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Finite-capacity $M/M/2$ machine repair model with impatient customers, triadic discipline, and two working vacation policies

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Abstract. In this paper, we model and analyze a machine repair system characterized as an $M/M/2$ queue with finite source L, operating under the triadic policy $(0,Q,N,M)$, considering impatience, and both multiple and single working vacations. The two servers can be active, on working vacation, or dormant depending on the number of failed machines in the system, following the triadic policy $(0,Q,N,M)$. We analyze the system's steady-state using the matrix-geometric method. Various performance measures are numerically presented and accurately interpreted. Finally, the Quadratic Fit Search method is employed to determine the optimal service rate μ_v^* and the optimal expected cost. Additionally, the effect of system parameters on the cost function is investigated. This study offers a comprehensive analytical framework for complex queueing environments, informing decision-making and operational efficiency across various industrial sectors.

Keywords: Queueing system, servers vacation, impatience, matrix-geometric method, reliability, optimization.
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1 Introduction

Optimal performance in industrial systems management hinges on the effective collaboration between humans and machines. This is particularly evident when system failures occur, necessitating immediate

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repairs. Queueing systems have emerged as a significant tool in predicting queue characteristics across various sectors, including manufacturing systems and call centers [10, 14, 15, 19].

Key concepts in queueing theory such as server vacations (servers are temporarily unavailable due to a variety of reasons such as maintenance, rest periods, or system updates), working vacation (a period during which the service is provided at a slower rate, rather than being completely stopped), customer balking (choosing not to join the queue), and reneging (leaving before service) are critical in understanding and optimizing these systems. Several studies, including those by [1–5, 7, 18], have delved into these concepts.

Triadic policy in queueing models has garnered considerable interest among researchers. Rhee [16] pioneered the introduction of the triadic policy in the context of the $M/M/2$ queueing system model. The controllable $M/M/2$ system, when operated under the triadic $(0,Q,N,M)$ policy, was studied by Wang et al. [20]. A comparative numerical analysis for various cases of mixed standbys (using the Runge-Kutta/ANFIS methods) for a single unreliable server system was presented by Jain et al. [6]. Lin et al. [12] examined an $M/M/2$ system with infinite capacity, deriving analytical closed-form solutions for the queueing system operating under the triadic $(0,Q,N,M)$ policy, and conducted a cost optimization to determine the optimal operating parameters $(0,Q,N,M)$. Additionally, Laxmi et al. [11] analyzed an $M/M/2$ model with the triadic $(0,Q,N,M)$ policy and a second optional service using the matrix-geometric method. Liou et al. [13] employed a recursive method to obtain analytical steady-state solutions for the controllable $M/M/2$ system under the triadic $(0,Q,N,M)$ policy. They formulated an optimization problem to ascertain the minimum cost for the machine repair system and subsequently identified the optimal operating parameters for $(0,Q,N,M)$. Later, Ketema [8] considered an $M/M/2$ machine repair system with multiple working vacations (MWV), impatient customers, and the triadic policy $(0,Q,N,M)$. Ketema et al. [9] implemented the triadic policy $(0,Q,N,M)$ in the $M/M/2$ machine repair problem, identifying the optimal working vacation service rate η^*, along with the optimal operating triadic parameters $(0,Q^*,N^*,M^*)$. Recently, Sharma et al. [17] applied the genetic algorithm, the artificial bee colony algorithm, and particle swarm optimization to obtain the optimum cost function for an $M/M/2/L$ machine repair system with triadic policy and discouragement.

Working vacation queues, triadic policy, balking, and reneging often coexist in real-world scenarios across various sectors, including industry, healthcare, construction, and infrastructure. Although previous studies have investigated the impact of multiple working vacations under triadic policies on system performance in machine repair systems, there remains a critical gap regarding the economic and cost implications of integrating triadic policies with working vacations (both single and multiple), balking, and reneging in machine repair contexts.

This paper addresses this gap by proposing a comprehensive machine repair model incorporating features such as impatience timers, balking, single and multiple working vacations, finite capacity, triadic policy, and two repairmen. While some of these aspects have been discussed separately in the machine repair queueing literature, no previous work has combined all these features into a single model, even in recent studies. The queueing literature reveals that there are only a few articles focused on the queueing model with vacations and a triadic policy. Moreover, it is important to note that most research on multi-server vacation queueing models assumes an infinite customer source capacity. However, the analysis of finite-source vacation systems is often more practical and insightful than that of infinite vacation queues. The implications of finite system capacity include the loss of incoming customers due to queue saturation. We use the matrix-geometric method to derive steady-state probabilities and various performance measures and reliability indices of the queueing system, such as the expected number of failed and operating machines, mean queue length, and average rates of balking, reneging, and customer loss.

Additionally, we construct an expected cost function and formulate an optimization problem to determine the optimal cost. The Quadratic Fit Search (QFS) method is then employed to solve this problem and identify the optimal service rate during vacation periods, aiming to minimize the expected cost. Finally, we present numerical examples to illustrate how different parameters of the model affect the stationary characteristics of the system. This study has the potential to significantly impact industry and technology systems, particularly in terms of cost optimization and operational efficiency across diverse industrial sectors.

The remainder of the paper is structured as follows: Section 2 presents the mathematical description of the proposed model. In Section 3, we establish the balance equations and transition matrix, followed by the presentation of steady-state results for the queueing model. Section 4 delves into the use of the matrix-geometric method to provide detailed steady-state results. Performance measures are discussed in Section 5. Section 6 exhibits some significant cases relevant to our proposed model. Section 7 showcases numerical results depicting various system performance measures and an analysis of the optimal cost function. Finally, the paper concludes with a summary in Section 8.

2 Model description

We consider a machine repair model with capacity L operating machines maintained by a two repairman. The assumptions of the model are built up as follows:

1. If the operating machine fails, it joins the system to repair. The inter-arrivals for the failed machine accrued according to an exponential process with rate λ.

2. Failed machine decides either to join the queue with probability β_i, or balk with probability $1-\beta_i$, $0 \le i \le L$, where: $\beta_0 = 1$, $0 < \beta_{i+1} \le \beta_i \le 1$, $1 \le i \le L-1$ and $\beta_L = 0$.

3. Service times follows an exponential distribution with rate μ in busy period and μ_v in the vacation period $(\mu_v < \mu)$. The First-In-First-Out (FIFO) service discipline is adopted.

4. When the failed machine enters the system, it activates a timer T_0 (respectively T_1). This time T_0 (resp. T_1) is a random variable exponentially distributed with rate ξ_0 in the dormant and working vacation period (resp. ξ_1 in the busy period). Failed machine leaves the queue with probability α, he can return to the system with probability $1-\alpha$.

5. The triadic policy:

   - It takes the systemic bellow:
     - ◇ When the queue of machines waiting for service reaches N, one server will immediately begin the busy period.
     - ◇ After some time, if the queue of machines waiting for service reaches surpasses level M $(M > N)$, the second server becomes operational.
     - ◇ When the total number of machines in the system drops to Q $(Q < N)$, and both servers are actively serving, the server that has just finished servicing becomes inactive.

   We define a policy denoted as the triadic $(0, Q, N, M)$ policy, where Q, N, and M serve as key parameters. Additionally, when the system experiences a complete depletion of machines while one server remains active, all servers enter a working vacation period denoted as (WV).

- Within a WV period, one of the servers attends to incoming machines at a rate typically lower than the regular service rate. Upon completing the vacation, if the system size is N, both servers switches to a normal busy period, operating under a triadic policy. However, if the system size falls short of N, they embark on another WV cycle, repeating this process until the system reaches size N by the end of the vacation period. This policy is referred to as the multiple working vacation (MWV) policy.
- In a single working vacation (SWV) policy, servers take precisely one WV when the system becomes empty. If there are at least N machines remaining at the end of the vacation period, both servers transition to a regular busy period under a triadic policy. Otherwise, they remain idle in the system, awaiting the arrival of N machines rather than initiating another WV.

6. We consider vacation durations, assuming they follow an exponential distribution with rates ϕ. Our approach combines the analysis of SWV and MWV models. To facilitate this, we introduce an indicator function denoted by γ, where

$$\gamma = \begin{cases} 0, & \text{for the MWV results,} \\ 1, & \text{for the SWV results.} \end{cases}$$

All parameters are respectively independent.

Before continuing with the analysis, we briefly summarize the assumptions and notations used in our queueing model:

L	System capacity (Number of operating machines)
M, N *and* Q	Threshold machines values $(M > N > Q)$
λ	Rate by which the arriving failed machines joins the system
μ	Rate of service in the regular busy periods
μ_v	Rate of service in working vacation period
$\xi_i, i = 0; 1$	Impatience time rate of failed machines
ϕ	Vacation time rate of server
$1 - \beta_i$	Balking probability of failed machines
α	Reneging probability of failed machines from the queue

We also denote that for $0 \leq i \leq L$:

$$\lambda_i = (L-i)\lambda\beta i,$$
$$\zeta_{m,i} = i\alpha\xi_m, \quad m = \{0, 1\}.$$

3 Steady-state analysis

We can define the Quasi-Birth-and-Death process of $\{N(t), J(t)\}$ with the state space Ω, where,

$$\Omega = \{(0,0) \cup (n, j) : 0 \leq n \leq L, j = \{0,1,2,3\}\},$$

$N(t) \equiv$ Number of customers in the system at time t, and

$$J(t) = \begin{cases} 0, & \text{if the servers in WV period,} \\ 1, & \text{if one server is active during regular busy period,} \\ 2, & \text{if both servers are active during regular busy period,} \\ 3, & \text{if both servers are dormant.} \end{cases}$$

Figure 1 represents the state-transition diagram for the queueing model.

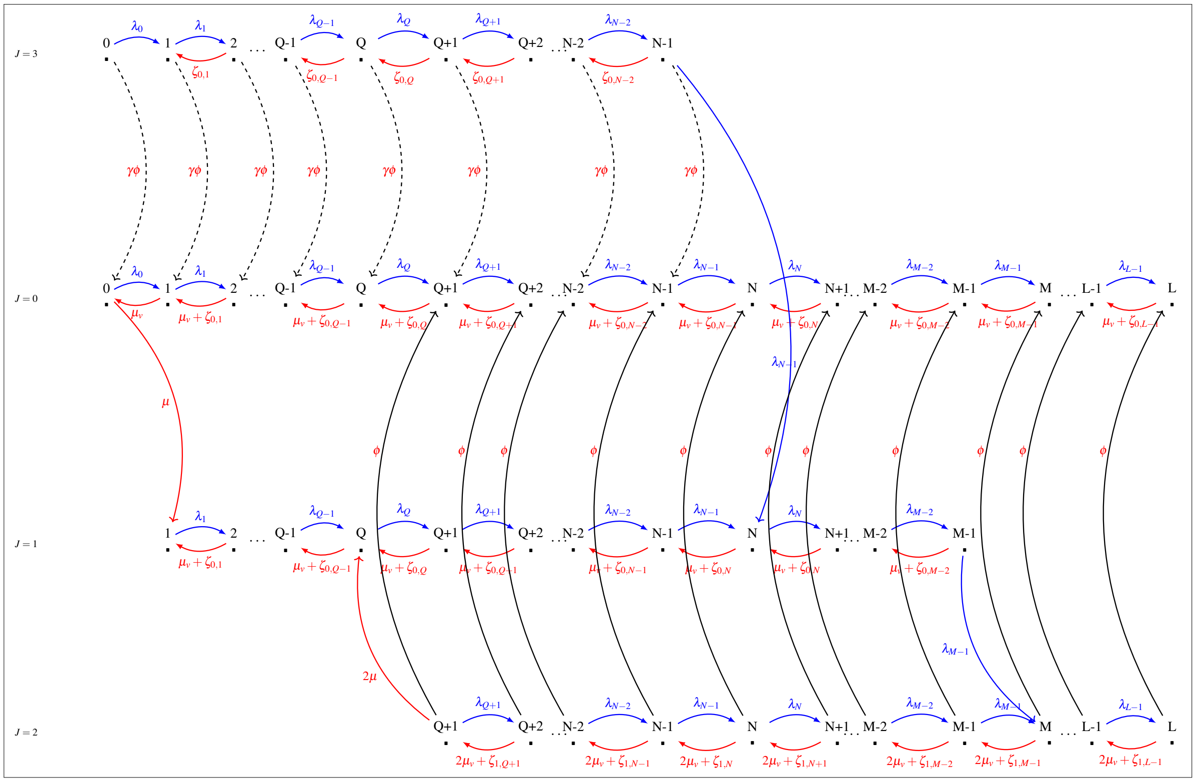

Figure 1: State-transition diagram.

3.1 Balance equations

The balance equations of the model are:
For $J(t) = 0$:

$$(\lambda_0 + \gamma\phi)P_{0,0} = \mu_v P_{0,1} + \mu P_{1,1}, \qquad n = 0, \tag{1}$$
$$[\lambda_n + \mu_v + \zeta_{0,n-1} + \gamma\phi]P_{0,n} = \lambda_{n-1}P_{0,n-1} + (\mu_v + \zeta_{0,n})P_{0,n+1}, \qquad 1 \le n \le N-1, \tag{2}$$
$$[\lambda_N + \mu_v + \zeta_{0,N-1} + \phi]P_{0,N} = \lambda_{N-1}P_{0,N-1} + (\mu_v + \zeta_{0,N})P_{0,N+1}, \qquad n = N, \tag{3}$$
$$[\lambda_n + \mu_v + \zeta_{0,n-1} + \phi]P_{0,n} = \lambda_{n-1}P_{0,n-1} + (\mu_v + \zeta_{0,n})P_{0,n+1}, \qquad N+1 \le n \le L-1, \tag{4}$$
$$[\zeta_{0,L-1} + \mu_v + \phi]P_{0,L} = \lambda_{L-1}P_{L-1}, \qquad n = L. \tag{5}$$

For $J(t) = 1$:

$$[\mu + \lambda_1]P_{1,1} = (\mu + \zeta_{1,1})P_{1,2}, \qquad n = 1, \tag{6}$$
$$[\lambda_n + \mu + \zeta_{1,n-1}]P_{1,n} = \lambda_{n-1}P_{1,n-1} + (\mu + \zeta_{1,n})P_{1,n+1}, \qquad 2 \le n \le Q-1, \tag{7}$$
$$[\lambda_Q + \mu + \zeta_{1,Q-1}]P_{1,Q} = \lambda_{Q-1}P_{1,Q-1} + (\mu + \zeta_{1,Q})P_{1,Q+1} + 2\mu P_{2,Q+1}, \qquad n = Q, \tag{8}$$
$$[\lambda_n + \mu + \zeta_{1,n-1}]P_{1,n} = \lambda_{n-1}P_{1,n-1} + (\mu + \zeta_{1,n})P_{1,n+1}, \qquad Q+1 \le n \le N-1, \tag{9}$$
$$[\lambda_n + \mu + \zeta_{1,n-1}]P_{1,n} = \lambda_{n-1}P_{1,n-1} + (\mu + \zeta_{1,n})P_{1,n+1} + \lambda_{N-1}P_{3,N-1}, \qquad n = N, \tag{10}$$
$$[\lambda_n + \mu + \zeta_{1,n-1}]P_{1,n} = \lambda_{n-1}P_{1,n-1} + (\mu + \zeta_{1,n})P_{1,n+1}, \qquad N+1 \le n \le M-2, \tag{11}$$
$$[\lambda_{M-1} + \mu + \zeta_{1,M-2}]P_{1,M-1} = \lambda_{M-2}P_{1,M-2}, \qquad n = M-1. \tag{12}$$

For $J(t) = 2$:

$$\begin{aligned}
&[\lambda_{Q+1} + 2\mu]P_{2,Q+1} = [2\mu + \zeta_{1,Q+1}]P_{2,Q+2}, && n = Q+1, && (13)\\
&[\lambda_n + 2\mu + (n-1)\alpha\xi_1]P_{2,n} = \lambda_{n-1}P_{2,n-1} + (2\mu + \zeta_{1,n})P_{2,n+1}, && Q+2 \le n \le N-1, && (14)\\
&[\lambda_n + 2\mu + \zeta_{1,n-1}]P_{2,n} = \lambda_{n-1}P_{2,n-1} + (2\mu + \zeta_{1,n})P_{2,n+1} + \phi P_{0,n}, && N \le n \le M-1, && (15)\\
&[\lambda_M + 2\mu + \zeta_{1,M-1}]P_{2,M} = \lambda_{M-1}P_{2,M-1} + (2\mu + \zeta_{1,M})P_{2,M+1} && && \\
&\qquad + \phi P_{0,M} + \lambda_{M-1}P_{1,M-1}, && n = M, && (16)\\
&[\lambda_n + 2\mu + \zeta_{1,n-1}]P_{2,n} = \lambda_{n-1}P_{2,n-1} + (2\mu + \zeta_{1,n})P_{2,n+1} + \phi P_{0,n}, && M+1 \le n \le L-1, && (17)\\
&[2\mu + \zeta_{1,L-1}]P_{2,L} = \lambda_{L-1}P_{2,L-1} + \phi P_{0,L}, && n = L. && (18)
\end{aligned}$$

For $J(t) = 3$:

$$\begin{aligned}
&\lambda_0 P_{3,0} = \phi P_{0,0}, && n = 0, && (19)\\
&[\lambda_n + \zeta_{0,n-1}]P_{3,n} = \lambda_{n-1}P_{3,n-1} + \zeta_{0,n}P_{3,n+1} + \gamma\phi P_{0,n}, && 1 \le n \le N-2, && (20)\\
&[\lambda_{N-1} + \zeta_{0,N-2}]P_{3,N-1} = \lambda_{N-2}P_{3,N-2} + \gamma\phi P_{0,N-1}, && n = N-1. && (21)
\end{aligned}$$

To study the *MWV* and *SWV* cases, we multiply equations (19)–(21) by the indicator γ. So the dormant state being represented by the following equations:

$$\begin{aligned}
&\lambda_0 P_{3,0} = \gamma\phi P_{0,0}, && n = 0, && (22)\\
&[\lambda_n + \zeta_{0,n-1}]P_{3,n} = \gamma[\lambda_{n-1}P_{3,n-1} + \zeta_{0,n}P_{3,n+1} + \gamma\phi P_{0,n}], && 1 \le n \le N-2, && (23)\\
&[\lambda_{N-1} + \zeta_{0,N-2}]P_{3,N-1} = \gamma[\lambda_{N-2}P_{3,N-2} + \gamma\phi P_{0,N-1}], && n = N-1. && (24)
\end{aligned}$$

4 Resolution method

In this section, we use the matrix-geometric method to analyze the steady-state behavior of a machine repair problem incorporating Single Working Vacation (*SWV*), Multiple Working Vacation (*MWV*), reneging, balking, and a triadic policy. This method is chosen for its ability to efficiently handle high-dimensional systems with recursive structures and provide precise steady-state solutions for Markovian processes. Its strength in computing performance metrics such as queue lengths, server utilization, and availability rates, even with complex service policies, makes it ideal for our analysis. The matrix-geometric method's robustness and adaptability enhance the accuracy and relevance of our study's findings, offering valuable insights into the behavior and reliability of queueing systems. We define the steady-state probability vectors $\pi = [\pi_0, \pi_1, \ldots, \pi_L]$, where

$$\begin{aligned}
&\pi_0 = [P_0(0), P_3(0)], && && (25)\\
&\pi_j = [P_0(j), P_1(j), P_3(j)], && 1 \le j \le Q, && (26)\\
&\pi_j = [P_0(j), P_1(j), P_2(j), P_3(j)], && Q+1 \le j \le N-1, && (27)\\
&\pi_j = [P_0(j), P_1(j), P_2(j)], && N \le j \le M-1, && (28)\\
&\pi_j = [P_0(j), P_2(j)], && M \le j \le L. && (29)
\end{aligned}$$

4.1 Matrix-geometric representations

Now, we employ the matrix-geometric method to derive the obvious iterative expression for the steady-state probability vectors. The balance equations (1)–(24) can be written in the following matrix form:

$$\pi Q = 0, \tag{30}$$

where Q is the infinitesimal generator which can be written as:

$$Q = \begin{pmatrix} A_0 & B_0 & & & & & & \\ C_0 & A_1 & B_1 & & & & & \\ & C_1 & A_2 & B_2 & & & & \\ & & C_2 & A_3 & B_3 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & C_{L-2} & A_{L-2} & B_{L-2} & \\ & & & & & C_{L-1} & A_{L-1} & B_{L-1} \\ & & & & & & C_L & A_L \end{pmatrix}. \tag{31}$$

To formulate the set of transition matrices for the suggested model, starting with the notations below:

$$\begin{cases} \nu_{m,i} = \lambda_i + \zeta_{m,i-1}, \quad m = \{0,1\}, \\ \eta_i = -diag\left([\lambda_i + 2\mu + \zeta_{1,i-1}, 0]\right), \\ \varpi_i = -diag\left([\lambda_i + \mu_v + \zeta_{0,i-1} + \phi, \lambda_i + \mu + \zeta_{1,i-1}]\right). \end{cases}$$

The sub-transition matrices $B_0, A_0, C_1, \ldots, B_{L-1}, A_L, C_L$ of infinitesimal generator Q (see, equation (31)), are represented as follows:

$$A_0 = -B_0,$$

$$A_i = \begin{pmatrix} -[\nu_{0,i} + \gamma\phi + \mu_v] & 0 & 0 & \gamma\phi \\ 0 & -[\nu_{1,i} + \mu] & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu_{0,i} \end{pmatrix}, \; i = 1, \ldots, Q,$$

$$A_i = \begin{pmatrix} -[\nu_{0,i} + \gamma\phi + \mu_v] & 0 & 0 & \gamma\phi \\ 0 & -[\nu_{1,i} + \mu] & 0 & 0 \\ 0 & 0 & -[\nu_{1,i} + 2\mu] & 0 \\ 0 & 0 & 0 & -\nu_{0,i} \end{pmatrix}, \; i = Q+1, \ldots, N-1,$$

$$A_i = \begin{bmatrix} \varpi i & diag\left([\phi, 0]\right) \\ O_{2\times 2} & \eta_i \end{bmatrix}_{4\times 4}, \qquad i = N, \ldots, M-1,$$

$$A_i = \begin{bmatrix} -diag\left([\lambda_i + \mu_v + \zeta_{0,i-1} + \phi, 0]\right) & diag\left([\phi, 0]\right) \\ O_{2\times 2} & \eta_i \end{bmatrix}_{4\times 4}, \quad i = M, \ldots, L,$$

$$
\begin{array}{ll}
B_0 = diag\,([\lambda_0, 0, 0, \lambda_0])_{4\times 4}, & i = 0, \\
B_i = diag\,([\lambda_i, \lambda_i, 0, \lambda_i])_{4\times 4}, & i = 1, \ldots, Q, \\
B_i = diag\,([\lambda_i, \lambda_i, \lambda_i, \lambda_i])_{4\times 4}, & i = Q+1, \ldots, N-2, \\
B_i = diag\,([\lambda_i, \lambda_i, \lambda_i, 0])_{4\times 4}, & i = N-1, \ldots, M-2, \\
B_i = diag\,([\lambda_i, 0, \lambda_i, 0])_{4\times 4}, & i = M-1, \ldots, L-1, \\
C_1 = [\mu_v, \mu, 0, 0]^T_{4\times 1}, & i = 1, \\
C_i = diag\,([\mu_v + \zeta_{0,i-1}, \mu + \zeta_{1,i-1}, 0, \zeta_{0,i-1}])_{4\times 4}, & i = 2, \ldots, Q, \\
C_{Q+1} = diag\,([\mu_v + \zeta_{0,Q}, \mu + \zeta_{1,Q}, 2\mu, \zeta_{0,Q}])_{4\times 4}, & i = Q+1, \\
C_i = \begin{bmatrix} diag\,([\mu_v + \zeta_{0,i-1}, \mu + \zeta_{1,i-1}]) & diag\,([\phi, 0]) \\ O_{2\times 2} & diag\,([2\mu + \zeta_{1,i-1}, 0]) \end{bmatrix}_{4\times 4}, & i = N, \ldots, M-1, \\
C_i = \begin{bmatrix} diag\,([\mu_v + \zeta_{0,i-1}, 0]) & diag\,([\phi, 0]) \\ O_{2\times 2} & diag\,([2\mu + \zeta_{1,i-1}, 0]) \end{bmatrix}_{4\times 4}, & i = M, \ldots, L.
\end{array}
$$

4.2 Steady-state probabilities

The steady-state probabilities can be determined by solving equations (1)–(24) using the matrix-geometric method recursively as follows:
Starting with equation (30), we have

$$
\begin{cases}
\pi_0 A_0 + \pi_1 C_1 = 0, \\
\pi_0 B_0 + \pi_1 A_1 + \pi_2 C_2 = 0, \\
\qquad \vdots \\
\pi_{L-2} B_{L-2} + \pi_{L-1} A_{L-1} + \pi_L C_L = 0, \\
\pi_{L-1} B_{L-1} + \pi_L A_L = 0.
\end{cases} \tag{32}
$$

Using system of equations (32), we obtain the general term as:

$$\pi_0 A_0 + \pi_1 C_1 = 0, \tag{33}$$
$$\pi_{i-1} B_{i-1} + \pi_i A_i + \pi_{i+1} C_{i+1} = 0, \quad 1 \le i \le L-1, \tag{34}$$
$$\pi_{L-1} B_{L-1} + \pi_L A_L = 0. \tag{35}$$

Using equation (35), we get recursively π_L:

$$\pi_L = \pi_{L-1} B_{L-1} (A_L)^{-1} = \pi_{L-1} \psi_L, \tag{36}$$

where $\psi_L = B_{L-1}(A_L)^{-1}$, $i = L-1$.
Combining (34) and (36), we obtain

$$\pi_{L-1} = \pi_{L-2}(-B_{L-1})(A_L + \psi_L C_L)^{-1} = \pi_{L-2} \psi_{L-1},$$

where $\psi_{L-1} = (-B_{L-1})(A_L + \psi_L C_L)^{-1}$.
Recursively, we obtain π_1:

$$\pi_1 = \pi_0 \psi_1, \quad \text{where} \quad \psi_1 = (-B_0)(A_1 + \psi_2 C_2)^{-1}.$$

Next, from equation (33), we get:

$$\pi_0(A_0+\psi_1 C_1)=0.$$

To obtain π_0, we have to solve the equation:

$$\pi_0(A_0+(-B_0)(A_1+\psi_2 C_2)^{-1}C_1)=0.$$

Then, we can obtain the general expression of π_i based on ψ_i, $1\le i\le L$:

$$\pi_L=\pi_{L-1}\psi_L=\cdots=\pi_0(\psi_1\psi_2\psi_3\ldots\psi_L)=\pi_0\prod_{i=1}^{L}\psi_i,$$

$$\text{where } \psi_i=\begin{cases}(-B_{i-1})[A_i+(\psi_{i+1}C_{i+1})]^{-1}, & 1\le i\le L-1,\\ (-B_{i-1})(A_i)^{-1},\ i=L.\end{cases}$$

After obtaining $\pi_0,\pi_1,\ldots,\pi_L$, the normalization equation gives

$$\sum_{i=0}^{L}\pi e_4=1,$$

$$\pi_0\left[e_4+\sum_{i=1}^{L}\left(\prod_{r=1}^{i}\psi_r\right)e_4\right]=1,\ \text{ where }\ e_4=(1,1,1,1)^t.$$

5 Performance measures

In this section, we obtain different performance measures and reliability indices, which are presented in what follows.

1. The expected number of failed machines in the system:

$$E[N]=\sum_{n=0}^{L}n\pi_{0,n}+\sum_{n=1}^{M-1}n\pi_{1,n}+\sum_{n=Q+1}^{L}n\pi_{2,n}+\gamma\sum_{n=1}^{N-1}n\pi_{3,n}.$$

2. The expected number of operating machines in the system:

$$E[OP]=L-E(N).$$

3. The expected number to have only one server is busy during the WV period:

$$E[B_0]=\sum_{i=0}^{L}\pi_{0,n}. \tag{37}$$

4. The expected number to have only one server is busy during the regular busy period:

$$E[B_1]=\sum_{i=1}^{M-1}\pi_{1,n}. \tag{38}$$

5. The expected number to have both servers are busy during the regular busy period:

$$E[B_2] = 2\sum_{i=Q+1}^{L} \pi_{2,n}. \tag{39}$$

6. The expected number to have only one server is busy server during the dormant period:

$$E[B_3] = \gamma \sum_{i=0}^{N-1} \pi_{3,n}. \tag{40}$$

7. The expected queue length:

$$E_q = \sum_{i=1}^{L}(n-1)\pi_{0,n} + \sum_{i=1}^{M-1}(n-1)\pi_{1,n} + \sum_{i=Q+1}^{L}(n-2)\pi_{2,n} + \gamma\sum_{i=1}^{N-1}(n-1)\pi_{3,n}. \tag{41}$$

8. The expected number of idle servers in the system:

$$E[I] = 2\pi_{0,0} + \sum_{i=1}^{L}\pi_{0,n} + \sum_{i=1}^{M-1}\pi_{1,n} + 2\gamma\sum_{i=1}^{N-1}\pi_{3,n}. \tag{42}$$

9. Machine availability:

$$AV = \frac{E[OP]}{L}. \tag{43}$$

10. Operative utilization (the fraction of busy servers):

$$BS = \frac{E[B_0]+E[B_1]+E[B_2]+E[B_3]}{2}. \tag{44}$$

11. The average balking rate:

$$\begin{aligned} BR = &\sum_{i=1}^{L}(n-1)\lambda(1-b_n)\pi_{0,n} + \sum_{i=1}^{M-1}(n-1)\lambda(1-b_n)\pi_{2,n} \\ &+ \sum_{i=Q+1}^{L}(n-2)\lambda(1-b_n)\pi_{2,n} + \gamma\sum_{i=1}^{N-1}(n-1)\lambda(1-b_n)\pi_{3,n}. \end{aligned} \tag{45}$$

12. The average reneging rate:

$$\begin{aligned} AR = &\,\xi_0\sum_{i=1}^{L}(n-1)\pi_{0,n} + \xi_1\sum_{i=1}^{M-1}(n-1)\pi_{1,n} \\ &+ \xi_1\sum_{i=Q+1}^{L}(n-2)\pi_{2,n} + \gamma\xi_0\sum_{i=1}^{N-1}(n-1)\pi_{3,n}. \end{aligned} \tag{46}$$

13. The average customer loss rate:

$$LR = BR + AR. \tag{47}$$

6 Special cases

Now, we show some particular cases of the analyzed machine repair model:

Cases	Vacation type	Authors	Models	Resolution method
$\phi \to \infty$, $\mu = \mu_v$, $\xi_0 = \xi_1 = 0$, $\beta_i = 1$ $\alpha = 1$ and $L = \infty$	No vacation	Liou et al. [13]	$M/M/2$ with triadic policy	Recursive method
$\gamma = 1$, $Q = N = 1$, $M \to \infty$, $\xi_0 = \xi_1 = 0$, $\beta_i = 1$ and $\alpha = 1$	*SWV* policy	He et al. [4]	$M/M/1$ queue	Q-matrix method
$\gamma = 0$, $\xi_0 = \xi_1 = 1$ and customers may be impatient before the service begins	*MWV* policy	Sahrma et al. [17]	$M/M/2$ with triadic policy	Recursive method
$\gamma = 0$ and $\xi_0 = \xi_1$	*MWV* policy	Ketema [8]	$M/M/2$ with triadic policy	Recursive method
$\gamma = 0$, $\xi_0 = \xi_1 = 0$, $\beta_i = 1$ and $\alpha = 1$	*MWV* policy	Ketema et al. [9]	$M/M/2$ with triadic policy	Recursive method
$\gamma = 0$, $\xi_0 = \xi_1 = 0$, $\beta_i = 1$, $\alpha = 1$, $Q = 1$, $M = \infty$, $L = \infty$, system without repairable machines	*MWV* policy	Wang et al. [19]	$M/M/1$ with N-policy	Recursive method

7 Numerical results

We present the numerical outcomes to investigate how different parameters impact system performance. To perform the computations, we have implemented the matrix-geometric method, using R-software. These computations allow us to determine the system's performance indices. We fix arbitrary some system parameters as: $\lambda = 1.9$, $\mu = 2$, $\mu_v = 1.5$, $\phi = 2$, $\xi_0 = 2.5$ and $\xi_1 = 1$.
In what follows, we provide interpretations of how various parameters in our model affect the system's performance.

- Figure 2 (resp. Figure 3) suggests that a higher rate of machine failures (λ) results in more machines reneging (resp. balking). It emphasizes that as customer demand increases, reneging and balking also rise. We observe that the *MWV* case is more affected.

- Figure 4 (resp. Figure 5) indicates that the rise in E_q is a direct consequence of the increased machine failure rate λ. Furthermore, an increase in λ leads to a longer queue, as depicted. This increasing rate also results in more customer loss, as shown in Figure 9.

- Figure 6 indicates that a higher balking rate (β) results in fewer machines balking (resp. reneging). A high customer strength in joining the queue may provoke queued failed machines to leave without service.

- Figure 7 suggests that more frequent vacations (ϕ) result in fewer busy servers and a lower likelihood of a server being on a working vacation. When the server rarely switches to vacation mode, customers may not become impatient or renege, leading to a lower mean number of lost customers due to the absence of reneging.

- A higher reneging rate (α) leads to more machines reneging, especially when ξ_0 or ξ_1 are high, as shown in Figures 10–11. This increase in α also results in a longer queue and more customer loss due to more machines joining and leaving the queue (see Figures 8–9).

- Figure 12 indicates that a faster service rate (μ) results in fewer idle servers. If α is high, more machines may decide to leave the queue before being serviced. This can lead to a decrease in queue length and customer loss (see Figure 13), as there are fewer machines in the queue and fewer machines leaving without service.

7.1 Cost profit analysis

In this subsection, we investigate the changes on the optimum values of μ_v^* via the system parameters on the expected cost. To this aim, we fix for:

- SWV results: $C_i = 2$, $C_0 = 5$, $C_1 = 10$, $C_2 = 25$, $C_u = 3$, and $C_v = 15$,

- MWV results: $C_i = 20$, $C_0 = 50$, $C_1 = 10$, $C_2 = 25$, $C_u = 80$, and $C_v = 15$, where

C_i $\equiv$ Cost per unit time of idle server in the system.

C_0 $\equiv$ Cost per unit time of one busy server during WV.

C_1 $\equiv$ Cost per unit time of one busy server during the regular busy period.

C_2 $\equiv$ Cost per unit time of two busy servers during the regular busy period.

C_u $\equiv$ Cost for using μ as a service rate.

C_v $\equiv$ Cost for using μ_v as a service rate.

By linking this set of cost elements to the performance measures, the total expected cost function is expressed as:

$$F(\mu_v) = E[I]C_i + E[B_0]C_0 + E[B_1]C_1 + E[B_2]C_2 + \mu C_u + \mu_v C_c. \tag{48}$$

The set of system parameters are fixed as follows: $\lambda = 0.7$, $\mu = 2.5$, $\xi_0 = 2.5$, $\xi_1 = 1.5$, $\beta = 0.2$, $\alpha = 0.7$, $\phi = 1.5$, $Q = 5$, $N = 13$ and $M = 18$.

7.2 The QFS method

In this section, we focus on determining the optimal service rate during vacation periods (μ_v) to minimize the expected cost (F) in a complex queueing system. Due to the intricate nature of the cost function, we utilize QFS method to tackle this optimization problem. The QFS method is highly regarded for its robustness in queueing system optimization, offering several key advantages. It delivers efficient, accurate, and flexible solutions by effectively approximating and minimizing complex cost functions. This approach not only facilitates precise cost optimization but also enhances decision-making in practical applications. Its versatility in adapting to various parameters and scenarios makes it particularly effective for addressing diverse and challenging queueing issues, ensuring improved resource management and operational efficiency. By applying the QFS method, our study makes a significant contribution to

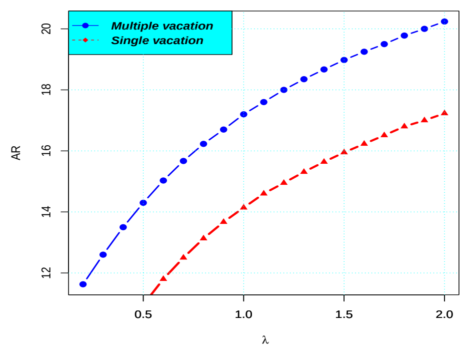

Figure 2: The effect of λ vs. AR, for $\beta = 0.6$, $\alpha = 0.1$, $\phi = 2$, $L = 10$, $M = 8$, $N = 5$ and $Q = 2$.

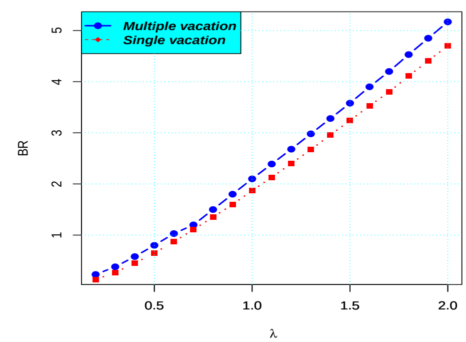

Figure 3: The effect of λ vs. BR, for $\beta = 0.6$, $\alpha = 0.1$, $L = 10$, $M = 8$, $N = 5$ and $Q = 2$.

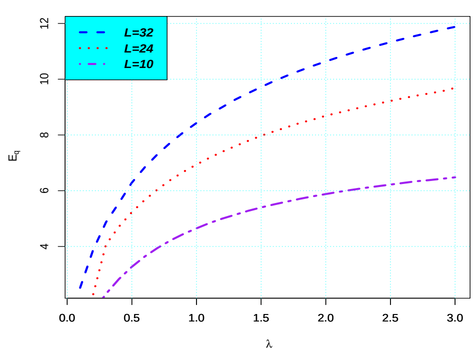

Figure 4: E_q vs. λ and L, when $\beta = 0.9$, $\alpha = 0.4$, $M = 8$, $N = 5$, $Q = 2$, under MWV.

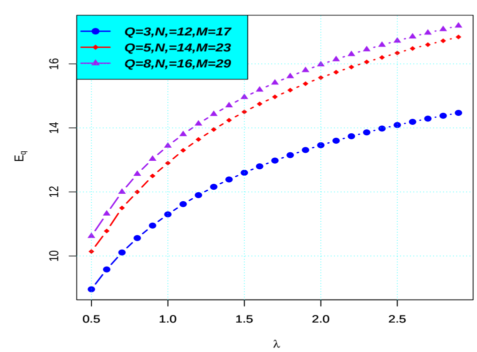

Figure 5: E_q vs. Q, N and M, when $\beta = 0.9$ and $\alpha = 0.6$, under SWV.

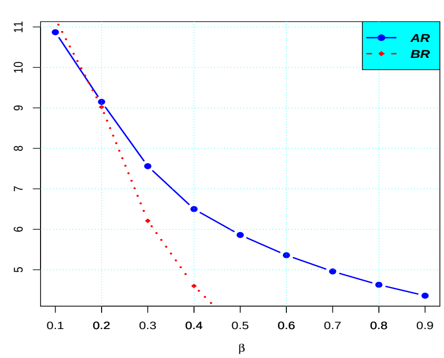

Figure 6: AR and BR vs. β, for $\alpha = 0.6$, $L = 10$, $M = 8$, $N = 5$ and $Q = 2$ under MWV.

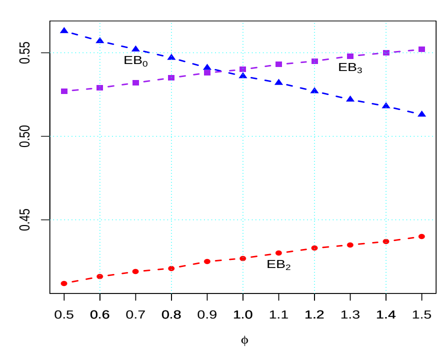

Figure 7: $E[B_0]$, $E[B_2]$ and $E[B_3]$ vs. ϕ, for $\beta = 0.9$, $\alpha = 0.6$, $L = 10$, $M = 8$, $N = 5$ and $Q = 2$ under SWV.

optimizing system performance and cost management in real-world contexts. For this analysis, all system parameters are assumed to be fixed, with the service rate during vacation periods (μ_v) being the only

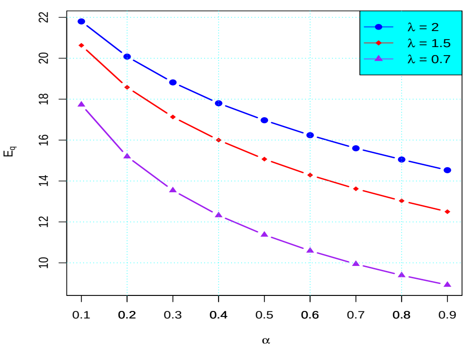

Figure 8: The impact of α on E_q, for $\beta = 0.6$, $L = 32$, $M = 28$, $N = 15$ and $Q = 8$ under SWV.

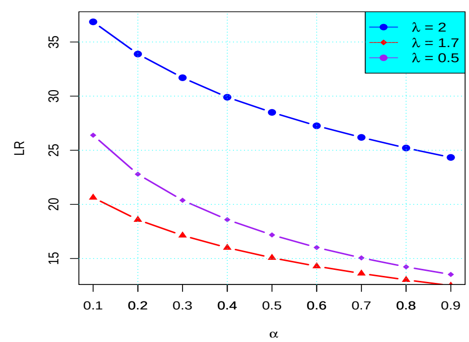

Figure 9: The impact of α on LR, for $\beta = 0.6$, $L = 32$, $M = 28$, $N = 15$ and $Q = 8$ under SWV.

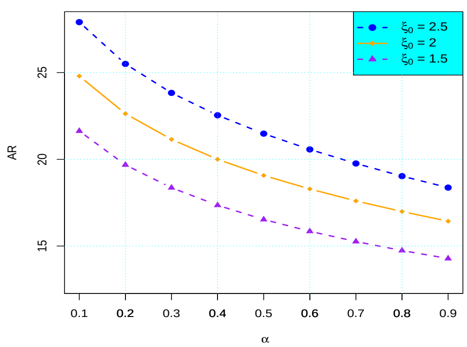

Figure 10: The impact of α and ξ_0 vs. AR, for $\beta = 0.6$, $L = 24$, $M = 25$, $N = 14$, $Q = 5$ under SWV.

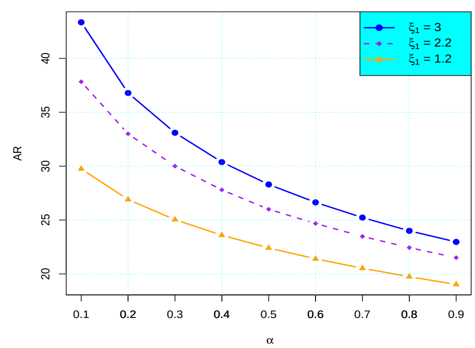

Figure 11: The impact of α and ξ_1 on AR, for $\beta = 0.6$, $L = 24$, $M = 23$, $N = 14$, $Q = 5$ under SWV.

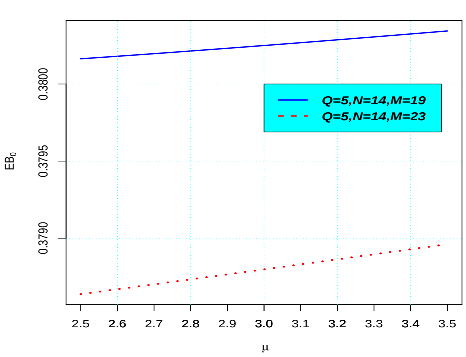

Figure 12: The impact of μ and M vs. EB_0, for $\beta = 0.9$, $\alpha = 0.4$ and $L = 24$ under MWV.

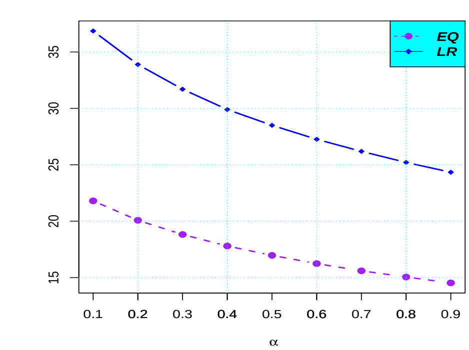

Figure 13: The impact of α on E_q and LR, for $\beta = 0.9$, $L = 32$, $M = 28$, $N = 15$ and $Q = 8$ under SWV.

variable under control, so we define:

$$x^q = \frac{1}{2}\frac{F(x^l)((x^m)^2 - (x^u)^2) + F(x^m)((x^u)^2 - (x^l)^2) + F(x^u)((x^l)^2 - (x^m)^2)}{F(x^l)(x^m - x^u) + F(x^m)(x^u - x^l) + F(x^u)(x^l - x^m)},$$

where, x^l, x^u and x^m are 3-point pattern one of this pattern are replaced with x^q.

After applying the QFS method, we obtain the numerical results of the optimum value of μ_v, and the optimum cost F^*, which are represented in tables and figures below: From Tables 1–5, we can draw

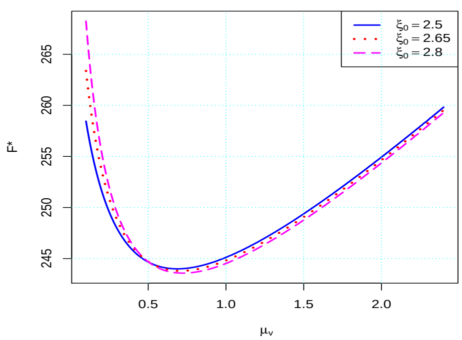

Figure 14: The optimum μ_v^* for different values of ξ_0 under MWV.

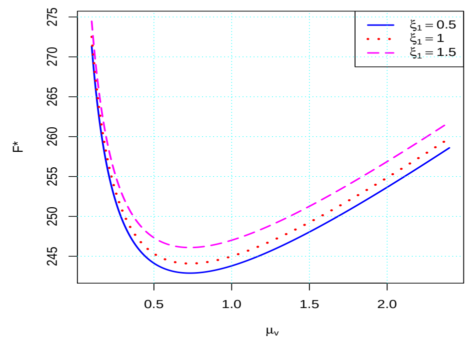

Figure 15: The optimum μ_v^* for different values of ξ_1 under SWV.

Table 1: The expected optimum μ_v^* and F^* corresponding to different values of ξ_0.

	$\xi_0 = 2.5$		$\xi_0 = 2.65$		$\xi_0 = 2.8$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ_v^*	0.931	0.687	1.678	0.709	2.36	0.723
F^*	110.3456	243.9955	153.8345	204.7978	212.2325	243.5766

Table 2: The expected optimum μ_v^* and F^* for different values of ξ_1.

	$\xi_1 = 0.5$		$\xi_1 = 1$		$\xi_1 = 1.5$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ_v^*	1.365	0.7209	1.366	0.7315	1.367	0.7320
F^*	134.4433	246.0976	133.3141	244.0929	132.64	242.8901

Table 3: The expected optimum μ_v^* and F^* for different values of μ.

	$\mu = 2.5$		$\mu = 3$		$\mu = 3.5$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ_v^*	1.365	0.7741	1.367	0.7743	1.370	0.7744
F^*	132.6503	243.5736	133.9659	283.2434	135.3053	322.9609

Table 4: The expected optimum μ_v^* and F^* for different values of ϕ.

	$\phi = 0.2$		$\phi = 0.4$		$\phi = 0.5$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ_v^*	1.36512	0.7647	1.3451	0.7646	1.2933	0.7644
F^*	133.4822	248.2272	130.6013	248.2169	121.5498	248.2117

the following conclusions:

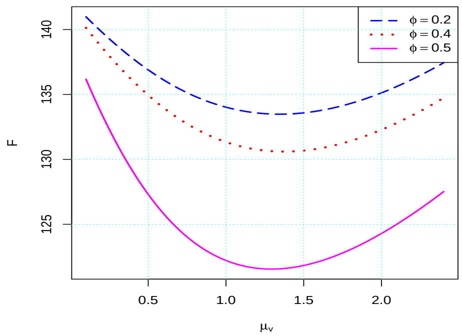

Figure 16: The optimum μ_v^* for different values of ϕ under SWV.

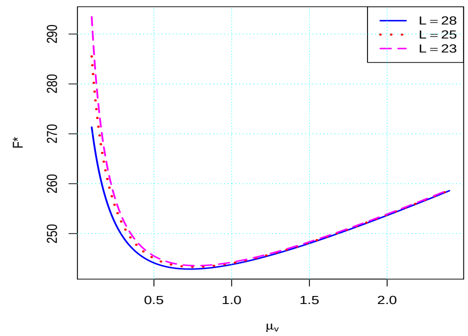

Figure 17: The optimum μ_v^* for different values of L under MWV.

Table 5: The expected optimum μ_v^* and F^* for different values of L.

	$L=21$		$L=23$		$L=25$	
	SWV	MWV	SWV	MWV	SWV	MWV
μ_v^*	1.4225	0.7838	1.0762	0.7741	0.75216	0.7625
F^*	335.656	243.8514	305.5713	243.5737	273.924	243.3177

– As shown in Table 1 (and similarly in Table 2), as ξ_0 (or ξ_1) increases, both the optimum expected cost F^* and the optimum values μ_v^* increase significantly. This is due to the increased number of failed machines leaving the queue without service during both the working vacation and dormant periods. Essentially, the system becomes less efficient as more machines leave without service, leading to higher costs.

– Table 3 reveals that an increase in the service rate μ results in a decrease in the values of the service rate μ_v^*. However, the expected cost values F^* increase in response to this. This indicates that while faster service rates can reduce the need for service during vacation periods, they also increase operational costs.

– According to Table 4, an increase in the vacation rate ϕ leads to a decrease in both the values of the service rate μ_v^* and F^*. This suggests that more frequent vacations can reduce the need for service during these periods and lower operational costs.

– Table 5 demonstrates that as L (the number of operating machines) increases, both the optimum expected cost F^* and the optimum value μ_v^* decrease. This is because, with more operating machines, the load on each machine decreases, reducing the likelihood of failure and thus the cost of repairs.

8 Conclusion and future perspectives

In this paper, we analyzed a machine repair queueing model with a triadic policy $(0,Q,N,M)$, which incorporates features such as balking (machines choosing not to join the queue), reneging (machines leaving the queue before service), and two types of working vacation policies: Single Working Vacation and Multiple Working Vacation. The primary aim of the triadic policy in the design and analysis of machining systems is to mitigate congestion issues, ensure optimal reliability of repairmen in industrial, manufacturing, and production sectors, and maintain cost efficiency. To evaluate the steady-state probabilities of the queueing system, we employed the matrix-geometric method, specifically leveraging its recursive approach. We derived and presented several key performance measures and reliability indices, including the expected number of failed and operating machines, machine availability, queue length, and various rates of balking, reneging, and customer loss. Using the matrix-geometric method implemented in *R* software, we accurately calculated these metrics and applied the Quadratic Fit Search method to determine the optimal service rate during vacation periods, to minimize the expected cost. The results, detailed through comprehensive tables and figures, highlighted how variations in the service rate impact system performance and cost.

The results obtained can be used as performance assessment measures for the system in question, applicable to various congestion scenarios encountered in a range of practical domains, such as customer service centers, communication and network systems, production systems, and science and technology. Based on the steady-state analysis, there is potential for extending the current model to include multi-heterogeneous repairmen, encompassing both fully and partially unreliable repairmen, along with retention and feedback policies. Currently, this study has not undertaken a transient analysis, which represents the next phase of research.

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