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Weak cotorsion modules with respect to an integer and a semidualizing bimodule

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Abstract. Let R and S be rings, $C = {}_SC_R$ a faithfully semidualizing bimodule, and m a non-negative integer. In this paper, we present some homological properties relative to the class of R-modules of C-weak injective dimension at most m and the class of S-modules of C-weak flat dimension at most m. We introduce and study weak cotorsion modules with respect to m and C by using the class of modules of C-weak flat dimension at most m.

Keywords: C-m-weak cotorsion modules, C-weak flat dimensions, C-weak injective dimensions, Semidualizing bimodules.
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1 Introduction

Throughout this paper, m is a non-negative integer, n is a positive integer, R and S are fixed associative rings with unites, and all R- or S-modules are understood to be unital left R- or S-modules (unless specified otherwise). ${}_SM$ (resp. M_R) is used to denote that M is a left S-module (resp. right R-module). Also, ${}_SM_R$ is used to denote that M is an (S, R)-bimodule which means that M is both a left S-module and a right R-module, and these structures are compatible. Right R- or S-modules are identified with left modules over the opposite rings R^{op} and S^{op}.

Enochs in [8] introduced the concept of cotorsion modules as a generalization of cotorsion abelian groups. It differs from Matlis' definition in [19] whose concern was with domains. Since then, the investigation of these modules have become a vigorously active area of research. We refer the reader to [3, 8, 17] for background on cotorsion modules. In [18], Mao and Ding

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introduced the concept of m-cotorsion modules as a new classification of cotorsion modules by using of the class of modules of flat dimension at most m.

Gao and Wang in [12] introduced the concept of weak injective and weak flat modules by using super finitely presented modules. They investigated the properties of modules with weak injective and weak flat dimension at most m. In [25], Zhao investigated homological properties of modules with finite weak injective and weak flat dimensions. Gao and Zhao in [13] introduced the concept of C-weak injective and C-weak flat modules by using a semidualizing bimodule $C = {}_SC_R$. Throughout this paper, $C = {}_SC_R$ stands for a faithfully semidualizing bimodule.

Selvaraj and Prabakaran in [21] introduced a particular case of m-cotorsion modules and called them m-weak cotorsion modules by using of the class of modules of weak flat dimension at most m instead of the class of modules of flat dimension at most m.

In recent years, homological algebra on weak injective and weak flat modules have been researched extensively by many authors (see, for example, [1,2,12,13,21,25]). In this paper, we introduce and study C-m-weak cotorsion modules by using of the class of modules of C-weak flat dimension at most m.

In Section 2, some fundamental concepts and some preliminary results are stated. In Section 3, we give some homological relationships between the classes $\mathcal{WI}(S)_{\leq m}$, $\mathcal{WF}(R)_{\leq m}$, $\mathcal{WI}_C(R)_{\leq m}$, $\mathcal{WF}_C(S)_{\leq m}$, $\mathcal{A}_C(R)$, and $\mathcal{B}_C(S)$, where these classes are the class of S-modules of weak injective dimension at most m, the class of R-modules of weak flat dimension at most m, the class of R-modules of C-weak injective dimension at most m, the class of S-modules of C-weak flat dimension at most m, the Auslander class under C, and the Bass class under C, respectively. Among other results, we prove that (i) for an R-module M (resp. S-module N), $M \in \mathcal{WI}_C(R)_{\leq m}$ (resp. $N \in \mathcal{WF}_C(S)_{\leq m}$) if and only if $M \in \mathcal{A}_C(R)$ (resp. $N \in \mathcal{B}_C(S)$) and $C \otimes_R M \in \mathcal{WI}(S)_{\leq m}$ (resp. $\mathrm{Hom}_S(C, N) \in \mathcal{WF}(R)_{\leq m}$), and (ii) the classes $\mathcal{WI}_C(R)_{\leq m}$ and $\mathcal{WF}_C(S)_{\leq m}$ are preenveloping and covering. Section 4 is devoted to introduce an study C-m-weak cotorsion modules. We investigate the relationship between C-m-weak cotorsion modules and reduced C-m-weak cotorsion modules. For an R^{op}-module M, we show that if ${}_RR \in \mathcal{WI}_C(R)_{\leq m}$ and ${}_{R^{op}}R^{op} \in \mathcal{WF}_C(R^{op})_{\leq m}$, then M is a C-m-weak cotorsion R^{op}-module if and only if M is a direct sum of an injective R^{op}-module and a reduced C-m-weak cotorsion R^{op}-module. We also find that when every module is C-m-weak cotorsion.

2 Preliminaries

In this section, some fundamental concepts are recalled and notations are stated.

Definition 1. (see [4, Section 1 and Definitions 3.1 and 3.2], [11, Definition 2.1], [12, Definitions 2.1 and 3.2], [8, Section 2], [18, Definition 4.1], [21, Definition 1], [13, 1.2, 1.3, and 1.4, and Definition 2.1], and [23, Definition 3.1])

(i) An R-module M is called *finitely n-presented* if there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

such that F_i is a finitely generated free (equivalently, finitely generated projective) R-module for all $0 \leq i \leq n$. An R-module M is called *FP_n-injective* or *$(n, 0)$-injective* (resp. *FP_n-flat* or *$(n,0)$-flat*) if $\mathrm{Ext}^1_R(L,M)=0$ (resp. $\mathrm{Tor}^R_1(L,M)=0$) for any finitely n-presented R-module (resp. R^{op}-module) L;

(ii) An R-module M is called *super finitely presented* if there exists an exact sequence

$$\cdots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

such that F_i is a finitely generated free (equivalently, finitely generated projective) R-module for all $i \geq 0$. An R-module M is called *weak injective* (resp. *weak flat*) if $\mathrm{Ext}^1_R(L,M)=0$ (resp. $\mathrm{Tor}^R_1(L,M)=0$) for any super finitely presented R-module (resp. R^{op}-module) L. We denote the class of all weak injective (resp. weak flat) R-modules by $\mathcal{WI}(R)$ (resp. $\mathcal{WF}(R)$);

(iii) For any R-module M, the *weak injective dimension* (resp. *weak flat dimension*) of M, denoted by $\mathrm{wid}_R(M)$ (resp. $\mathrm{wfd}_R(M)$), is defined to be the smallest non-negative integer k such that $\mathrm{Ext}^{k+1}_R(L,M)=0$ (resp. $\mathrm{Tor}^R_{k+1}(L,M)=0$) for all super finitely presented R-modules (resp. R^{op}-module) L. If no such k exists, set $\mathrm{wid}_R(M)=\infty$ (resp. $\mathrm{wfd}_R(M)=\infty$). We denote the class of all R-modules with weak injective (resp. weak flat) dimension less than or equal to m by $\mathcal{WI}(R)_{\leq m}$ (resp. $\mathcal{WF}(R)_{\leq m}$);

(iv) An R-module M is said to be *cotorsion* if $\mathrm{Ext}^1_R(L,M)=0$ for any flat R-module L. An R-module M is called *m-cotorsion* if $\mathrm{Ext}^1_R(L,M)=0$ for any R-module $L \in \mathcal{F}(R)_{\leq m}$, where the symbol $\mathcal{F}(R)_{\leq m}$ denotes the class of all R-modules with flat dimension less than or equal to m. An R-module M is said to be *m-weak cotorsion* if $\mathrm{Ext}^1_R(L,M)=0$ for any R-module $L \in \mathcal{WF}(R)_{\leq m}$. We denote the class of all m-cotorsion (resp. m-weak cotorsion) R-modules by $\mathcal{C}_m(R)$ (resp. $\mathcal{WC}_m(R)$);

(v) A *degreewise finite projective resolution* of an R-module M is a projective resolution of M

$$\cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

such that P_i is a finitely generated projective (equivalently, finitely generated free) R-module for all $i \geq 0$;

(vi) An (S,R)-bimodule $C = {}_SC_R$ is *semidualizing* if the following conditions hold:

(a_1) ${}_SC$ admits a degreewise finite S-projective resolution;

(a_2) C_R admits a degreewise finite R^{op}-projective resolution;

(b_1) The homothety map ${}_S\gamma : {}_SS_S \longrightarrow \mathrm{Hom}_{R^{op}}(C,C)$ is an isomorphism;

(b_2) The homothety map $\gamma_R : {}_RR_R \longrightarrow \mathrm{Hom}_S(C,C)$ is an isomorphism;

(c_1) $\mathrm{Ext}^i_S(C,C)=0$ for all $i \geq 1$;

(c_2) $\mathrm{Ext}^i_{R^{op}}(C,C)=0$ for all $i \geq 1$.

A semidualizing bimodule ${}_SC_R$ is *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_SN$ and M_R:

(1) If $\mathrm{Hom}_S(C,N)=0$, then $N=0$;

(2) If $\operatorname{Hom}_{R^{op}}(C, M) = 0$, then $M = 0$.

By [15, Proposition 3.2], there exist many examples of faithfully semidualizing bimodules were provided over a wide class of non-commutative rings;

(vii) *The Auslander class $\mathcal{A}_C(R)$ with respect to C consists of all R-modules M satisfying the following conditions:*

(A_1) $\operatorname{Tor}_i^R(C, M) = 0$ *for all* $i \geq 1$;

(A_2) $\operatorname{Ext}_S^i(C, C \otimes_R M) = 0$ *for all* $i \geq 1$;

(A_3) *The natural evaluation homomorphism* $\mu_M : M \longrightarrow \operatorname{Hom}_S(C, C \otimes_R M)$ *is an isomorphism (of R-modules).*

The Bass class $\mathcal{B}_C(S)$ with respect to C consists of all S-modules N satisfying the following conditions:

(B_1) $\operatorname{Ext}_S^i(C, N) = 0$ *for all* $i \geq 1$;

(B_2) $\operatorname{Tor}_i^R(C, \operatorname{Hom}_S(C, N)) = 0$ *for all* $i \geq 1$;

(B_3) *The natural evaluation homomorphism* $\nu_N : C \otimes_R \operatorname{Hom}_S(C, N) \longrightarrow N$ *is an isomorphism (of S-modules).*

It is an important property of Auslander and Bass classes that they are equivalent under the pair of functors:

$$\mathcal{A}_C(R) \underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\underset{\sim}{\rightleftarrows}}} \mathcal{B}_C(S)$$

(see [15, Proposition 4.1]);

(viii) An R-module is called C-FP_n-injective if it has the form $\operatorname{Hom}_S(C, E)$ for some FP_n-injective S-module E. An S-module is called C-FP_n-flat if it has the form $C \otimes_R F$ for some FP_n-flat R-module F;

(ix) An R-module is called C-weak injective if it has the form $\operatorname{Hom}_S(C, E)$ for some weak injective S-module E. An S-module is called C-weak flat if it has the form $C \otimes_R F$ for some weak flat R-module F. We denote the class of all C-weak injective R-modules by $\mathcal{WI}_C(R)$ and the class of C-weak flat S-modules by $\mathcal{WF}_C(S)$. Therefore $\mathcal{WI}_C(R) = \{\operatorname{Hom}_S(C, E) : E \in \mathcal{WI}(S)\}$ and $\mathcal{WF}_C(S) = \{C \otimes_R F : F \in \mathcal{WF}(R)\}$;

(x) The C-weak injective dimension of an R-module M is defined that C-$\operatorname{wid}_R(M) \leq m$ if and only if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{m-1} \longrightarrow E_m \longrightarrow 0$$

of R-modules such that $E_i \in \mathcal{WI}_C(R)$ for all $0 \leq i \leq m$. If no such exact sequence exists, set C-$\operatorname{wid}_R(M) = \infty$. Also, the C-weak flat dimension of an S-module N is defined that C-$\operatorname{wfd}_S(N) \leq m$ if and only if there exists an exact sequence

$$0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

of S-modules such that $F_i \in \mathcal{WF}_C(S)$ for all $0 \leq i \leq m$. If no such exact sequence exists, set C-$\mathrm{wfd}_S(N) = \infty$. We denote the class of all R-modules with C-weak injective (resp. C-weak flat) dimension less than or equal to m by $\mathcal{WI}_C(R)_{\leq m}$ (resp. $\mathcal{WF}_C(R)_{\leq m}$).

The following propositions, which are needed in the next section, show that the classes $\mathcal{WI}(R)_{\leq m}$ and $\mathcal{WF}(R)_{\leq m}$ are closed under direct summands, direct products, direct sums, direct limits, and extensions.

Proposition 1. *The following assertions hold:*

- (i) *The class $\mathcal{WI}(R)_{\leq m}$ is closed under direct summands, direct products, direct sums, and direct limits;*
- (ii) *The class $\mathcal{WF}(R)_{\leq m}$ is closed under direct summands, direct products, direct sums, and direct limits.*

Proof. (i). Let $M \in \mathcal{WI}(R)_{\leq m}$, let M' be a summand of M, and let L be a super finitely presented R-module. There exists an R-module M'' such that $M \cong M' \oplus M''$. Since $\mathrm{Ext}_R^{m+1}(L, M) = 0$ and $\mathrm{Ext}_R^{m+1}(L, M) \cong \mathrm{Ext}_R^{m+1}(L, M') \oplus \mathrm{Ext}_R^{m+1}(L, M'')$ by [20, Proposition 7.22], we have $\mathrm{Ext}_R^{m+1}(L, M') = 0$. Thus $M' \in \mathcal{WI}(R)_{\leq m}$. Now, let $\{M_j\}_{j \in J}$ be a family of R-modules (resp. direct system of R-modules with J directed) such that $M_j \in \mathcal{WI}(R)_{\leq m}$ for all $j \in J$, and L a super finitely presented R-module. Since $\mathrm{Ext}_R^{m+1}(L, M_j) = 0$ for all $j \in J$, $\mathrm{Ext}_R^{m+1}(L, \prod_{j \in J} M_j) \cong \prod_{j \in J} \mathrm{Ext}_R^{m+1}(L, M_j)$ from [20, Proposition 7.22], and $\mathrm{Ext}_R^{m+1}(L, \bigoplus_{j \in J} M_j) \cong \bigoplus_{j \in J} \mathrm{Ext}_R^{m+1}(L, M_j)$ (resp. $\mathrm{Ext}_R^{m+1}(L, \varinjlim_{j \in J} M_j) \cong \varinjlim_{j \in J} \mathrm{Ext}_R^{m+1}(L, M_j)$) by [5, Lemma 2.9(2)], we get $\mathrm{Ext}_R^{m+1}(L, \prod_{j \in J} M_j) = 0$ and $\mathrm{Ext}_R^{m+1}(L, \bigoplus_{j \in J} M_j) = 0$ (resp. $\mathrm{Ext}_R^{m+1}(L, \varinjlim_{j \in J} M_j) = 0$). Hence $\prod_{j \in J} M_j \in \mathcal{WI}(R)_{\leq m}$ and $\bigoplus_{j \in J} M_j \in \mathcal{WI}(R)_{\leq m}$ (resp. $\varinjlim_{j \in J} M_j \in \mathcal{WI}(R)_{\leq m}$).

(ii). By using [20, Propositions 7.6 and 7.8] and [5, Lemma 2.10(2)], the proof is similar to that of (i). □

Proposition 2. *The following statements hold true:*

- (i) *The class $\mathcal{WI}(R)_{\leq m}$ is closed under extensions;*
- (ii) *The class $\mathcal{WF}(R)_{\leq m}$ is closed under extensions.*

Proof. (i). Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of R-modules with $M', M'' \in \mathcal{WI}(R)_{\leq m}$. By putting $n = 0$ in [2, Proposition 3.2], there exist the exact sequences

$$0 \longrightarrow M' \longrightarrow E'_0 \longrightarrow E'_1 \longrightarrow \cdots \longrightarrow E'_{m-1} \longrightarrow E'_m \longrightarrow 0$$

and

$$0 \longrightarrow M'' \longrightarrow E''_0 \longrightarrow E''_1 \longrightarrow \cdots \longrightarrow E''_{m-1} \longrightarrow E''_m \longrightarrow 0$$

such that E'_i and E''_i are injective for all $0 \leq i \leq m-1$, and E'_m and E''_m are in $\mathcal{WI}(R)$. From Horseshoe lemma, we have the exact sequences

$$0 \longrightarrow M \longrightarrow E'_0 \oplus E''_0 \longrightarrow \cdots \longrightarrow E'_{m-1} \oplus E''_{m-1} \longrightarrow M_m \longrightarrow 0$$

and

$$0 \longrightarrow E'_m \longrightarrow M_m \longrightarrow E''_m \longrightarrow 0.$$

For all $0 \leq i \leq m-1$, $E'_i \oplus E''_i$ is in $\mathcal{WI}(R)$ by [12, Proposition 2.3(1)] and M_m is in $\mathcal{WI}(R)$ from [10, Proposition 2.6(1)]. Thus, again by taking $n = 0$ in [2, Proposition 3.2], $M \in \mathcal{WI}(R)_{\leq m}$.

(ii). This is similar to that of (i). □

Remark 1. *Every C-FP_n-injective (resp. C-FP_n-flat) module is C-weak injective (resp. C-weak flat).*

The following example shows that the converse of Remark 1 is not always true. Recall that a ring R is said to be an *$(n, 0)$-ring* or *n-regular ring* if every finitely n-presented R-module is projective (see [16, Section 1] and [26, Definition 3.7]).

Example 1. Let K be a field, E a K-vector space with infinite rank, and A a Noetherian ring of global dimension 0. Set $B = K \ltimes E$ the trivial extension of K by E and $R = A \times B$ the direct product of A and B. By *[16, Theorem 3.4(3)]*, R is a $(2, 0)$-ring which is not a $(1, 0)$-ring. Since every super finitely presented R-module is finitely 2-presented, it follows that, for every R-module M and every super finitely presented R-module L, $\mathrm{Ext}^1_R(L, M) = 0$ (resp. $\mathrm{Tor}^R_1(L, M) = 0$). Hence every R-module is weak injective (resp. weak flat). If $C = R = S$, then every R-module is C-weak injective (resp. C-weak flat). On the other hand, there exists an R-module which is not C-FP_1-injective (resp. C-FP_1-flat), since if every R-module is C-FP_1-injective (resp. C-FP_1-flat), *[26, Theorem 3.9]* implies that R is a $(1, 0)$-ring and this is a contradiction.

3 Homological properties of modules with C-weak injective and C-weak flat dimensions at most m

In this section, we give some homological relationships between the classes $\mathcal{WI}(S)_{\leq m}$, $\mathcal{WF}(R)_{\leq m}$, $\mathcal{WI}_C(R)_{\leq m}$, $\mathcal{WF}_C(S)_{\leq m}$, $\mathcal{A}_C(R)$, and $\mathcal{B}_C(S)$. From here to the end of the article, m is a non-negative integer and $C = {}_SC_R$ is a faithfully semidualizing bimodule.

The first main result of this section shows that, for an R-module M (resp. S-module N), $M \in \mathcal{WI}_C(R)_{\leq m}$ (resp. $N \in \mathcal{WF}_C(S)_{\leq m}$) if and only if $M \in \mathcal{A}_C(R)$ (resp. $N \in \mathcal{B}_C(S)$) and $C \otimes_R M \in \mathcal{WI}(S)_{\leq m}$ (resp. $\mathrm{Hom}_S(C, N) \in \mathcal{WF}(R)_{\leq m}$).

Theorem 1. *Let M be an R-module and N an S-module. Then the following statements hold true:*

(i) *C-$\mathrm{wid}_R(M) \leq m$ if and only if $M \in \mathcal{A}_C(R)$ and $\mathrm{wid}_S(C \otimes_R M) \leq m$;*

(ii) *C-$\mathrm{wfd}_S(N) \leq m$ if and only if $N \in \mathcal{B}_C(S)$ and $\mathrm{wfd}_R(\mathrm{Hom}_S(C, N)) \leq m$.*

Proof. (i). ($\Rightarrow$) Assume that C-$\mathrm{wid}_R(M) \leq m$. Then $M \in \mathcal{A}_C(R)$ by [13, Proposition 3.3(2)] and $\mathrm{wid}_S(C \otimes_R M) \leq m$ from [13, Proposition 3.2].

($\Leftarrow$) Assume that $M \in \mathcal{A}_C(R)$ and $\mathrm{wid}_S(C \otimes_R M) \leq m$. Thus $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ and so C-$\mathrm{wid}_R(M) \leq m$ by [13, Proposition 3.2].

(ii). This is similar to the first part. □

In the following corollaries, we prove that the classes $\mathcal{WI}_C(R)_{\leq m}$ and $\mathcal{WF}_C(S)_{\leq m}$ are closed under direct summands, direct products, direct sums, direct limits, and extensions.

Corollary 1. *The following assertions hold:*

(i) *The class $\mathcal{WI}_C(R)_{\leq m}$ is closed under direct summands, direct products, direct sums, and direct limits;*

(ii) *The class $\mathcal{WF}_C(S)_{\leq m}$ is closed under direct summands, direct products, direct sums, and direct limits.*

Proof. (i). Let $M \in \mathcal{WI}_C(R)_{\leq m}$ and let M' be a summand of M. Then, from Theorem 1(i), $M \in \mathcal{A}_C(R)$ and $C\otimes_R M \in \mathcal{WI}(S)_{\leq m}$, and also there exists an R-module M'' such that $M \cong M'\oplus M''$. By [15, Proposition 4.2(a)], it follows that $M' \in \mathcal{A}_C(R)$. Also, from [20, Theorem 2.65], we have $C \otimes_R M \cong (C \otimes_R M') \oplus (C \otimes_R M'')$ which shows by Proposition 1(i) that $C \otimes_R M' \in \mathcal{WI}(S)_{\leq m}$. Hence $M' \in \mathcal{WI}_C(R)_{\leq m}$ from Theorem 1(i). Now, let $\{M_j\}_{j\in J}$ be a family of R-modules (resp. direct system of R-modules with J directed) such that $M_j \in \mathcal{WI}_C(R)_{\leq m}$ for all $j \in J$. Then, by Theorem 1(i), $M_j \in \mathcal{A}_C(R)$ and $C \otimes_R M_j \in \mathcal{WI}(S)_{\leq m}$ for all $j \in J$. Hence, from [15, Proposition 4.2(a)], $\prod_{j\in J} M_j \in \mathcal{A}_C(R)$ and $\bigoplus_{j\in J} M_j \in \mathcal{A}_C(R)$ (resp. $\varinjlim_{j\in J} M_j \in \mathcal{A}_C(R)$) and, by Proposition 1(i), $\prod_{j\in J}(C \otimes_R M_j) \in \mathcal{WI}(S)_{\leq m}$ and $\bigoplus_{j\in J}(C \otimes_R M_j) \in \mathcal{WI}(S)_{\leq m}$ (resp. $\varinjlim_{j\in J}(C \otimes_R M_j) \in \mathcal{WI}(S)_{\leq m}$) and so $C \otimes_R (\prod_{j\in J} M_j) \in \mathcal{WI}(S)_{\leq m}$ from [5, Lemma 2.10(2)] and $C \otimes_R (\bigoplus_{j\in J} M_j) \in \mathcal{WI}(S)_{\leq m}$ by [20, Proposition 7.6] (resp. $C \otimes_R (\varinjlim_{j\in J} M_j) \in \mathcal{WI}(S)_{\leq m}$ from [20, Proposition 7.8]). Thus $\prod_{j\in J} M_j \in \mathcal{WI}_C(R)_{\leq m}$ and $\bigoplus_{j\in J} M_j \in \mathcal{WI}_C(R)_{\leq m}$ (resp. $\varinjlim_{j\in J} M_j \in \mathcal{WI}_C(R)_{\leq m}$) by Theorem 1(i).

(ii). By using [20, Proposition 7.22] and [5, Lemma 2.9(2)], the proof is similar to that of (i). □

Corollary 2. *The following statements hold true:*

(i) *The class $\mathcal{WI}_C(R)_{\leq m}$ is closed under extensions;*

(ii) *The class $\mathcal{WF}_C(S)_{\leq m}$ is closed under extensions.*

Proof. (i). Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of R-modules with $M', M'' \in \mathcal{WI}_C(R)_{\leq m}$. Then, from Theorem 1(i), M' and M'' are in $\mathcal{A}_C(R)$ (and so $\mathrm{Tor}_1^R(C, M'') = 0$), and $C \otimes_R M'$ and $C \otimes_R M''$ are in $\mathcal{WI}(S)_{\leq m}$. Thus, $M \in \mathcal{A}_C(R)$ from [15, Corollary 6.3] and

$$0 \longrightarrow C \otimes_R M' \longrightarrow C \otimes_R M \longrightarrow C \otimes_R M'' \longrightarrow 0$$

is a short exact sequence of S-modules by applying the functor $C \otimes_R -$ to the above short exact sequence. Hence, $C \otimes_R M \in \mathcal{WI}(S)_{\leq m}$ from Proposition 2(i). Therefore, $M \in \mathcal{WI}_C(R)_{\leq m}$ by Theorem 1(i).

(ii). This is similar to that of (i). □

Assume that M' is an R-submodule of M. We say that M' is a *pure submodule* of M and M/M' is a *pure quotient* of M if

$$0 \longrightarrow A \otimes_R M' \longrightarrow A \otimes_R M \longrightarrow A \otimes_R M/M' \longrightarrow 0$$

is an exact sequence for all R^{op}-modules A, equivalently, if

$$0 \longrightarrow \mathrm{Hom}_R(B, M') \longrightarrow \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(B, M/M') \longrightarrow 0$$

is an exact sequence for all finitely 1-presented R-modules B [9, Definition 5.3.6].

In the next corollary, we prove that the classes $\mathcal{WI}_C(R)_{\leq m}$ and $\mathcal{WF}_C(S)_{\leq m}$ are closed under pure submodules and pure quotients.

Corollary 3. *Let M' be a pure submodule of R-module M and let N' be a pure submodule of S-module N. Then the following statements hold true:*

(i) *$M' \in \mathcal{WI}_C(R)_{\leq m}$ and $M/M' \in \mathcal{WI}_C(R)_{\leq m}$ whenever $M \in \mathcal{WI}_C(R)_{\leq m}$;*

(ii) *$N' \in \mathcal{WF}_C(S)_{\leq m}$ and $N/N' \in \mathcal{WF}_C(S)_{\leq m}$ whenever $N \in \mathcal{WF}_C(S)_{\leq m}$.*

Proof. (i). Since M' is a pure submodule of R-module M,

$$0 \longrightarrow C \otimes_R M' \longrightarrow C \otimes_R M \longrightarrow C \otimes_R M/M' \longrightarrow 0$$

is an exact sequence and $C \otimes_R M'$ is a pure submodule of $C \otimes_R M$ by [20, Proposition 2.57]. Assume that $M \in \mathcal{WI}_C(R)_{\leq m}$. Then $M \in \mathcal{A}_C(R)$ and $C \otimes_R M \in \mathcal{WI}(S)_{\leq m}$ from Theorem 1(i). By putting $n = 0$ in [2, Proposition 3.7(1)], we deduce that $C \otimes_R M' \in \mathcal{WI}(S)_{\leq m}$ and $C \otimes_R M/M' \in \mathcal{WI}(S)_{\leq m}$. Thus [13, Corollary 2.3] implies that $C \otimes_R M'$ and $C \otimes_R M/M'$ are in $\mathcal{B}_C(S)$. Hence, by [13, Lemma 2.9(2)], M' and M/M' are in $\mathcal{A}_C(R)$. Therefore M' and M/M' are in $\mathcal{WI}_C(R)_{\leq m}$ from Theorem 1(i).

(ii). By using [20, Theorem 2.76], the proof is similar to that of (i). □

The following result is another application of Theorem 1.

Corollary 4. *Let $M \in \mathcal{WI}_C(R)_{\leq m}$ and let*

$$0 \longrightarrow M \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{m-1} \longrightarrow M_m \longrightarrow 0$$

be an exact sequence of R-modules with $M_0, M_1, \ldots, M_{m-1} \in \mathcal{WI}_C(R)$. Then $M_m \in \mathcal{WI}_C(R)$.

Proof. By Theorem 1(i), $C \otimes_R M$ is in $\mathcal{WI}(S)_{\leq m}$, $C \otimes_R M_i$ is in $\mathcal{WI}(S)$ for all $0 \leq i \leq m-1$, and $M, M_0, M_1, \ldots, M_{m-1}$ are in $\mathcal{A}_C(R)$ and so M_m is in $\mathcal{A}_C(R)$ from [15, Corollary 6.3]. Thus $\mathrm{Tor}_j^R(C, M) = 0$ and $\mathrm{Tor}_j^R(C, M_i) = 0$ for all $j \geq 1$ and all $0 \leq i \leq m$. Hence, by applying the functor $C \otimes_R -$ to the above exact sequence, we obtain the exact sequence of S-modules

$$0 \longrightarrow C \otimes_R M \longrightarrow C \otimes_R M_0 \longrightarrow \cdots \longrightarrow C \otimes_R M_{m-1} \longrightarrow C \otimes_R M_m \longrightarrow 0.$$

By putting $n = 0$ in [2, Proposition 3.2], we deduce that $C \otimes_R M_m \in \mathcal{WI}(S)$. Therefore, $M_m \in \mathcal{WI}_C(R)$ from Theorem 1(i). □

The next corollary will be needed in the next section.

Corollary 5. *Let $N \in \mathcal{WF}_C(S)_{\leq m}$ and let*

$$0 \longrightarrow N_m \longrightarrow N_{m-1} \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow N \longrightarrow 0$$

be an exact sequence of S-modules with $N_0, N_1, \ldots, N_{m-1} \in \mathcal{WF}_C(S)$. Then $N_m \in \mathcal{WF}_C(S)$.

Proof. By using [2, Proposition 3.3], this is sufficiently similar to that of Corollary 4 to be omitted. We leave the proof to the reader. □

In the course of the remaining parts of the paper, we denote the character module of M by $M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ [20, Page 135].

Proposition 3. *Let M be an R-module and N an S-module. Then the following statements hold:*

(i) *$M \in \mathcal{WI}_C(R)_{\leq m}$ if and only if $M^* \in \mathcal{WF}_C(R^{op})_{\leq m}$;*

(ii) *$N \in \mathcal{WF}_C(S)_{\leq m}$ if and only if $N^* \in \mathcal{WI}_C(S^{op})_{\leq m}$.*

Proof. (i). ($\Rightarrow$) Assume that $M \in \mathcal{WI}_C(R)_{\leq m}$. Then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{m-1} \longrightarrow E_m \longrightarrow 0$$

of R-modules such that $E_i \in \mathcal{WI}_C(R)$ for all $0 \leq i \leq m$. Thus, by [20, Lemma 3.53],

$$0 \longrightarrow E_m^* \longrightarrow E_{m-1}^* \longrightarrow \cdots \longrightarrow E_1^* \longrightarrow E_0^* \longrightarrow M^* \longrightarrow 0$$

is an exact sequence of R^{op}-modules and, from [13, Proposition 2.6(2)], $E_i^* \in \mathcal{WF}_C(R^{op})$ for all $0 \leq i \leq m$. Hence $M^* \in \mathcal{WF}_C(R^{op})_{\leq m}$ as we desired.

($\Leftarrow$) Assume that $M^* \in \mathcal{WF}_C(R^{op})_{\leq m}$. Then there exists an exact sequence

$$0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M^* \longrightarrow 0$$

of R^{op}-modules such that $F_i \in \mathcal{WF}_C(R^{op})$ for all $0 \leq i \leq m$. Therefore, by [20, Lemma 3.53],

$$0 \longrightarrow M^{**} \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_{m-1}^* \longrightarrow F_m^* \longrightarrow 0$$

is an exact sequence of R-modules and, from [13, Proposition 2.6(1)], $F_i^* \in \mathcal{WI}_C(R)$ for all $0 \leq i \leq m$. Thus $M^{**} \in \mathcal{WI}_C(R)_{\leq m}$. On the other hand, M is a pure submodule of M^{**} by [24, Proposition 2.3.5] and hence $M \in \mathcal{WI}_C(R)_{\leq m}$ from Proposition 3(i).

(ii). This is similar to the first part. □

Let $\mathcal{F}$ be a class of R-modules and let M be an R-module. A morphism $f : F \longrightarrow M$ (resp. $f : M \longrightarrow F$) with $F \in \mathcal{F}$ is called an *$\mathcal{F}$-precover* (resp. *$\mathcal{F}$-preenvelope*) of M when $\mathrm{Hom}_R(F', F) \longrightarrow \mathrm{Hom}_R(F', M) \longrightarrow 0$ (resp. $\mathrm{Hom}_R(F, F') \longrightarrow \mathrm{Hom}_R(M, F') \longrightarrow 0$) is exact for all $F' \in \mathcal{F}$. Assume that $f : F \longrightarrow M$ (resp. $f : M \longrightarrow F$) is an $\mathcal{F}$-precover (resp. $\mathcal{F}$-preenvelope) of M. Then f is called an *$\mathcal{F}$-cover* (resp. *$\mathcal{F}$-envelope*) of M if every morphism $g : F \longrightarrow F$ such that $fg = f$ (resp. $gf = f$) is an isomorphism. The class $\mathcal{F}$ is called (*pre*)*covering* (resp. (*pre*)*enveloping*) if each R-module has an $\mathcal{F}$-(pre)cover (resp. $\mathcal{F}$-(pre)envelope) (see [9, Definitions 5.1.1 and 6.1.1]).

A *duality pair* over R is a pair $(\mathcal{M}, \mathcal{N})$ such that $\mathcal{M}$ is a class of R-modules and $\mathcal{N}$ is a class of R^{op}-modules, subject to the following conditions:

(i) For an R-module M, one has $M \in \mathcal{M}$ if and only if $M^* \in \mathcal{N}$;

(ii) $\mathcal{N}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{M}, \mathcal{N})$ is called *(co)product-closed* if the class $\mathcal{M}$ is closed under (co)products in the category of all R-modules (see [14, Definition 2.1]).

In the second main result of this section, we show that $\mathcal{WI}_C(R)_{\leq m}$ and $\mathcal{WF}_C(S)_{\leq m}$ are preenveloping and covering.

Theorem 2. *The following statements hold:*

(i) $(\mathcal{WI}_C(R)_{\leq m}, \mathcal{WF}_C(R^{op})_{\leq m})$ *is a duality pair and the class* $\mathcal{WI}_C(R)_{\leq m}$ *is preenveloping and covering;*

(ii) $(\mathcal{WF}_C(S)_{\leq m}, \mathcal{WI}_C(S^{op})_{\leq m})$ *is a duality pair and the class* $\mathcal{WF}_C(S)_{\leq m}$ *is preenveloping and covering.*

Proof. (i). By Proposition 3(i), an R-module M is in $\mathcal{WI}_C(R)_{\leq m}$ if and only if M^* is in $\mathcal{WF}_C(R^{op})_{\leq m}$. Also, from Corollary 1(ii), $\mathcal{WF}_C(R^{op})_{\leq m}$ is closed under direct summands and direct sums. Thus $(\mathcal{WI}_C(R)_{\leq m}, \mathcal{WF}_C(R^{op})_{\leq m})$ is a duality pair. For the last part, from Corollary 1(i), the class $\mathcal{WI}_C(R)_{\leq m}$ is closed under direct products and direct sums. Therefore, by [14, Theorem 3.1], the class $\mathcal{WI}_C(R)_{\leq m}$ is preenveloping and covering.

(ii). This is similar to that of (i). $\square$

Recall that, an injective R-module E is said to be an *injective cogenerator* for R-modules if for each R-module M and non-zero element $m \in M$, there is $f \in \mathrm{Hom}_R(M, E)$ such that $f(m) \neq 0$ (equivalently, $\mathrm{Hom}_R(M, E) \neq 0$ for any module $M \neq 0$). It is well-known that R^* is an injective cogenerator for R^{op}-modules [9, Definition 3.2.7].

Corollary 6. *The following assertions are equivalent:*

(i) ${}_RR$ *is in* $\mathcal{WI}_C(R)_{\leq m}$*;*

(ii) *Every* R^{op}*-module has a monic* $\mathcal{WF}_C(R^{op})_{\leq m}$*-preenvelope;*

(iii) *Every injective* R^{op}*-module is in* $\mathcal{WF}_C(R^{op})_{\leq m}$*;*

(iv) *Every flat* R*-module is in* $\mathcal{WI}_C(R)_{\leq m}$*;*

(v) *Every projective* R*-module is in* $\mathcal{WI}_C(R)_{\leq m}$*;*

(vi) *Every* R*-module has an epic* $\mathcal{WI}_C(R)_{\leq m}$*-cover.*

Proof. (i)⇒(ii). From Theorem 2(ii), every R^{op}-module M has a $\mathcal{WF}_C(R^{op})_{\leq m}$-preenvelope $f : M \longrightarrow F$. By Proposition 3(i), $R^* \in \mathcal{WF}_C(R^{op})_{\leq m}$, and so $\prod_{j \in J} R^* \in \mathcal{WF}_C(R^{op})_{\leq m}$ from Corollary 1(ii). Also, R^* is an injective cogenerator for R^{op}-modules. Thus we have the exact sequence $0 \longrightarrow M \xrightarrow{g} \prod_{j \in J} R^*$, and hence there exists a morphism $h : F \longrightarrow \prod_{j \in J} R^*$ such that $hf = g$. Since g is monic, we deduce that f is also monic.

(ii)⇒(iii). Let E be an injective R^{op}-module. By assumption, E has a monic $\mathcal{WF}_C(R^{op})_{\leq m}$-preenvelope $f : E \longrightarrow F$. Therefore, the exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow F/E \longrightarrow 0$$

is split, and so $F \cong E \oplus F/E$. Hence, from Corollary 1(ii), E is in $\mathcal{WF}_C(R^{op})_{\leq m}$.

(iii)⇒(iv). Let F be a flat R-module. Then, by [20, Poroposition 3.54], F^* is an injective R^{op}-module and so F^* is in $\mathcal{WF}_C(R^{op})_{\leq m}$ from assumption, and hence F is in $\mathcal{WI}_C(R)_{\leq m}$ by Proposition 3(i).

(iv)⇒(v). This is clear.

(v)⇒(i). It is clear.

(i)⇒(vi). From Theorem 2(i), every R-module M has a $\mathcal{WI}_C(R)_{\leq m}$-cover $f : F \longrightarrow M$. Also, there is a short exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow F' \xrightarrow{g} M \longrightarrow 0$$

where F' is free. Since R is in $\mathcal{WI}_C(R)_{\leq m}$, $F' \cong \bigoplus_{j\in J} R \in \mathcal{WI}_C(R)_{\leq m}$ by Corollary 1(i). So there exists a map $h : F' \to F$ such that $fh = g$. Since g is epic, we deduce that f is also epic.

(vi)⇒(i). By assumption, ${}_RR$ has an epic $\mathcal{WI}_C(R)_{\leq m}$-cover $f : F \longrightarrow R$. Therefore, the exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow F \longrightarrow R \longrightarrow 0$$

is split, and so $F \cong \mathrm{Ker}(f) \oplus R$. Hence, from Corollary 1(i), ${}_RR$ is in $\mathcal{WI}_C(R)_{\leq m}$. □

For a class of R-modules $\mathcal{F}$, we denote the class of all R-modules M such that $\mathrm{Ext}^1_R(M, F) = 0$ (resp. $\mathrm{Ext}^1_R(F, M) = 0$) for all $F \in \mathcal{F}$ by ${}^{\perp}\mathcal{F}$ (resp. $\mathcal{F}^{\perp}$). For a class of R-modules (resp. R^{op}-modules) $\mathcal{F}$, we denote the class of all R^{op}-modules (resp. R-modules) M such that $\mathrm{Tor}^R_1(M, F) = 0$ (resp. $\mathrm{Tor}^R_1(F, M) = 0$) for all $F \in \mathcal{F}$ by ${}^{\top}\mathcal{F}$ (resp. $\mathcal{F}^{\top}$) (see [9, Definition 7.1.1] and [22, Definition 1.10]).

Here, we give more properties of $\mathcal{WI}_C(R)_{\leq m}$-preenvelopes and $\mathcal{WF}_C(S)_{\leq m}$-preenvelopes.

Proposition 4. *Let M be an R-module and N an S-module. Let $f : M \longrightarrow F$ be a $\mathcal{WI}_C(R)_{\leq m}$-preenvelope of M where F is flat and $g : N \longrightarrow G$ a $\mathcal{WF}_C(S)_{\leq m}$-preenvelope of N where G is flat. Then the following statements hold:*

(i) $\mathrm{Coker}(f) \in \mathcal{WF}_C(R^{op})^{\top}_{\leq m}$;

(ii) $\mathrm{Coker}(g) \in \mathcal{WI}_C(S^{op})^{\top}_{\leq m}$.

Proof. (i). Assume that $I := \mathrm{Im}(f)$, $D := \mathrm{Coker}(f)$, and $L\mathcal{WF}_C(R^{op})_{\leq m}$. Then, we have the short exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow F \longrightarrow D \longrightarrow 0$$

and, from Proposition 3(ii), $L^* \in \mathcal{WI}_C(R)_{\leq m}$. Thus $\mathrm{Hom}_R(F, L^*) \longrightarrow \mathrm{Hom}_R(M, L^*) \longrightarrow 0$ is exact and so $\mathrm{Hom}_R(F, L^*) \longrightarrow \mathrm{Hom}_R(I, L^*) \longrightarrow 0$ is exact. Hence, by [20, Theorem 2.76], $(L \otimes_R F)^* \longrightarrow (L \otimes_R I)^* \longrightarrow 0$ is exact. Therefore $0 \longrightarrow L \otimes_R I \longrightarrow L \otimes_R F$ is exact from [20, Lemma 3.53]. On the other hand, we have $\mathrm{Tor}^R_1(L, F) = 0$ because F is flat. Thus, by the long exact sequence

$$\mathrm{Tor}^R_1(L, F) \longrightarrow \mathrm{Tor}^R_1(L, D) \longrightarrow L \otimes_R I \longrightarrow L \otimes_R F \longrightarrow L \otimes_R D \longrightarrow 0,$$

we get $\mathrm{Tor}^R_1(L, D) = 0$. Hence $D \in \mathcal{WF}_C(R^{op})^{\top}_{\leq m}$ as we desired.

(ii). This is similar to that of (i). □

Theorem 3. *Suppose that D is a super finitely presented R-module and that D' is a super finitely presented S-module. Then the following statements hold true:*

(i) *If ${}_RR \in \mathcal{WI}_C(R)_{\leq m}$, then $D \in \mathcal{WF}_C(R^{op})^{\top}_{\leq m}$ if and only if D is the cokernel of a $\mathcal{WI}_C(R)_{\leq m}$-preenvelope $f : M \longrightarrow F$ of an R-module M where F is free;*

(ii) *If ${}_SS \in \mathcal{WF}_C(S)_{\leq m}$, then $D' \in \mathcal{WI}_C(S^{op})^{\top}_{\leq m}$ if and only if D' is the cokernel of a $\mathcal{WF}_C(S)_{\leq m}$-preenvelope $g : N \longrightarrow G$ of an S-module N where G is free.*

Proof. (i). ($\Rightarrow$) Since D is a super finitely presented R-module, there exists an exact sequence

$$\cdots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow D \longrightarrow 0$$

such that F_i is a finitely generated free R-module for all $i \geq 0$. Set $F := F_0$ and $M := \mathrm{Ker}(F_0 \longrightarrow D)$, and let f be the inclusion map $M \longrightarrow F$. Then we have the short exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow F \longrightarrow D \longrightarrow 0.$$

Since ${}_RR \in \mathcal{WI}_C(R)_{\leq m}$, $F \cong \bigoplus_{j\in J} R \in \mathcal{WI}_C(R)_{\leq m}$ by Corollary 1(i). Assume that $F' \in \mathcal{WI}_C(R)_{\leq m}$. Then $F'^* \in \mathcal{WF}_C(R^{op})_{\leq m}$ from Proposition 3(i) and so $\mathrm{Tor}_1^R(F'^*, D) = 0$ because $D \in \mathcal{WF}_C(R^{op})^{\top}_{\leq m}$. Thus $0 \longrightarrow F'^* \otimes_R M \longrightarrow F'^* \otimes_R F$ is exact. Hence, by [20, Lemma 3.55 and Proposition 2.56], $0 \longrightarrow \mathrm{Hom}_R(M, F')^* \longrightarrow \mathrm{Hom}_R(F, F')^*$ is exact. Therefore $\mathrm{Hom}_R(F, F') \longrightarrow \mathrm{Hom}_R(M, F') \longrightarrow 0$ is exact from [20, Lemma 3.53].

($\Leftarrow$) It follows from Proposition 4(i).

(ii). This is similar to the first part. □

4 C-m-weak cotorsion modules

In this section, we introduce and study C-m-weak cotorsion modules. From here to the end of the article, we assume that $R = S$.

Definition 2. *An R-module M is called C-m-weak cotorsion if $\mathrm{Ext}^1_R(L, M) = 0$ for any R-module $L \in \mathcal{WF}_C(R)_{\leq m}$. We denote the class of all C-m-weak cotorsion R-modules by $\mathcal{CWC}_m(R)$.*

Remark 2. (i) $\mathcal{CWC}_m(R) = \mathcal{WF}_C(R)^{\perp}_{\leq m}$;

(ii) *Every injective module is C-m-weak cotorsion;*

(iii) *Every C-m-weak cotorsion module is C-m'-weak cotorsion for all $m' \leq m$ and so*

$$\cdots \subseteq \mathcal{CWC}_{m+1}(R) \subseteq \mathcal{CWC}_m(R) \subseteq \cdots \subseteq \mathcal{CWC}_1(R) \subseteq \mathcal{CWC}_0(R);$$

Proposition 5. *The following assertions hold:*

(i) *If $\{M_j\}_{j\in J}$ is a family of R-modules, then $\prod_{j\in J} M_j$ is C-m-weak cotorsion if and only if M_j is C-m-weak cotorsion for all $j \in J$;*

(ii) *$\mathcal{CWC}_m(R)$ is closed under direct summands;*

(iii) *$\mathcal{CWC}_m(R)$ is closed under extensions.*

Proof. (i). Assume that $L \in \mathcal{WF}_C(R)_{\leq m}$. By [20, Proposition 7.22], $\mathrm{Ext}^1_R(L, \prod_{j\in J} M_j) \cong \prod_{j\in J} \mathrm{Ext}^1_R(L, M_j)$ and so $\mathrm{Ext}^1_R(L, \prod_{j\in J} M_j) = 0$ if and only if $\mathrm{Ext}^1_R(L, M_j) = 0$ for all $j \in J$. Thus $\prod_{j\in J} M_j$ is C-m-weak cotorsion if and only if M_j is C-m-weak cotorsion for all $j \in J$.

(ii). Follows from the first part.

(iii). Assume that

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of R-modules with $M', M'' \in C\mathcal{WC}_m(R)$ and that $L \in \mathcal{WF}_C(R)_{\leq m}$. Then $\mathrm{Ext}^1_R(L, M') = 0$ and $\mathrm{Ext}^1_R(L, M'') = 0$. By applying the functor $\mathrm{Hom}_R(L, -)$ to the above exact sequence, we get the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_R(L, M') \longrightarrow \mathrm{Ext}^1_R(L, M) \longrightarrow \mathrm{Ext}^1_R(L, M'') \longrightarrow \cdots$$

which shows that $\mathrm{Ext}^1_R(L, M) = 0$. Hence $M \in C\mathcal{WC}_m(R)$. □

In the following results, we characterize C-m-weak cotorsion modules.

Proposition 6. *Let M be an R^{op}-module and let ${}_RR \in \mathcal{WI}_C(R)_{\leq m}$. Then the following conditions are equivalent:*

(i) *$M \in C\mathcal{WC}_m(R^{op})$;*

(ii) *For every short exact sequence of R^{op}-modules $0 \to M \to E \xrightarrow{f} D \to 0$ where E is injective, $f : E \longrightarrow D$ is a $\mathcal{WF}_C(R^{op})_{\leq m}$-precover of D;*

(iii) *M is the kernel of a $\mathcal{WF}_C(R^{op})_{\leq m}$-precover $f : E \longrightarrow D$ of an R^{op}-module D where E is injective.*

Proof. (i)⇒(ii). Let $0 \to M \to E \xrightarrow{f} D \to 0$ be a short exact sequence of R^{op}-modules where E is injective. From Corollary 6, $E \in \mathcal{WF}_C(R^{op})_{\leq m}$. Assume that $E' \in \mathcal{WF}_C(R^{op})_{\leq m}$. Then, by assumption, $\mathrm{Ext}^1_{R^{op}}(E', M) = 0$ and so, from applying the functor $\mathrm{Hom}_{R^{op}}(E', -)$ to the above short exact sequence, it follows that $\mathrm{Hom}_{R^{op}}(E', E) \to \mathrm{Hom}_{R^{op}}(E', D) \to 0$ is exact. Thus $f : E \longrightarrow D$ is a $\mathcal{WF}_C(R^{op})_{\leq m}$-precover of D.

(ii)⇒(iii). This is clear.

(iii)⇒(i). By assumption, there exists a $\mathcal{WF}_C(R^{op})_{\leq m}$-precover $f : E \longrightarrow D$ of an R^{op}-module D where E is injective and $M = \mathrm{Ker}(f)$. Set $I := \mathrm{Im}(f)$. Then we have the short exact sequence of R^{op}-modules $0 \to M \to E \to I \to 0$. Assume that $L \in \mathcal{WF}_C(R^{op})_{\leq m}$. Then, $\mathrm{Hom}_{R^{op}}(L, E) \to \mathrm{Hom}_{R^{op}}(L, D) \to 0$ is exact and so $\mathrm{Hom}_{R^{op}}(L, E) \to \mathrm{Hom}_{R^{op}}(L, I) \to 0$ is exact. By applying the functor $\mathrm{Hom}_{R^{op}}(L, -)$ to the above short exact sequence, we get the long exact sequence

$$\mathrm{Hom}_{R^{op}}(L, E) \longrightarrow \mathrm{Hom}_{R^{op}}(L, I) \longrightarrow \mathrm{Ext}^1_{R^{op}}(L, M) \longrightarrow \mathrm{Ext}^1_{R^{op}}(L, E)$$

which shows that $\mathrm{Ext}^1_{R^{op}}(L, M) = 0$. Therefore $M \in C\mathcal{WC}_m(R^{op})$. □

Recall that, an module is said to be *reduced* if it has no non-zero injective submodules (see [9, Remark 10.1.5]). In the next proposition, we characterize reduced C-m-weak cotorsion modules.

Proposition 7. *Let M be an R^{op}-module and let ${}_RR \in \mathcal{WI}_C(R)_{\leq m}$. Then M is a reduced C-m-weak cotorsion R^{op}-module if and only if M is the kernel of a $\mathcal{WF}_C(R^{op})_{\leq m}$-cover $f : E \longrightarrow D$ of an R^{op}-module D where E is injective.*

Proof. ($\Rightarrow$) Assume that M is a reduced C-m-weak cotorsion R^{op}-module. Assume also that E is an injective envelope of M and $D := E/M$. Then $0 \to M \to E \xrightarrow{f} D \to 0$ is a short exact sequence of R^{op}-modules where E is injective. Thus $f : E \longrightarrow D$ is a $\mathcal{WF}_C(R^{op})_{\leq m}$-precover of D from Proposition 6. Since M is reduced, E has no non-zero direct summand contained in M. Hence $f : E \longrightarrow D$ is a $\mathcal{WF}_C(R^{op})_{\leq m}$-cover of D by Theorem 2(ii) and [24, Corollary 1.2.8].

($\Leftarrow$) Assume that there exists a $\mathcal{WF}_C(R^{op})_{\leq m}$-cover $f : E \longrightarrow D$ of an R^{op}-module D where E is injective and $M = \mathrm{Ker}(f)$. Therefore, from Proposition 6, M is C-m-weak cotorsion. Assume that M' is an injective submodule of M. Thus there is a submodule of E' of E where $E = M' \oplus E'$. Assume that $i : E' \longrightarrow E$ is the injection map and $\pi : E \longrightarrow E'$ is the projection map. Then $f(M') = 0$ and so $fi\pi = f$. Hence $i\pi$ is an isomorphism and so i is an epimorphism. Therefore $E = E'$ and so $M' = 0$. Thus M is reduced as we desired. □

We are now ready to state and prove the first main result of this section.

Theorem 4. *Let M be an R^{op}-module, let ${}_RR \in \mathcal{WI}_C(R)_{\leq m}$, and let ${}_{R^{op}}R^{op} \in \mathcal{WF}_C(R^{op})_{\leq m}$. Then M is a C-m-weak cotorsion R^{op}-module if and only if M is a direct sum of an injective R^{op}-module and a reduced C-m-weak cotorsion R^{op}-module.*

Proof. ($\Rightarrow$) Assume that M is a C-m-weak cotorsion R^{op}-module. Assume also that E is an injective envelope of M and $D := E/M$. Then $0 \to M \to E \xrightarrow{f} D \to 0$ is a short exact sequence of R^{op}-modules where E is injective. By Proposition 6. $f : E \longrightarrow D$ is a $\mathcal{WF}_C(R^{op})_{\leq m}$-precover of D. From Theorem 2(ii), D has a $\mathcal{WF}_C(R^{op})_{\leq m}$-cover $g : F \longrightarrow D$. Also, there is a short exact sequence of R^{op}-modules $0 \to K' \to F' \xrightarrow{g'} D \to 0$ where F' is free. Since ${}_{R^{op}}R^{op} \in \mathcal{WF}_C(R^{op})_{\leq m}$, $F' \cong \bigoplus_{j\in J} R^{op} \in \mathcal{WF}_C(R^{op})_{\leq m}$ by Corollary 1(ii). So there exists a map $h : F' \to F$ such that $gh = g'$. Since g' is epic, we deduce that g is also epic. Set $K := \mathrm{Ker}(g)$. Then, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "j"] \arrow[d, "h_2"] & F \arrow[r, "g"] \arrow[d, "h_1"] & D \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "k_2"] & E \arrow[r, "f"] \arrow[d, "k_1"] & D \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "j"] & F \arrow[r, "g"] & D \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Set $K_1 := \mathrm{Ker}(k_1)$ and $K_2 := \mathrm{Ker}(k_2)$. Since $gk_1h_1 = g$, k_1h_1 is an isomorphism and so $E = K_1 \oplus \mathrm{Im}(h_1)$ and $F \cong \mathrm{Im}(h_1)$. Thus F and K_1 are injective. Hence K is a reduced C-m-weak cotorsion R^{op}-module from Proposition 7. On the other hand, by Five Lemma, k_2h_2

is an isomorphism and so $M = K_2 \oplus \mathrm{Im}(h_2)$ and $K \cong \mathrm{Im}(h_2)$. We have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_2 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "k_2"] & E \arrow[r, "f"] \arrow[d, "k_1"] & D \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "j"] \arrow[d] & F \arrow[r, "g"] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which shows that $K_2 \cong K_1$ by [20, Exercise 2.32]. Hence we have $M = K_2 \oplus K$ where K_2 is an injective R^{op}-module and K is a reduced C-m-weak cotorsion R^{op}-module.

($\Leftarrow$) It follows from Remark 2(ii) and Proposition 5(i). $\square$

The following lemma is needed in the proof of the second main result of this section.

Lemma 1. *Let ${}_RR \in \mathcal{WF}_C(R)$, let M be a C-m-weak cotorsion R-module, and let k be a non-negative integer. Then the following assertions hold true:*

(i) $\mathrm{Ext}_R^{k+1}(L, M) = 0$ *for any R-module $L \in \mathcal{WF}_C(R)_{\leq m+k}$;*

(ii) $\mathrm{Ext}_R^{j}(L, M) = 0$ *for all $j \geq 1$ and any R-module $L \in \mathcal{WF}_C(R)_{\leq m}$;*

(iii) *The $(k-1)$th cosyzygy of M is a C-$k+m$-weak cotorsion R-module.*

Proof. (i). Assume that $L \in \mathcal{WF}_C(R)_{\leq m+k}$. Since ${}_RR \in \mathcal{WF}_C(R)$, every projective R-module is in $\mathcal{WF}_C(R)$ by Corollary 1. Thus, from Corollary 5, there is an exact sequence of R-modules

$$0 \longrightarrow P_{m+k} \longrightarrow P_{m+k-1} \longrightarrow \cdots \longrightarrow P_{k-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow L \longrightarrow 0$$

such that P_i is projective for all $0 \leq i \leq m+k-1$ and $P_{m+k} \in \mathcal{WF}_C(R)$. Set $K := \mathrm{Ker}(P_{k-1} \longrightarrow P_{k-2})$. It is clear that $K \in \mathcal{WF}_C(R)_{\leq m}$. Thus $\mathrm{Ext}_R^1(K, M) = 0$ and so $\mathrm{Ext}_R^{k+1}(L, M) = 0$ because we have $\mathrm{Ext}_R^{k+1}(L, M) \cong \mathrm{Ext}_R^{k}(K_0, M) \cong \mathrm{Ext}_R^{k-1}(K_1, M) \cong \cdots \cong \mathrm{Ext}_R^{2}(K_{k-2}, M) \cong \mathrm{Ext}_R^{1}(K, M)$ from applying the derived functors of $\mathrm{Hom}_R(-, M)$ to the short exact sequences

$$0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow L \longrightarrow 0,$$

$$0 \longrightarrow K_i \longrightarrow P_i \longrightarrow K_{i-1} \longrightarrow 0$$

for all $1 \leq i \leq k-2$, and

$$0 \longrightarrow K \longrightarrow P_{k-1} \longrightarrow K_{k-2} \longrightarrow 0$$

where $K_0 = \mathrm{Ker}(P_0 \longrightarrow L)$ and $K_i = \mathrm{Ker}(P_i \longrightarrow P_{i-1})$ for all $1 \leq i \leq k-2$.

(ii). This follows from the first part.

(iii). Assume that V^{k-1} is the $(k-1)$th cosyzygy of M. By [20, Proposition 8.10(iii)] and the first part, $\mathrm{Ext}_R^1(L, V^{k-1}) = 0$ for any R-module $L \in \mathcal{WF}_C(R)_{\leq m+k}$. Therefore V^{k-1} is a C-$k+m$-weak cotorsion R-module. $\square$

Let $\mathcal{F}$ be a class of R-modules, let M be an R-module, and let $f : M \longrightarrow F$ be an $\mathcal{F}$-envelope of M. We say that f has the *unique mapping property* if for any homomorphism $g : M \longrightarrow F'$ with $F' \in \mathcal{F}$, there is a unique homomorphism $h : F \longrightarrow F'$ such that $hf = g$ (see [6, Section 1]).

We end this paper with a result that shows when every module is C-m-weak cotorsion.

Theorem 5. *The following conditions are equivalent:*

(i) *Every R-module is in $C\mathcal{WC}_m(R)$;*

(ii) *Every R-module in $\mathcal{WF}_C(R)_{\leq m}$ is projective;*

(iii) $\mathrm{Ext}^1_R(L, M) = 0$ *for all R-modules $L \in \mathcal{WF}_C(R)_{\leq m}$ and all R-modules M.*

Moreover, if ${}_RR \in \mathcal{WF}_C(R)$, then the above conditions are also equivalent to:

(iv) *For any integer k, $\mathrm{Ext}^{k+1}_R(L, M) = 0$ for all R-modules $L \in \mathcal{WF}_C(R)_{\leq m+k}$ and all R-modules M;*

(v) *Every R-module in $\mathcal{WF}_C(R)_{\leq m}$ is in $C\mathcal{WC}_m(R)$;*

(vi) *Every R-module M has a $\mathcal{WF}_C(R)^{\perp}_{\leq m}$-envelope with the unique mapping property.*

Proof. (i)⇔(ii). This is clear.

(i)⇔(iii). It is clear.

(i)⇒(iv). Follows from Lemma 1(i).

(iv)⇒(iii). Take $k = 0$ in (iv).

(i)⇒(v). This is clear.

(v)⇒(i). Since ${}_RR \in \mathcal{WF}_C(R)_{\leq m}$, by Corollary 1(ii), we deduce that every free R-module is in $\mathcal{WF}_C(R)_{\leq m}$. Thus, from Corollary 1(ii), $\mathcal{WF}_C(R)_{\leq m}$ contains all projective R-module because every projective R-module is a direct summand of a free R-module. Also, $\mathcal{WF}_C(R)_{\leq m}$ is closed under direct limits and extensions by Corollaries 1(ii) and 2(ii). Assume that M is an R-module and that $L \in \mathcal{WF}_C(R)_{\leq m}$. By Theorem 2(ii), M has a $\mathcal{WF}_C(R)_{\leq m}$-cover. From [7, Theorem 5], we deduce that M has an epic $\mathcal{WF}_C(R)_{\leq m}$-cover $f : F \longrightarrow M$ such that $K := \mathrm{Ker}(f) \in \mathcal{WF}_C(R)^{\perp}_{\leq m}$. We have the short exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0.$$

By applying the derived functors of $\mathrm{Hom}_R(L, -)$ to the above short exact sequence, we get the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_R(L, F) \longrightarrow \mathrm{Ext}^1_R(L, M) \longrightarrow \mathrm{Ext}^2_R(L, K) \longrightarrow \cdots$$

Since $F \in \mathcal{WF}_C(R)_{\leq m}$, $F \in C\mathcal{WC}_m(R)$ by assumption, and so $\mathrm{Ext}^1_R(L, F) = 0$. Also, from Lemma 1(ii), $\mathrm{Ext}^2_R(L, K) = 0$. Hence $\mathrm{Ext}^1_R(L, M) = 0$. Therefore $M \in C\mathcal{WC}_m(R)$.

(i)⇒(vi). It is clear.

(vi)⇒(v). Assume that $M \in \mathcal{WF}_C(R)_{\leq m}$. By assumption, M has a $\mathcal{WF}_C(R)^{\perp}_{\leq m}$-envelope $f : M \longrightarrow F$ with the unique mapping property. Assume that E is an injective envelope of M, $D := \mathrm{Coker}(f)$, and E' is an injective envelope of D. Then, we have the exact sequences of R-modules $0 \to M \xrightarrow{i} E$, $M \xrightarrow{f} F \xrightarrow{h} D \to 0$, and $0 \to D \xrightarrow{g} E'$. Since f has the unique mapping property and $E \in \mathcal{WF}_C(R)^{\perp}_{\leq m}$, there is a unique homomorphism $f' : F \longrightarrow E$ such that $f'f = i$. Thus f is monic and so

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "f"] \arrow[dr, "0"'] & F \arrow[r, "h"] \arrow[d, "gh"] & D \arrow[r] \arrow[dl, "g"] & 0 \\
 & & E' & &
\end{tikzcd}$$

is a commutative diagram of R-modules with exact row. Since $ghf = 0$, it follows that $gh = 0$ because f has the unique mapping property and $E' \in \mathcal{WF}_C(R)^{\perp}_{\leq m}$. Hence $\mathrm{Im}(h) \subseteq \ker(g)$. Therefore $D = 0$ and so $M \cong F$. Thus $M \in \mathcal{WF}_C(R)^{\perp}_{\leq m}$ and so $M \in C\mathcal{WC}_m(R)$. $\square$

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