$. Hence $f(1) + t \geq f(a) + t > 1$, i.e., $1 \in (f,t)_q$. Let $x, y \in X$ be such that $x \in (f,t)_q$ and $x * y \in (f,t)_q$. Then $\frac{x}{t}\, q\, f$ and $\frac{x*y}{t}\, q\, f$, that is, $f(x) + t > 1$ and $f(x*y) + t > 1$. It follows that

$$f(y) + t \geq \min\{f(x), f(x*y)\} + t = \min\{f(x) + t, f(x*y) + t\} > 1.$$

Hence $\frac{y}{t}\, q\, f$, and therefore $y \in (f,t)_q$. Consequently, $(f,t)_q$ is a GE-filter of X for all $t \in (0,1]$. □

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References

1. R. K. Bandaru, A. Borumand Saeid and Y. B. Jun, *On GE-algebras*, Bulletin of the Section of Logic, **50** (2021), no. 1, 81–96.
2. R. A. Borzooei and J. Shohani, *On generalized Hilbert algebras*, Ital. J. Pure Appl. Math., **29** (2012), 71–86.
3. I. Chajda and R. Halas, *Congruences and idealas in Hilbert algebras*, Kyungpook Math. J., **39** (1999), 429–432.
4. I. Chajda, R. Halas and Y.B. Jun, *Annihilators and deductive systems in commutative Hilbert algebras*, Comment. Math. Univ. Carolinae, (3) **43** (2002), 407–417.
5. A. Diego, *Sur les algebres de Hilbert*, Collection de Logique Mathematique, Edition Hermann, Serie A, XXI, 1966.
6. Y. B. Jun, *Commutative Hilbert algebras*, Soochow Journal of Mathematics, (4) **22** (1996), 477–484.
7. Y. B. Jun and K. H. Kim, *H-filters of Hilbert algebras*, Sci. Math. Jpn. **e-2005**, 231–236.
8. A. S. Nasab and A. Borumand Saeid, *Semi maximal filter in Hilbert algebras*, Journal of Intelligent & Fuzzy Systems, **30** (2016), 7–15.
9. A. S. Nasab and A. Borumand Saeid, *Stonean Hilbert algebra*, Journal of Intelligent & Fuzzy Systems, **30** (2016), 485–492.
10. A. S. Nasab and A. Borumand Saeid, *Study of Hilbert algebras in point of filters*, An St. Univ. Ovidius Constanta, (2) **24** (2016), 221–251.
11. P. M. Pu and Y. M. Liu, *Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl. **76** (1980), 571–599.
12. L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353.

Ravikumar Bandaru
Department of Mathematics, School of Advanced Sciences, VIT-AP University, Amaravati 522237, Andhra Pradesh, India
Email: ravimaths83@gmail.com

Teferi Getachew Alemayehu
Department of Mathematics, Debre Berhan University, Debre Berhan 445 Ethiopia
Email: teferigetachew3@gmail.com

Young Bae Jun
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea
Email: skywine@gmail.com