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Legendre-collocation method to solve the second kind Cauchy integral equations

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Abstract. A numerical solution for the second kind singular integral equations with Cauchy kernel is developed using the collocation method. To achieve this, we approximate the Cauchy integral equation using the collocation method and Legendre orthogonal polynomial expansions. The accuracy of our proposed method is assessed through convergence and error analysis. Finally, several numerical examples are presented to demonstrate the high efficiency of the method.

Keywords: Cauchy integral equation, Legendre polynomials, collocation method.
AMS Subject Classification 2010: 30E20, 45E05, 33C45, 65N35.

1 Introduction

Cauchy singular integral equations play a significant role in engineering and applied sciences, both in research and practical applications. These equations are commonly used to describe various phenomena, including electromagnetic radiation, unsteady aerodynamic behaviour, electron microscopy, control problems, viscoelasticity, thermoelasticity, and fluid dynamics [7, 8, 11]. Consider an integral equation that includes the unknown function f under the integral sign of an improper integral in the sense of Cauchy, represented as follows

$$f(\lambda) = g(\lambda) + \oint_{-1}^{1} \frac{f(x)}{\lambda - x} dx, \qquad -1 < \lambda < 1, \tag{1}$$

In this context, g is a known function, and the symbol $\oint$ denotes the Cauchy principal value integral, which is defined as follows

$$\oint_{-1}^{1} = \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{\lambda-\varepsilon} + \int_{\lambda+\varepsilon}^{1} \right).$$

In the special case of the problem mentioned above, the corresponding equation

$$\oint_{-1}^{1} \frac{f(x)}{\lambda - x} dx = -g(\lambda), \qquad -1 < \lambda < 1,$$

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represents the simplest singular integral equation of the first kind. This equation has a bounded solution, which is discussed in various contexts [3].

Now, consider Eq. (1) and assume that $f \in H_\alpha[-1,1]$, meaning that f is Hölder's continuous on the interval $[-1,1]$, with Hölder index α, where $0 < \alpha \leq 1$. For all points $l_1, l_2 \in [-1,1]$, the following condition holds

$$|f(l_2) - f(l_1)| \leq M\,|l_2 - l_1|^\alpha,$$

where M is independent of l_1 and l_2. General methods for solving this equation have been studied, particularly regarding its existence and uniqueness. According to references [4, 11], the function g belongs to $H_\alpha[-1,1]$ on $[-1,1]$ and we investigate solutions f that are Hölder continuous on any closed interval within the interior of $(-1,1)$ and are integrable at the endpoints -1 and $+1$.

In this work, the Collocation method is considered for solving the Cauchy type singular integral equations using Legendre polynomials. Let $\{p_k(x)\}_{k=0}^{\infty}$ denote the sequence of Legendre polynomials of degree k defined on the interval $[-1,1]$. These polynomials satisfy the following recurrence relation [15]

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x, \\ p_{k+1}(x) &= \frac{2k+1}{k+1} x p_k(x) - \frac{k}{k+1} p_{k-1}(x), \qquad k = 1,2,\ldots. \end{aligned} \tag{2}$$

Additionally, the Legendre polynomials form an orthogonal set, such that

$$\int_{-1}^{1} p_i(x) p_j(x)\,dx = h_i \delta_{ij}, \qquad i,j = 0,1,2,\ldots,$$

where δ_{ij} is the Kronecker delta and $h_i = 2/(2i+1)$ for any $i = 0,1,2,\ldots$. According to approximation theory, a function $f \in L^2([-1,1])$ can be approximated by a series of Legendre polynomials basis functions in the following form

$$f_n(x) = \sum_{j=0}^{n} c_j p_j(x), \tag{3}$$

where c_j are unknown constants that need to be determined. Let us define

$$I(f_n, \lambda) = \oint_{-1}^{1} \frac{f_n(x)}{\lambda - x}\,dx, \quad -1 < \lambda < 1. \tag{4}$$

From Eq. (3), we obtain

$$I(f_n, \lambda) = \sum_{j=0}^{n} c_j \oint_{-1}^{1} \frac{p_j(x)}{\lambda - x}\,dx = \sum_{j=0}^{n} c_j q_j(\lambda),$$

where

$$q_0(\lambda) = \oint_{-1}^{1} \frac{1}{\lambda - x}\,dx = \ln\left(\frac{1+\lambda}{1-\lambda}\right), \qquad q_1(\lambda) = \lambda q_0(\lambda) - 2,$$

and

$$q_j(\lambda) = \oint_{-1}^{1} \frac{p_j(x)}{\lambda - x}\,dx, \qquad j = 2,3,\ldots.$$

It can be easily shown that $q_j(x)$ satisfies the recurrence relation given by Eq. (2), specifically

$$q_{j+1}(\lambda) = \frac{2j+1}{j+1} \lambda q_j(\lambda) - \frac{j}{j+1} q_{j-1}(\lambda), \qquad j = 1,2,\ldots.$$

From [14], we have derived a recurrence relation concerning certain values of f represented as follows

$$I(f_n,\lambda) = \sum_{i=0}^{n} \frac{q_{n+1}(\lambda)-q_{n+1}(t_i)}{(\lambda-t_i)p'_{n+1}(t_i)} f(t_i), \tag{5}$$

$$c_j = \sum_{i=0}^{n} \frac{f(t_i)}{h_j\mu_i} p_j(t_i), \quad j=0,\ldots,n, \tag{6}$$

Here, t_i are the roots of the equation $p_{n+1}(x)=0$ and $\mu_i = \sum_{k=0}^{n} p_k^2(t_i)$ for each $i=0,1,2,\ldots,n$. Furthermore, several authors have studied Cauchy integral equations using various numerical methods. These methods include methods based on Bernstein polynomials [17, 18], the collocation method [5, 17], the Galerkin method [10], successive approximations [9], Chebyshev polynomial [13], the Adomian decomposition method [8], the differential transform method [1, 6], Gaussian elimination [12], Sinc approximations [2], GaussLegendre collocation [16], and so on.

The remainder of this paper is organized as follows. In Section 2, we describe the method for solving Cauchy-type singular integral equations. Error analysis and convergence of the method are discussed in Section 3. In Section 4, we demonstrate the efficiency of this method and present some numerical examples. Finally, Section 5 is devoted to the conclusion of this paper.

2 Algorithmic solution procedure

In this section, we approximate the unknown function f defined in Eq. (3) over the interval $(-1,1)$ using f_n, such that the following functional is minimized

$$\begin{aligned} R(f_n) &= \|g(\lambda)-f_n(\lambda)+\oint_{-1}^{1}\frac{f_n(x)}{\lambda-x}dx\|_{L_\infty[-1,1]} \\ &= \|g(\lambda)-\sum_{j=0}^{n}c_jp_j(\lambda)+\sum_{j=0}^{n}c_jq_j(\lambda)\|_{L_\infty[-1,1]}, \qquad -1<\lambda<1. \end{aligned} \tag{7}$$

Furthermore, by utilizing (5) we can express $R(f_n)$ as follows

$$\begin{aligned} R(f_n) &= \|f_n(\lambda)-g(\lambda)-\sum_{i=0}^{n}\frac{q_{n+1}(\lambda)-q_{n+1}(t_i)}{(\lambda-t_i)p'_{n+1}(t_i)}f(t_i)\|_{L_\infty[-1,1]} \\ &= \|f_n(\lambda)-g(\lambda)-\sum_{i=0}^{n}V_i(\lambda)f(t_i)\|_{L_\infty[-1,1]}, \end{aligned}$$

where

$$V_i(\lambda) = \frac{q_{n+1}(\lambda)-q_{n+1}(t_i)}{(\lambda-t_i)p'_{n+1}(t_i)}.$$

From Eqs. (3), (6), and (7) we obtain

$$\begin{aligned} R(f_n) &= \|\sum_{j=0}^{n}\sum_{i=0}^{n}\frac{1}{\mu_ih_j}p_j(t_i)f(t_i)p_j(\lambda)-\sum_{i=0}^{n}f(t_i)V_i(\lambda)-g(\lambda)\|_{L_\infty[-1,1]} \\ &= \|\sum_{i=0}^{n}\left(\frac{1}{\mu_i}\sum_{j=0}^{n}\frac{1}{h_j}p_j(t_i)p_j(\lambda)-V_i(\lambda)\right)f(t_i)-g(\lambda)\|_{L_\infty[-1,1]}, \ -1<\lambda<1, \end{aligned} \tag{8}$$

where $f(t_i)$, $i = 0,\dots,n$ are unknowns. Clearly, if $g \in \pi_n$, where π_n is the set of polynomials of degree at most n, then by selecting any $n+1$ arbitrary values from the interval $(-1,1)$, we have $R(f_n) = 0$. Consequently, the minimization problem described in (8) has a unique solution. Additionally, if $g \in L_2[-1,1]$, we can easily determine the minimum of $R(f_n)$.

In addition, if $g \in C^{n+1}[-1,1]$, we can uniquely determine the unknown values $f(t_i)$ for $i = 0,\dots,n$ by interpolating g with a polynomial of degree at most n. To achieve this, we assume that

$$\lambda_m = -1-\varepsilon+mk, \qquad m = 0,1,\dots,n. \tag{9}$$

where $\varepsilon > 0$ is arbitrary and $k = \frac{2-2\varepsilon}{n}$. This choice ensures that the singular integral at the endpoint remains finite. By substituting these points into Eq. (8), we arrive at the conclusion

$$\sum_{j=0}^{n} c_j^* \left(p_j(\lambda_m) - \sum_{i=0}^{n} V_i(\lambda_m) p_j(t_i) \right) = g(\lambda_m), \qquad m = 0,1,\dots,n. \tag{10}$$

Due to the orthogonality of the basis $\{p_n(x)\}$, the system of equations in Eq. (10) possesses nonsingular properties, allowing us to obtain a unique solution for the unknowns. Consequently, by solving this system of $n+1$ equations with $n+1$ unknowns, we can compute the coefficients c_j^*. Finally, we define

$$f_n^*(x) = \sum_{j=0}^{n} c_j^* p_j(x). \tag{11}$$

The following algorithm demonstrate the step-by-step process of the proposed scheme through an algorithmic solution procedure.

Algorithm 1.
Input: Read n and $\varepsilon > 0$.
Step 1: Calculate $k = (2-2\varepsilon)/n$ and $\lambda_m = -1-\varepsilon+mk$, where $m = 0,1,\dots,n$. Next, we consider the Legendre polynomials given in (2). The values t_i, which are the roots of the Legendre polynomials, are obtained by solving the equation $p_{n+1}(x) = 0$.
Step 2: Compute

$$\mu_i = \sum_{k=0}^{n} p_k^2(t_i), \text{ and } h_j = \frac{2}{2j+1}, \quad i,j = 0,1,2,\dots.$$

Step 3: Calculate the recurrence relations given by the following equations

$$\begin{aligned} q_0(\lambda) &= \ln\left(\frac{1+\lambda}{1-\lambda}\right), \qquad q_1(\lambda) = \lambda q_0(\lambda) - 2, \\ q_{j+1}(\lambda) &= \frac{2j+1}{j+1}\lambda q_j(\lambda) - \frac{j}{j+1} q_{j-1}(\lambda), \qquad j = 1,2,\dots. \end{aligned}$$

Step 4: Solve Eq. (10) using any numerical method to find the solutions for the unknowns c_j^*.
Output: By solving the $n+1$ equations with $n+1$ unknowns, you will obtain the values of c_j^*. Finally, compute $f_n^*(x)$ using (11).

3 Error analysis

we first establish the decay rates of the coefficients in the Legendre series expansion. We then demonstrate the error bounds of the truncated Legendre series in the uniform norm through two theorems. Finally, we derive the error analysis and convergence of the collocation method. We analyze the decay rates of the Legendre coefficients using the following theorem.

Theorem 1. *If f is analytic inside and on the Bernstein ellipse ε_ρ with foci at ± 1 and with the major and minor semi-axis summing to $\rho > 1$, then for each $n \geq 0$,*

$$|f(x) - f_n(x)| \leq \frac{(2n\rho + 3\rho - 2n - 1)\ell(\varepsilon_\rho)M}{\pi\rho^{n+1}(\rho-1)^2(1-\rho^{-2})} = M_n, \qquad x \in [-1,1], \tag{12}$$

where $M = \max_{z\in\varepsilon_\rho} |f(z)|$ and $\ell(\varepsilon_\rho)$ denotes the length of the circumference of ε_ρ.

Proof. See [19] □

Theorem 2. *Assuming that the conditions of Theorem 1 are satisfied, we have $f_n^* \to f$ as $n \to \infty$.*

Proof. To obtain the error estimation for the proposed approximation, let $g_{p,n}$ be the interpolation polynomial of g at $n+1$ distinct nodes $\lambda_m, m = 0, 1, \ldots, n$. Then, we get

$$\sum_{j=0}^{n} c_j^* p_j(\lambda) = g_{p,n}(\lambda) + \oint_{-1}^{1} \frac{\sum_{j=0}^{n} c_j^* p_j(x)}{\lambda - x} dx.$$

Thus,

$$\sum_{j=0}^{n} c_j^*(p_j(\lambda) - q_j(\lambda)) = g_{p,n}(\lambda). \tag{13}$$

Now, let

$$\delta_n(\lambda) = f(\lambda) - f_n^*(\lambda) = f(\lambda) - \sum_{j=0}^{n} c_j^* p_j(\lambda).$$

Then, Eqs. (1) and (13) yield

$$\begin{aligned}
\delta_n(\lambda) &= g(\lambda) + \oint_{-1}^{1} \frac{f(x)}{\lambda - x} dx - \sum_{j=0}^{n} c_j^* p_j(\lambda) \\
&= \left(g(\lambda) + \oint_{-1}^{1} \frac{f(x)}{\lambda - x} dx\right) - \left(g_{p,n}(\lambda) + \oint_{-1}^{1} \frac{\sum_{j=0}^{n} c_j^* p_j(x)}{\lambda - x} dx\right) \\
&= (g(\lambda) - g_{p,n}(\lambda)) + \oint_{-1}^{1} \frac{\delta_n(x)}{\lambda - x} dx.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\| \delta_n(\lambda) - \oint_{-1}^{1} \frac{\delta_n(x)}{\lambda - x} dx \|_\infty &= \| g(\lambda) - g_{p,n}(\lambda) \|_\infty \\
&\leq \frac{|g^{(n+1)}(\xi(\lambda))|}{(n+1)!} \prod_{i=0}^{n} |\lambda^2 - \lambda_i^2| \\
&= \frac{M_{n+1}}{(n+1)!},
\end{aligned}$$

where M_{n+1} is an upper bound for $|g^{(n+1)}|$ in $[-1,1]$. Clearly, we have

$$\lim_{n\to\infty} \parallel \delta_n(\lambda) - \oint_{-1}^{1} \frac{\delta_n(x)}{\lambda - x} dx \parallel_\infty = 0.$$

The uniqueness theorem for solutions to equation (1) gives us

$$\lim_{n\to\infty} \parallel \delta_n(\lambda) \parallel_\infty = 0,$$

and thus, the proof is complete. □

4 Numerical examples

To illustrate the performance of our method, we dedicate this section to presenting numerical results through several examples. In these computations, each table displays the absolute errors of our approximate solutions.

Example 1. Consider the integral equation

$$f(\lambda) = \frac{\lambda + 2Ln3}{\lambda + 2} - \frac{\lambda}{\lambda + 2} Ln\left|\frac{\lambda + 1}{\lambda - 1}\right| + \int_{-1}^{1} \frac{f(x)}{\lambda - x} dx,$$

where the exact solution is given by $f(\lambda) = \frac{\lambda}{\lambda+2}$.

Table 1: Absolute errors for Example 1.

N	$\varepsilon = 0.001$	$\varepsilon = 0.0001$
6	2.289E-3	1.252E-3
8	3.704E-4	1.759E-4
10	6.241E-5	2.513E-5
12	1.098E-5	3.625E-6

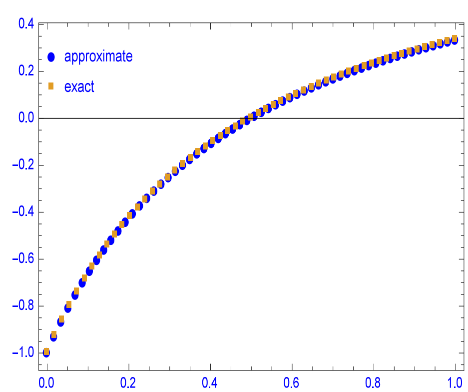

Figure 1: Comparison between the exact and approximate solutions of Example 1 with $N = 12$.

Example 2. Consider the integral equation as described in [3]

$$k\varphi = \varphi(t) + \int_L \frac{\varphi(\tau)}{\tau - t} dt = f(t),$$

where

$$f(t) = \frac{1}{1+t^2}\left(t^3 Ln(\frac{1-t}{1+t}) + t^3 + 2t^2 + \frac{4-\pi}{2}\right), \quad k(t,\tau) = 0,$$

and $L = [-1,1]$. The exact solution is given by $\varphi(t) = \frac{t^3}{1+t^2}$.

Table 2: Comparison of absolute errors [3] with the present method considering $\varepsilon = 0.0001$.

N	Present method	Method used in [3]
4	1.488E-1	1.129E-1
8	1.162E-2	3.338E-2
16	2.710E-4	4.933E-3

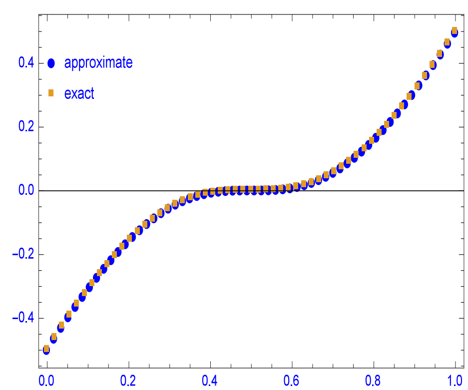

Figure 2: Comparison between the exact and approximate solutions of Example 2 with $N = 16$

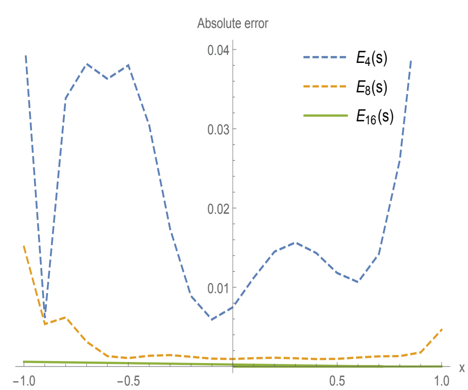

Figure 3: Absolute errors for Example 2 with different values of N considering $\varepsilon = 0.0001$.

Example 3. Consider the integral equation from [16]

$$\lambda x(s)+\mu\int_{-1}^{1}\frac{x(t)}{t-s}dt=\xi(s),\ s\in(-1,1),$$

where $\lambda=\mu=1$, and

$$\xi(s) = \sinh(s)-chi(-s-1)\sinh(s)+chi(1-s)\sinh(s)+Ln(-1-s)\sinh(s)-Ln(1-s)\sinh(s) +Ln(\frac{1-s}{1+s})\sinh(s)+\cosh(s)shi(1-s)+\cosh(s)shi(1+s).$$

In this context, *chi* and *shi* refer to the hyperbolic cosine integral and hyperbolic sine integral, respectively. The exact solution is given by $x(s)=\sinh(s)$.

Table 3: Comparison of absolute errors between the present method considering $\varepsilon=0.0001$ and the method in [16].

	Presented method			Method of [16]		
Node	$N=3$	$N=5$	$N=7$	$N=3$	$N=5$	$N=7$
-1.0	$2.21e-4$	$3.29e-6$	$2.34e-8$	$1.91e-3$	$1.81e-5$	$8.90e-8$
-0.8	$6.80e-4$	$5.97e-7$	$2.89e-8$	$3.27e-4$	$2.71e-6$	$1.03e-8$
-0.6	$2.80e-4$	$1.27e-6$	$1.56e-9$	$4.56e-4$	$3.09e-6$	$2.75e-10$
-0.4	$1.81e-4$	$2.89e-6$	$1.61e-8$	$7.13e-4$	$4.86e-7$	$4.69e-9$
-0.2	$8.84e-5$	$3.93e-6$	$6.74e-9$	$5.31e-4$	$1.50e-6$	$1.59e-8$
-0.2	$4.31e-4$	$1.15e-6$	$1.13e-9$	$7.62e-6$	$2.35e-6$	$5.44e-10$
0.0	$7.10e-4$	$6.16e-7$	$1.28e-8$	$4.36e-4$	$5.16e-6$	$7.95e-9$
0.4	$7.88e-4$	$2.84e-6$	$3.03e-9$	$3.80e-4$	$1.85e-6$	$1.33e-8$
0.6	$2.21e-4$	$6.49e-7$	$1.93e-8$	$2.72e-4$	$2.78e-6$	$1.17e-8$
0.8	$5.63e-4$	$3.64e-7$	$2.43e-9$	$9.57e-4$	$1.95e-6$	$7.49e-9$
1.0	$8.19e-5$	$5.14e-7$	$2.23e-10$	$8.32e-5$	$2.08e-6$	$1.23e-8$

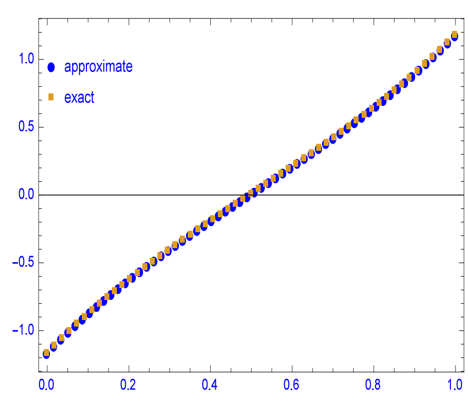

Figure 4: Comparison between the exact and approximate solutions for Example 3 with $N=7$.

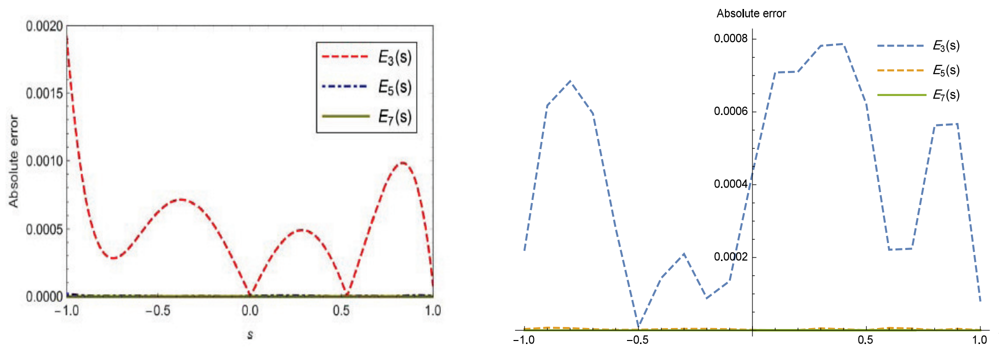

Figure 5: Absolute errors for Example 3 in [16] (left) and the absolute errors of the solutions obtained in this paper considering $\varepsilon = 0.0001$ (right).

This study employs Legendre orthogonal polynomial expansions, offering a robust framework for approximating solutions due to their inherent orthogonality. This approach significantly reduces the computational effort compared to methods that utilize Bessel basis polynomials and Gauss-Legendre collocation points, making it a more efficient option for solving these equations.

5 Conclusion

We have used an expansion approach for the unknown function and the Cauchy kernel in this study. The unknown coefficients were defined by applying the collocation method. Using a system of linear equations, we can compute an approximate solution and eliminate singularities with this method. Through the provided examples, we show that, although our numerical solution is straightforward and quick, it is also very accurate and efficient when compared to other approaches used in related studies.

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