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On the regularity theory for quasilinear elliptic systems with the application of Leray-Schauder method

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Abstract. In this article, we consider an elliptic system of partial differential equations in the general form

$$\sum_{i=1,\ldots,n} \frac{d}{dx_i} A_i\left(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u}\right) + B\left(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u}\right) = 0$$

under fair general conditions on its structural coefficients. We study the regularity properties of the solutions to this system, and we establish the existence of a Holder solution by the modified Leray-Schauder fixed-point method and the application of the apriori estimations obtained with utilization of form-boundary conditions.

Keywords: Quasilinear Partial Differential Equation, Holder solution, regularity theory, Leray Schauder theorem, form-bounded, elliptic system.
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1 Introduction

In this paper, we consider elliptic systems of partial differential equations presented in the form

$$\sum_{i=1,\ldots,n} \frac{d}{dx_i} A_i\left(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u}\right) + B\left(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u}\right) = 0$$

for the unknown vector-function $\overrightarrow{u} : \Omega \to R^N$, where Ω is a bounded, Lipschitz smooth domain in R^n.

We consider the modification of the Leray-Schauder fixed-point method, which provides a possibility to prove the existence of solutions to elliptic partial differential equations and elliptic systems by means

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of apriori estimates. The developments presented in this paper in quasilinear theory become possible due to the recent breakthrough in the linear perturbation theory of elliptic and parabolic operators, which is based on the classical works of DeGiorgi, Moser, and Nash [5, 6, 11, 13], who studied the non-perturbed cases of elliptic and parabolic equations. Namely, an elliptic equation which is given by

$$\nabla_i a_{ij} \nabla_j u(x) = 0,$$

and a parabolic equation

$$\partial_t u(x,\, t) - \nabla_i a_{ij} \nabla_j u(x,\, t) = 0.$$

In recent years, the perturbation theory of the classical DeGiorgi-Moser-Nash results was developed by many authors [1–4, 7–10, 12, 14–28]. There were studied the properties of general linear operators

$$-\nabla \cdot a \cdot \nabla_j + b \cdot \nabla + \nabla \cdot \tilde{b} + V$$

with measurable uniformly elliptic matrix, the estimations of heat kernel were established under the rather general assumptions on the structural coefficients. Such conditions on coefficients are formulated in terms of functional Kato, Gevrey, and Nash classes, and form-boundary conditions, see [11]. Generally, these kinds of problems are considered with an application of some variants of Duhamels principle and the Lie product formula for propagators. Let us consider an example of the elliptic equation with the Gilbarg-Sirrin matrix presented in the form

$$\zeta u(x) - \nabla_i (a_{ij}(x) \nabla_j u(x)) + \nabla_i a_{ij}(x) \circ \nabla_j u(x) = 0$$

where

$$a_{ij} = \delta_{ij} + b \frac{\nabla_i x \nabla_j x}{|x|^2}, \quad b = -1 + \frac{n-1}{1-\lambda},$$

$\lambda < 1$, $n \geq 3$, and $\zeta > 0$, $f \in L^1 \cap L^p$, $p > 1$. The form boundary condition is given in the explicit form

$$\left\| b^2 (1+b)^{-1} \left(\frac{n-1}{|x|} \right)^2 \varphi \right\|_2^2 \leq \beta \left\| \nabla \overrightarrow{\varphi} \right\|_2^2 + c(\beta) \left\| \overrightarrow{\varphi} \right\|_2^2$$

with form-boundary constant

$$\beta = 4 \left(\frac{b}{b+1} \cdot \frac{n-1}{n-2} \right)^2 = 4 \left(1 + \frac{\lambda}{n-2} \right)^2.$$

Thus, the equation

$$a_{ij} \nabla_i \nabla_j u = 0$$

has always two solutions for all $\beta > 4$, and assuming $p > q$, $2 \leq q < p$ then $|x|^{\lambda} \notin L_{loc}^{\frac{pn}{n-2}}(R^n)$ with $\lambda = \frac{n-2}{-q}$ that excludes unbounded solutions. For $\beta > 4$ the equation $a_{ij} \nabla_i \nabla_j u = 0$ has always two licitus solutions [13].

The same situation relative to singularities of the structural coefficients appears in the case of systems, however, there is an additional complication connected with the growth of structural coefficients, which can be explained in the following example [13]. The pair of functions $u(x)=\cos(\chi x)$ and $v(x)=\sin(\chi x)$ satisfies the E. Heinz elliptic system

$$\frac{d^2u}{dx^2}=-\left(\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dx}\right)^2\right)u,$$

$$\frac{d^2v}{dx^2}=-\left(\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dx}\right)^2\right)v.$$

The E. Heinz example shows the impossibility of obtaining apriori estimates of

$$\max_{[0,\,2\pi]}\left\{\left|\frac{du}{dx}\right|,\ \left|\frac{dv}{dx}\right|\right\},|u|_{\alpha,\,\Omega},\quad |v|_{\alpha,\,\Omega}$$

in terms of $\max\limits_{[0,\,2\pi]}\{|u|,\ |v|\}$. The E. Heinz accentuates the difference between equations and systems, in the case of equations such estimates are possible [13].

In this paper, we establish sufficient conditions on the structural coefficients under which generalized solutions $\overrightarrow{u}\in W_1^p(\Omega)\cap L^q(\Omega)$, $\tilde{M}_1=ess\max\limits_{\partial\Omega}|\overrightarrow{u}|<\infty$ to the system

$$\frac{d}{dx_i}A_i(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})+B(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})=0$$

estimated by number depending only on $\tilde{M}_1,\ \tilde{\nu},\ \beta,\ mes(\Omega),\ \|\overrightarrow{u}\|_{L^q},\ \varepsilon.$ This solution $\overrightarrow{u}$ satisfies the Holder continuity condition of order α, where constant $\alpha>0$ depends only on $\tilde{M}_1,\ \tilde{\nu},\ \beta,$ $mes(\Omega),\ \|\overrightarrow{u}\|_{L^q},\ \varepsilon.$

2 Notations and the Leray-Schauder approach

Let $x=(x_1,\ ...,\ x_n)$ be a n-dimensional real vector, i.e. $x\in R^n$, $\overrightarrow{u}$ be a vector-function $\overrightarrow{u}(x)=\left(u^1(x),....,u^N(x)\right)$ defined and measurable in a bounded simply connected domain $\Omega\subset R^n$.

We study the solvability of a quasilinear elliptic system given by

$$\frac{d}{dx_i}A_i(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})+B(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})=0, \tag{1}$$

where A_i and B is N-dimensional vector-functions for each $i=1\ ,...,\ n$. The boundary condition is given by $\overrightarrow{u}|_{\partial\Omega}=\psi(x)|_{\partial\Omega}$.

We assume that vector-functions A_i and B satisfy the following conditions

$$A_i\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)\overrightarrow{k}_i\geq\nu(|\overrightarrow{u}|)\left|\overrightarrow{k}\right|^p-\mu(|\overrightarrow{u}|), \tag{2}$$

$$\sum_{i=1,\dots,n}\left|A_i\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)\right|\left(1+\left|\overrightarrow{k}\right|\right)+\left|B\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)\right|\leq\mu(|\overrightarrow{u}|)\left(1+\left|\overrightarrow{k}\right|\right)^p, \tag{3}$$

where ν and μ are positive monotone functions, and $1<p<n$.

We denote

$$a_{ij}(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})\nabla_i\nabla_j\overrightarrow{u}=\left(\nabla_i\nabla_j u^k\right)\frac{\partial A_i(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})}{\partial\nabla_j u^k} \tag{4}$$

and

$$\overrightarrow{b}\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)=\frac{\partial A_i\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)}{\partial u^k}\nabla_i u^k+\frac{\partial A_i\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)}{\partial x_i}+B\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right). \tag{5}$$

Then, we rewrite system (1) in the form

$$A(\overrightarrow{u})=a_{ij}(x,\ \overrightarrow{u})\nabla_i\nabla_j\overrightarrow{u}+\overrightarrow{b}(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})=0. \tag{6}$$

We assume

$$\overrightarrow{b}\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)\overrightarrow{u}\le-\tilde{\gamma}_1(x)|\overrightarrow{u}|^2+\tilde{\gamma}_2(x) \tag{7}$$

and

$$\tilde{\nu}\xi^2\le a_{ij}(x,\ \overrightarrow{u})\xi_i\xi_j\le\tilde{\mu}\xi^2, \tag{8}$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are positive continuous functions.

In order to apply the Leray-Schauder approach to establish the existence of the solutions to the system (1), we study a family of differential operators for all $\tau\in(0,\ 1)$

$$A^\tau(\overrightarrow{u})=\tau A(\overrightarrow{u})+(1-\tau)(\theta_1\Delta\overrightarrow{u}-\theta_2\overrightarrow{u}), \tag{9}$$

where we denote

$$A(\overrightarrow{u})=a_{ij}(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})\nabla_i\nabla_j\overrightarrow{u}+\overrightarrow{b}(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u}) \tag{10}$$

for arbitrary numbers θ_1 and θ_2.

We multiply the system $A^\tau(\overrightarrow{u})=0$ by the vector $2\overrightarrow{u}$ and obtain the equation

$$((1-\tau)\delta_{ij}\theta_1+\tau a_{ij})\left(\nabla_i\nabla_j\left(|\overrightarrow{u}|^2\right)-2(\nabla_i\overrightarrow{u},\ \nabla_j\overrightarrow{u})\right)-2\left((1-\tau)\theta_2|\overrightarrow{u}|^2-\tau\overrightarrow{b}\overrightarrow{u}\right)=0. \tag{11}$$

If the function $|\overrightarrow{u}|^2$ achieves its maximum at point $x_0\in\Omega$, then

$$\left[((1-\tau)\delta_{ij}\theta_1+\tau a_{ij})\left(\nabla_i\nabla_j\left(|\overrightarrow{u}|^2\right)-2(\nabla_i\overrightarrow{u},\ \nabla_j\overrightarrow{u})\right)\right]_{x=x_0}\le 0.$$

Therefore, we have

$$(1-\tau)\theta_2|\overrightarrow{u}|^2-\tau\overrightarrow{b}\overrightarrow{u}\le 0$$

and

$$(1-\tau)\theta_2|\overrightarrow{u}|^2+\tau\left(\tilde{\gamma}_1|\overrightarrow{u}|^2-\tilde{\gamma}_2\right)\le 0,$$

so

$$|\overrightarrow{u}|^2\le\frac{\tilde{\gamma}_2}{\min\{\theta_2,\ \tilde{\gamma}_1\}}.$$

From the last inequality, we have that the inequality

$$M_1=\max_{\Omega}|\overrightarrow{u}|\le\max\left\{\max_{\partial\Omega}|\overrightarrow{u}|,\ \left(\frac{\tilde{\gamma}_2}{\min\{\theta_2,\ \tilde{\gamma}_1\}}\right)^{\frac{1}{2}}\right\}$$

holds for all classical solutions for system (1).

3 The estimation of the $|\mathrm{vec}u|_{\alpha,\,\Omega}$

We assume that the structural coefficients of system (6) satisfy the following conditions

$$\tilde{\nu}(M_1)\,\xi^2 \le a_{ij}(x,\,\overrightarrow{u})\,\xi_i\xi_j \le \tilde{\mu}(M_1)\,\xi^2, \tag{12}$$

$$\left|\overrightarrow{b}\left(x,\,\overrightarrow{u},\,\overrightarrow{k}\right)\right| \le \left(\varepsilon(M_1)+\zeta\left(M_1,\,\overrightarrow{k}\right)\right)\left(1+\left|\overrightarrow{k}\right|^2\right), \tag{13}$$

where the value $\varepsilon(M_1)$ is a small enough constant and $\lim\limits_{\left|\overrightarrow{k}\right|\to\infty}\zeta\left(M_1,\,\overrightarrow{k}\right)=0$. The derivatives satisfy the conditions

$$\left|\frac{\partial a_{ij}(x,\,\overrightarrow{u})}{\partial x_m}\right| \le \tilde{\mu}(M_1), \quad \left|\frac{\partial a_{ij}(x,\,\overrightarrow{u})}{\partial u^k}\right| \le \tilde{\mu}(M_1). \tag{14}$$

Let $\overrightarrow{u}$ be a vector-function defined and measurable on the set Ω. Then, a function $\overrightarrow{u}$ is said to belong to the class $B^p_{N_1}$ if there exist N_1 functions $\varphi^1\left(u^1,\,...,\,u^N\right),\ldots,\varphi^{N_1}\left(u^1,\,...,\,u^N\right)$, which are continuously differentiable on the domain $|\overrightarrow{u}|\le M_1$ and satisfy the following conditions:

1) all functions are essentially bounded, namely $ess\max\limits_{\Omega}\left|\varphi^l(x)\right| \le M_1,\ \varphi^l\in W^p_1(\Omega)$ for all $l=1,\,...,\,N_1$;

2) for all concentric balls $B(r),\,B(2r)\subset\Omega$ there exists a number s_0 such that

$$osc\left\{\varphi^{s_0}(x),\quad B(2r)\right\} \ge \delta_1 \max_{k=1,\ldots,N} osc\left\{u^k,\quad B(2r)\right\}$$

and

$$mes\left\{\varphi^{s_0}(x) \le \max_{B(2r)}\varphi^{s_0}(x) - \delta_2 osc\left\{\varphi^{s_0}(x),\quad B(2r)\right\}\right\} \ge (1-\delta_3)\,c(n)\,r^n$$

for some positive numbers $\delta_1,\,\delta_2,\,\delta_3$ and such that $\delta_2,\,\delta_3\in(0,\,1)$, where $osc\{u(x),\quad\Omega\}$ means the oscillation of the function u in the domain Ω, which is defined by

$$osc\{u(x),\quad\Omega\} = ess\max_{\Omega}(u(x)) - ess\min_{\Omega}(u(x)),$$

where the Lebesgue measure is denoted by *mes*;

3) for each function $\varphi^l\in W^p_1(\Omega)$, $l=1,\,...,\,N_1$ and all balls $B(r)\subset\Omega$, the inequality

$$\int_{E(L,\,r(1-\sigma))}\left|\nabla\varphi^l\right|^p dx \le \varpi\left(\frac{1}{\sigma^p r^{p\left(1-\frac{n}{t}\right)}}\max_{E(L,\,r)}\left\{\left(\varphi^l-L\right)^p\right\}+1\right)\left(mes\left(E\left(L,\,r\right)\right)\right)^{\left(1-\frac{p}{t}\right)}$$

holds for all numbers L such that

$$\max_{E(L,\,r)}\left\{\varphi^l\right\} - L \le \delta$$

for $\sigma\in(0,\,1)$ and $1<p\le n$, where we denote

$$E(L,\,r) = \left\{x\in B(r):\ \varphi^l(x) > L\right\}.$$

Some of the properties of the functions of $B^p_{N_1}$-classes are given in the following theorem.

Theorem 1. *Let the vector-function $\overrightarrow{u}$ belongs to $\mathrm{B}_p^{N_1}$ and let φ^l , $l=1,...,N_1$ on $\partial\Omega$ satisfy the Holder conditions*

$$osc\left\{\varphi^l(x), \quad B(r)\cap\partial\Omega\right\}\le Cr^{\varepsilon} \quad l=1,...,N_1.$$

Then, the inequality

$$osc\left\{u^l(x), \quad B(r)\cap\Omega\right\}\le \hat{C}r^{\alpha} \quad l=1,...,N$$

holds for some constants $\hat{C}$ and α is determined by the class $\mathrm{B}_p^{N_1}$.

Now, we assume that the function $\overrightarrow{u}\in C_2(\Omega)$, $ess\max_{\Omega}|\overrightarrow{u}(x)|\le M_1$ is a solution of the system (6).

By multiplying system (6) by the vector-function $\xi\in W_1^p(\Omega)\cap C_0^{\infty}(\Omega)$, we obtain the scalar equality

$$\int_{\Omega}a_{ij}(x,\ \overrightarrow{u})\nabla_j\overrightarrow{u}\nabla_i\xi dx+\int_{\Omega}\frac{\partial a_{ij}(x,\ \overrightarrow{u})}{\partial u^k}\nabla_j\overrightarrow{u}\nabla_i u^k\xi dx$$
$$+\int_{\Omega}\frac{\partial a_{ij}(x,\ \overrightarrow{u})}{\partial x_i}\nabla_j\overrightarrow{u}\xi dx+\int_{\Omega}\overrightarrow{b}\xi dx=0,$$

here, we assumed the conditions $\left|\frac{\partial a_{ij}(x,\ \overrightarrow{u})}{\partial x_m}\right|\le\tilde{\mu}(M_1)$, $\left|\frac{\partial a_{ij}(x,\ \overrightarrow{u})}{\partial u^k}\right|\le\tilde{\mu}(M_1)$.

We select the function $\xi=\left(2\overrightarrow{u}+\check{c}N\overrightarrow{e}^l\right)\eta$ where the vector $\overrightarrow{e}^l$ is the unit vector in R^N the l-component of which is not zero, and $\eta\in W_1^p\left(\tilde{\Omega}\right)\cap C_0^{\infty}\left(\tilde{\Omega}\right)$. So, we have

$$\int_{\Omega}2a_{ij}\eta\nabla_j\overrightarrow{u}\nabla_i\overrightarrow{u}dx+\int_{\Omega}a_{ij}\eta\nabla_j\varphi_+^l\nabla_i\eta dx+\int_{\Omega}\frac{\partial a_{ij}}{\partial u^k}\nabla_i u^k\nabla_j\varphi_+^l\eta dx$$
$$+\int_{\Omega}\frac{\partial a_{ij}}{\partial x_i}\eta\nabla_j\varphi_+^l dx-\int_{\Omega}\left(2\overrightarrow{b}\overrightarrow{u}+\check{c}N\overrightarrow{b}\overrightarrow{e}^l\right)\eta dx=0,$$

where we denote $\overrightarrow{b}\overrightarrow{e}^l=b^l$, $\varphi_+^l(\overrightarrow{u})=\check{c}Nu^l+|\overrightarrow{u}|^2$ and $\varphi_-^l(\overrightarrow{u})=\check{c}N\left(1-u^l\right)+|\overrightarrow{u}|^2$. Let ϑ be a cutoff for the ball $B(r)$. We take $\eta(x)=\vartheta^2(x)\max\left\{\varphi_+^l(x)-L,\ 0\right\}$, and obtain

$$\int_{E(L,\ r)}2a_{ij}\vartheta^2\left(\varphi_+^l-L\right)\nabla_j\overrightarrow{u}\nabla_i\overrightarrow{u}dx+\int_{E(L,\ r)}a_{ij}\vartheta^2\nabla_j\varphi_+^l\nabla_i\varphi_+^l dx$$
$$+\int_{E(L,\ r)}2a_{ij}\left(\varphi_+^l-L\right)\vartheta\nabla_j\varphi_+^l\nabla_i\vartheta dx+\int_{E(L,\ r)}\frac{\partial a_{ij}}{\partial u^k}\vartheta^2\nabla_i u^k\nabla_j\varphi_+^l dx$$
$$+\int_{E(L,\ r)}\frac{\partial a_{ij}}{\partial x_i}\vartheta^2\nabla_j\varphi_+^l dx=\int_{\Omega}\left(2\overrightarrow{b}\overrightarrow{u}+\check{c}N\overrightarrow{b}\overrightarrow{e}^l\right)\vartheta^2 dx,$$

where we denote $E(L,\ r)=\left\{x\in B(r):\ {\varphi^l}_+(x)>L\right\}$. Next, we estimate

$$2\nu\int_{E(L,\ r)}\vartheta^2\left(\varphi_+^l-L\right)|\nabla\overrightarrow{u}|^2dx+\nu\int_{E(L,\ r)}\vartheta^2\left|\nabla\varphi_+^l\right|^2dx$$
$$\le 2\mu\int_{E(L,\ r)}\left(\varphi_+^l-L\right)\vartheta\left|\nabla\varphi_+^l\right||\nabla\vartheta|dx+\mu\int_{E(L,\ r)}\vartheta^2\left|\nabla u^k\right|\left|\nabla\varphi_+^l\right|dx$$
$$+\mu\int_{E(L,\ r)}\vartheta^2\left|\nabla\varphi_+^l\right|dx+\int_{\Omega}\left(2\overrightarrow{b}\overrightarrow{u}+\check{c}N\overrightarrow{b}\overrightarrow{e}^l\right)\vartheta^2dx.$$

By applying conditions (13), we have

$$\left|\left(2\overrightarrow{b}\overrightarrow{u}+\check{c}N\overrightarrow{b}\overrightarrow{e}^l\right)\vartheta^2\right|\le(2M_1+\check{c}N)\left(\varepsilon(M_1)+\zeta\left(M_1,\ \overrightarrow{k}\right)\right)\left(1+\left|\overrightarrow{k}\right|^2\right),$$

we take $(2M_1 + \check{c}N)\,\varepsilon(M_1) \le \nu$.

Therefore, there exists a positive constant $\hat{c}$ such that the inequality

$$\nu \int_{E(L,\,r)} \vartheta^2 \left|\nabla \varphi^l_+\right|^2 dx \le \hat{c} \int_{E(L,\,r)} \left(\varphi^l_+ - L\right)^2 |\nabla\vartheta|^2 dx + \hat{c} \int_{E(L,\,r)} \left(\vartheta^2 \left(\varphi^l_+ - L\right)^2 \left|\nabla \varphi^l_+\right|^2 + 1\right) dx$$

holds for all L, which satisfies the inequality $\max\limits_{B(r)} \left\{\varphi^l_+ - L\right\} \le \delta$ for small enough positive numbers δ.

Similarly, we estimate the value $\left|\nabla \varphi^l_-\right|$ for all φ^l, $\quad l = 1, \ldots, N$. Thus, we obtain that $\overrightarrow{u} \in \mathrm{B}^{2N}_2$.

Theorem 2. *Let $\overrightarrow{u} \in C_2(\Omega)$ be a solution of system (6) whose coefficients satisfy (13)–(14) for an internal subset $\tilde{\Omega} \subset \Omega$, $\max\limits_{\Omega} |\overrightarrow{u}(x)| \le M_1$ and let the boundary $\partial\Omega$ be smooth enough. Then, the Holder norm $|\overrightarrow{u}|_{\alpha,\,\Omega}$ can be estimated by the constant depending on $\nu(M_1)$, $\mu(M_1)$, $\varepsilon(M_1)$, $\zeta\left(M_1, \overrightarrow{k}\right)$, M_1, n, N.*

Proof. To obtain an estimation of the Holder norm $|\overrightarrow{u}|_{\alpha,\,\Omega}$ in Ω, we consider a ball $B(r)$ that intersects the boundary $\partial\Omega$. In this ball, we obtain

$$\begin{aligned}
&\int_{E(L,\,r)} 2a_{ij}\vartheta^2 \left(\omega^l - L\right) \nabla_j \overrightarrow{u} \nabla_i \overrightarrow{u}\, dx + \int_{E(L,\,r)} a_{ij}\vartheta^2 \nabla_j \omega^l \nabla_i \omega^l dx \\
&\quad + \int_{E(L,\,r)} 2a_{ij} \left(\omega^l - L\right) \vartheta \nabla_j \omega^l \nabla_i \vartheta dx + \int_{E(L,\,r)} \frac{\partial a_{ij}}{\partial u^k} \vartheta^2 \nabla_i u^k \nabla_j \omega^l dx \\
&\quad + \int_{E(L,\,r)} \frac{\partial a_{ij}}{\partial x_i} \vartheta^2 \nabla_j \omega^l dx = \int_{\Omega} \left(2\overrightarrow{b}\,\overrightarrow{u} + \check{c}N\overrightarrow{b}\,\overrightarrow{e}^{\,l}\right) \vartheta^2 dx,
\end{aligned}$$

for both $\omega^l = \varphi^l_+$ and $\omega^l = \varphi^l_-$. Therefore, the estimation

$$\nu \int_{E(L,\,r)} \vartheta^2 \left|\nabla \omega^l\right|^2 dx \le \hat{c} \int_{E(L,\,r)} \left(\omega^l - L\right)^2 |\nabla\vartheta|^2 dx + \hat{c} \int_{E(L,\,r)} \left(\vartheta^2 \left(\omega^l - L\right)^2 \left|\nabla \omega^l\right|^2 + 1\right) dx$$

holds for $E(L,\,r) = \left\{x \in B(r):\ \omega^l(x) > L\right\}$ if $\max\limits_{B(r)\cap\partial\Omega} \omega^l(x) \le L$, $\max\limits_{B(r)\cap\Omega} \omega^l(x) \le L + \delta$. Thus, we conclude that $\overrightarrow{u} \in \mathrm{B}^{2N}_2$. $\square$

4 The estimation of the generalized solution to (1) under form-boundary conditions on its coefficients

Let $\overrightarrow{u} \in W^p_1(\Omega) \cap L^q(\Omega)$, $\frac{np}{n-p} \le q$, $p \le n$ and satisfies the integral equality

$$\int_{\Omega} A_i(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})\nabla_i \overrightarrow{\varphi}\, dx - \int_{\Omega} B(x,\ \overrightarrow{u},\ \nabla\overrightarrow{u})\,\overrightarrow{\varphi}\, dx = 0 \tag{15}$$

for all $\overrightarrow{\varphi} \in W^p_{1,0}(\Omega)$.

We assume that coefficients satisfy the following conditions

$$A_i\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)\overrightarrow{k}_i \ge \tilde{\nu}(|\overrightarrow{u}|)\left|\overrightarrow{k}\right|^p - \left(1 + |\overrightarrow{u}|^{\chi_1}\right)\tilde{\gamma}_0(x), \tag{16}$$

$$\left|B\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)\right| \leq \left(1+|\overrightarrow{u}|^{\chi_2}\right)\tilde{\gamma}_1(x)+\left(1+|\overrightarrow{u}|^{\chi_3}\right)\tilde{\gamma}_2(x)\left|\overrightarrow{k}\right|^{p-\varepsilon} \tag{17}$$

for $x\in clos\,[\Omega]\subset R^n$ and arbitrary $\overrightarrow{u}$ and $\overrightarrow{k}$, $\chi_1\geq 2,\ \chi_2\geq 1,\ \chi_3\geq \frac{2\varepsilon}{p}-1$ and $\tilde{\gamma}_0,\ \tilde{\gamma}_1,\ \tilde{\gamma}_2\in PK(\beta)$; where $\tilde{v}$ is a positive bounded continuous function.

In (15), we put $\varphi^l(x)=\max\left\{u^l(x)-L,\quad 0\right\},\quad L\geq \max\{M_1,\ 1\}$ and obtain the inequality

$$\begin{aligned}\tilde{v}\int_{E(L,\,r)}|\nabla\overrightarrow{u}|^p\,dx \leq& \tilde{c}\left(\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_1}\tilde{\gamma}_0dx+\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_2}\tilde{\gamma}_1|\overrightarrow{u}-L|\,dx\right.\\ &\left.+\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_3}\tilde{\gamma}_2|\overrightarrow{u}-L|\,|\nabla\overrightarrow{u}|^{p-\varepsilon}\,dx\right)\end{aligned} \tag{18}$$

where we assume $\varepsilon<p$.

Next, we estimate each term of (18) individually. By Young inequality, we have

$$|\overrightarrow{u}|^{\chi_3}\tilde{\gamma}_2|\overrightarrow{u}-L|\,|\nabla\overrightarrow{u}|^{p-\varepsilon} \leq \varepsilon p^{-1}\delta^{-\frac{p}{\varepsilon}}|\overrightarrow{u}|^{\frac{p\chi_3}{\varepsilon}}|\overrightarrow{u}-L|^{\frac{p}{\varepsilon}}\tilde{\gamma}_2^{\frac{p}{\varepsilon}}+\tfrac{(p-\varepsilon)}{p}\delta^{p-\varepsilon}|\nabla\overrightarrow{u}|^p$$

and assuming $\delta^{\frac{p}{p-\varepsilon}}(p-\varepsilon)\,p^{-1}=2^{-1}\tilde{v}$, we have the inequality

$$\int_{E(L,\,r)}|\nabla\overrightarrow{u}|^p\,dx \leq \tilde{c}_1\left(\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_1}\tilde{\gamma}_0dx+\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_2+1}\tilde{\gamma}_1dx+\int_{E(L,\,r)}|\overrightarrow{u}|^{\frac{p(\chi_3+1)}{\varepsilon}}\tilde{\gamma}_2^{\frac{p}{\varepsilon}}dx\right).$$

Applying the form-boundary condition, we have

$$\int_{E(L,\,r)}\left(|\overrightarrow{u}|^{\frac{\chi_1}{2}}\tilde{\gamma}_0^{\frac{1}{2}}\right)^2dx \leq \left(\frac{\chi_1}{2}\right)^2\beta\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_1-2}|\nabla\overrightarrow{u}|^2\,dx+c(\beta)\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_1}\,dx,$$

$$\int_{E(L,\,r)}\left(|\overrightarrow{u}|^{\frac{\chi_2+1}{2}}\tilde{\gamma}_1^{\frac{1}{2}}\right)^2dx \leq \left(\frac{\chi_2+1}{2}\right)^2\beta\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_2-1}|\nabla\overrightarrow{u}|^2\,dx+c(\beta)\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_2+1}\,dx$$

and

$$\begin{aligned}\int_{E(L,\,r)}\left(|\overrightarrow{u}|^{\frac{p}{2\varepsilon}(\chi_3+1)}\tilde{\gamma}_2^{\frac{1}{2}}\right)^2dx \leq& \left(\tfrac{p}{2\varepsilon}(\chi_3+1)\right)^2\beta\int_{E(L,\,r)}|\overrightarrow{u}|^{\frac{p}{\varepsilon}(\chi_3+1)-2}|\nabla\overrightarrow{u}|^2\,dx\\ &+c(\beta)\int_{E(L,\,r)}|\overrightarrow{u}|^{\frac{p}{\varepsilon}(\chi_3+1)}\,dx.\end{aligned}$$

By the Holder inequality, we obtain

$$\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_1-2}|\nabla\overrightarrow{u}|^2\,dx \leq \varepsilon_1^{-\frac{p-2}{p}}\frac{p-2}{p}\int_{E(L,\,r)}|\overrightarrow{u}|^{(\chi_1-2)\frac{p}{p-2}}\,dx+\frac{p}{2}\varepsilon_1^{\frac{p}{2}}\int_{E(L,\,r)}|\nabla\overrightarrow{u}|^p\,dx,$$

$$\int_{E(L,\,r)}|\overrightarrow{u}|^{\chi_2-1}|\nabla\overrightarrow{u}|^2\,dx \leq \varepsilon_2^{-\frac{p-2}{p}}\frac{p-2}{p}\int_{E(L,\,r)}|\overrightarrow{u}|^{(\chi_2-1)\frac{p}{p-2}}\,dx+\frac{p}{2}\varepsilon_2^{\frac{p}{2}}\int_{E(L,\,r)}|\nabla\overrightarrow{u}|^p\,dx,$$

and

$$\int_{E(L,\,r)}|\overrightarrow{u}|^{\frac{p}{\varepsilon}(\chi_3+1)-2}|\nabla\overrightarrow{u}|^2\,dx \leq \varepsilon_3^{-\frac{p-2}{p}}\frac{p-2}{p}\int_{E(L,\,r)}|\overrightarrow{u}|^{\left(\frac{p}{\varepsilon}(\chi_3+1)-2\right)\frac{p}{p-2}}\,dx+\frac{p}{2}\varepsilon_3^{\frac{p}{2}}\int_{E(L,\,r)}|\nabla\overrightarrow{u}|^p\,dx.$$

Next, we apply the inequality of the type

$$\int_{E(L,\,r)} |\overrightarrow{u}|^{(\chi_1-2)\frac{p}{p-2}}\,dx \le \int_{E(L,\,r)} |\overrightarrow{u}-L|^{(\chi_1-2)\frac{p}{p-2}}\,dx + L^p mes\,(E\,(L,\,r))$$

and applying the Holder estimation

$$\|\overrightarrow{u}-L\|^{\chi}_{L^{\chi}} \le \left(\int_{E(L,\,r)} |\overrightarrow{u}-L|^{a}\,dx\right)^{\frac{\chi}{a}} (mes\,(E\,(L,\,r)))^{\frac{a-\chi}{a}},$$

we obtain

$$\int_{E(L,\,r)} |\nabla\overrightarrow{u}|^{p}\,dx \le \tilde{c}_2\left(\left(\int_{E(L,\,r)} |\overrightarrow{u}-L|^{a}\,dx\right)^{\frac{\chi}{a}} + \sum_{i=1,\ldots,6} L^{\varsigma_i}\,(mes\,(E\,(L,\,r)))\right)$$

where $\zeta_i,\quad i=1\,,\ldots,\,6$ depend on $\chi_i,\quad i=1,..,3$.

We are going to use the following statement.

Statement 1. Let $f\in W_1^p\,(\Omega)\cap L^q\,(\Omega)$, $ess\max\{f\,(x)\}<\infty$. Also, the estimation

$$\int_{E(L,\,r)} |\nabla f|^{p}\,dx \le c\left(\left(\int_{E(L,\,r)} (f-L)^{a}\,dx\right)^{\frac{p}{a}} + \sum_{i=1,\ldots,n_1} L^{\varsigma_i}\,(mes\,(E\,(L,\,r)))^{1-\frac{p}{m}+\varepsilon_i}\right)$$

holds for some positive constants $c,\ a,\ \varsigma_i,\ \varepsilon_i,\ n_1$ such that $\frac{n-p}{n}<\frac{p}{a}$, $\varepsilon_i>0$ for all $i=1,\ \ldots,\ n_1$. Then, the value $\underset{\Omega}{ess\max}\{f\,(x)\}$ can be estimated above by constant depending only on $c,\ a,\ \varsigma_i,\ \varepsilon_i,\ n_1,p,\ n,\ q$, $mes\,(\Omega)$ and $\|f\|_{L^1(E(L,\,r))}$.

So, we obtained the statement about the boundedness of the generalized solutions.

Theorem 3. *Let $\overrightarrow{u}\in W_1^p\,(\Omega)\cap L^q\,(\Omega)$, $\tilde{M}_1=\underset{\partial\Omega}{ess\max}|\overrightarrow{u}|<\infty$, $\frac{np}{n-p}\le q$, $p\le n$ be a generalized solution of system (1) in the sense of (15). Let functions A_i, B satisfy conditions (16), (17) where $\tilde{\gamma}_0,\ \tilde{\gamma}_1,\ \tilde{\gamma}_2\in PK\,(\beta)$. Then, the value $M_1=\underset{\Omega}{ess\max}|\overrightarrow{u}|$ can be estimated by constant depending only on $\tilde{M}_1,\ \tilde{\nu},\ \beta$, $mes\,(\Omega)$, $\|\overrightarrow{u}\|_{L^q}$, ε; the function $\overrightarrow{u}$ satisfies the Holder condition of the order $\alpha>0$ depending on $\tilde{M}_1,\ \tilde{\nu},\ \beta,\ mes\,(\Omega)$, $\|\overrightarrow{u}\|_{L^q}$, ε. For any $\tilde{\Omega}\subset\Omega$, the value $|\overrightarrow{u}|_{\alpha,\,\Omega}$ is estimated by $\tilde{M}_1,\ \tilde{\nu},\ \beta,\ mes\,(\Omega)$, $\|\overrightarrow{u}\|_{L^q}$, ε and dist $\left(\tilde{\Omega},\,\Omega\right)$.*

5 The existence of the classical solutions

We consider system (6) under the conditions (7), (12) – (14). The estimations obtained above allow us to investigate the solvability of the first boundary problem for system (6). We assume that coefficients of system (11) satisfy the conditions $a_{ij}\,(x,\ \overrightarrow{u})\,\xi_i\xi_j\ge\tilde{\nu}\xi^2$ and $\overrightarrow{b}\left(x,\ \overrightarrow{u},\ \overrightarrow{k}\right)\overrightarrow{u}\le-\tilde{\gamma}_1\,(x)\,|\overrightarrow{u}|^2+\tilde{\gamma}_2\,(x)$ for all vectors $\overrightarrow{u}$ and $\overrightarrow{k}$ and all $x\in clos\,(\Omega)$. Then, the Leray-Schauder method yields the following theorem.

Theorem 4. *Let the coefficients of system (6) satisfy conditions (12) –(14) and the system*

$$\tau A\left(\overrightarrow{u}\right)+\left(1-\tau\right)\left(\theta_1\Delta\overrightarrow{u}-\theta_2\overrightarrow{u}\right)=0 \tag{19}$$

satisfy conditions (7), (8) and let $(2M_1+\check{c}N)\,\varepsilon(M_1)\leq\nu$ as in Theorem 1. If the function $\overrightarrow{b}$ belongs to the Holder class with $\hat{\alpha}$, in the domain $\left\{x\in clos\left(\Omega\right),\quad |\overrightarrow{u}|\leq M_1,\ \left|\overrightarrow{k}\right|\leq M_2\right\}$, and the boundary is smooth enough (at least$C_{2,\hat{\alpha}}$), then the boundary problem $\overrightarrow{u}|_{\partial\Omega}=0$ system (6) has a solution $\overrightarrow{u}$ in the functional class $C_{2,\hat{\alpha}}\left(\Omega\right)$.

Proof. We consider the parameterized system

$$\tau A\left(\overrightarrow{u}\right)+\left(1-\tau\right)\left(\theta_1\Delta\overrightarrow{u}-\theta_2\overrightarrow{u}\right)=0,$$

when we put $\theta_1=\theta_2=1$. If the parameter $\tau=0$, then system (19) breaks into separate equations the existence and uniqueness of the solution of which, we will study below. Thus, we assume that system (19) has a solution for $\tau=0$, then the statement of Theorem 4 immediately follows from Leray-Schauder theorem. □

In order to avoid misunderstanding, we remind our readers formulation of the Leray-Schauder theorem.

Theorem 5. ***(Lerey-Schauder)****. Let X be a complete Banach space, and let $clos\left(E\right)$ be the closure of arbitrarily connected open set $E\subset X$. Also, let $X\otimes[0,\ 1]$ be topological product of X and $[0,\ 1]$. Then, the equation $u=\Theta\left(u,\ \tau\right)$ has at least one solution in E for all $\tau\in[0,\ 1]$, if:*

1) mapping Θ is defined and continuous over $clos\left(E\right)\otimes[0,\ 1]$,

2) mapping Θ is uniformly continuous at $\tau\in[0,\ 1]$ on $clos\left(E\right)\otimes[0,\ 1]$,

3) the boundary ∂E does not contain any solution to $u=\Theta\left(u,\ \tau\right)$,

4) for $\tau=0$ the equation $u=\Theta\left(u,\ \tau\right)$ has a finite number of solutions with a summation index larger than zero.

6 The existence of the solution to the quasilinear equation

We consider the equation

$$A\left(u\right)=a_{ij}\left(x,\ u,\ \nabla u\right)\nabla_i\nabla_j u+b\left(x,\ u,\ \nabla u\right)=0, \tag{20}$$

where the a_{ij} is uniformly elliptic matrix and b satisfies the condition

$$b\left(x,\ u,\ 0\right)u\leq-\tilde{\gamma}_1\left(x\right)u^2+\tilde{\gamma}_2\left(x\right) \tag{21}$$

for $x\in\Omega$, where functions $\tilde{\gamma}_1,\ \tilde{\gamma}_2$ are positively determined and continuous.

For Eq. (20), the parametric family of differential operators is given by

$$A^{\tau}\left(u\right)=\tau A\left(u\right)+\left(1-\tau\right)\left(\nabla_i\left(\tilde{\theta}_1\left(1+|\nabla u|^2\right)^{\frac{p-2}{2}}\nabla_i u\right)-\tilde{\theta}_2 u\right),$$

for all $\tau \in [0,\ 1]$, where θ_1 and θ_2 are arbitrary numbers. The corresponding parametric family of equations is defined by

$$\begin{aligned} A^{\tau}(u) = \tau a_{ij}\nabla_i\nabla_j u + \tilde{\theta}_1(1-\tau)\left(1+|\nabla u|^2\right)^{\frac{p-2}{2}}\Delta u \\ +\tilde{\theta}_1(1-\tau)(p-2)\left(1+|\nabla u|^2\right)^{\frac{p}{2}-2}\nabla_i u\nabla_m u\nabla_i\nabla_m u + \tau b - (1-\tau)\tilde{\theta}_2 u = 0, \end{aligned} \tag{22}$$

and assume that its solution u achieves its maximum at point $x_0 \in \Omega$. Then, we estimate

$$\tau a_{ij}\nabla_i\nabla_j u \le 0,$$

$$\tilde{\theta}_1(1-\tau)\left(1+|\nabla u|^2\right)^{\frac{p-2}{2}}\Delta u \le 0$$

and

$$\tilde{\theta}_1(1-\tau)(p-2)\left(1+|\nabla u|^2\right)^{\frac{p}{2}-2}\nabla_i u\nabla_m u\nabla_i\nabla_m u = 0.$$

So

$$\tau b(x,\ u,\ 0) - (1-\tau)\tilde{\theta}_2 u \ge 0,$$

therefore, we conclude

$$-\tilde{\gamma}_1 u^2 - (1-\tau)\tilde{\theta}_2 u + \tilde{\gamma}_2\tau \ge 0$$

and $u(x_0,\ \tau) \le \left(\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1}\right)^{\frac{1}{2}}$.

Thus, for all classical solutions to the problem $A^{\tau}(u) = 0,\ u|_{\partial\Omega} = \tau\psi, \quad \tau \in [0,\ 1]$ the estimation

$$\max_{\Omega}|u(x,\ \tau)| \le \max\left\{\max_{\partial\Omega}|\psi|,\quad \left(\frac{\tilde{\gamma}_2}{\tilde{\gamma}_1}\right)^{\frac{1}{2}}\right\}$$

holds for all $\tau \in [0,\ 1]$.

Now, we formulate the theorem of the existence of the classical solutions to the boundary problem $u|_{\partial\Omega} = \psi|_{\partial\Omega}$ for Eq. (21).

Theorem 6. *Let a_{ij} be the element of a uniformly elliptic matrix, $\tilde{\nu}\xi^2 \le a_{ij}(x,\ \overrightarrow{u})\xi_i\xi_j \le \tilde{\mu}\xi^2$, b be a measurable and correctly defined function on set $\left\{x \in clos(\Omega),\quad |\overrightarrow{u}| \le M_1,\ \left|\overrightarrow{k}\right| \le M_2\right\}$ satisfies the estimation $b(x,\ u,\ 0)u \le -\tilde{\gamma}_1(x)u^2 + \tilde{\gamma}_2(x)$. Also, let conditions*

$$\left|\frac{\partial b}{\partial k_m}\right| \le \tilde{\mu}\left(1+|k|^2\right)^{\frac{1}{2}},$$

$$\left|\frac{\partial a_{ij}}{\partial u}\right|\left(1+|k|^2\right) + |b| + \left|\frac{\partial b}{\partial u}\right| \le \tilde{\mu}_1\left(1+|k|^2\right)$$

and

$$\left|\frac{\partial a_{ij}}{\partial x}\right|\left(1+|k|^2\right) + \left|\frac{\partial b}{\partial x}\right| \le \tilde{\mu}_1\left(1+|k|^2\right)^{\frac{p}{2}},$$

for all $x \in clos(\Omega)$, $|\overrightarrow{u}| \le M_1$, $\left|\overrightarrow{k}\right| \le M_2$ hold, where $\tilde{\mu}_1 = \varepsilon + \zeta(|k|)$ with continuous ζ that $\lim\limits_{|k|\to\infty}\zeta(|k|) = 0$, and the boundary be smooth enough. Then, the boundary problem $u|_{\partial\Omega} = \psi|_{\partial\Omega}$ for equation (20) has a solution in $C_{2,\alpha}(clos(\Omega))$.

Proof. The existence of a solution to the problem $u|_{\partial\Omega} = \psi|_{\partial\Omega}$ for Eq. (20) will follow from the Leray-Schauder theorem if we show that the summated index of solutions to the problem $A^{\tau}(u) = 0$, $u|_{\partial\Omega} = \tau\psi$, $\tau = 0$ does not equal zero. If $\tau = 0$, then the problem $A^{0}(u) = 0$, $u|_{\partial\Omega} = 0$ has one solution $\overrightarrow{u}(x,\ 0) \equiv 0$ since

$$\max_{\Omega} |u(x,\ 0)| \leq 0$$

and equality $0 = \Theta(w,\ 0) = v$ holds for all $w \in C_{1,\breve{\alpha}}(clos(\Omega))$ since

$$\tilde{\theta}_1 \left(1 + |\nabla w|^2\right)^{\frac{p-2}{2}} \Delta v + \tilde{\theta}_2 v + \tilde{\theta}_1 (1-\tau)(p-2) \left(1 + |\nabla w|^2\right)^{\frac{p}{2}-2} \nabla_i w \nabla_m w \nabla_i \nabla_m v = 0,$$

$$v|_{\partial\Omega} = 0.$$

Thus, its solution $v = 0$ for all fixed $w \in C_{1,\breve{\alpha}}(clos(\Omega))$ since $\tilde{\theta}_1,\ \tilde{\theta}_2 > 0$. Therefore, the mapping $w \mapsto w - \Theta(w,\ 0)$ is the identity mapping with an index equal to one, and the boundary problem $A^{0}(u) = 0$, $u|_{\partial\Omega} = 0$ has a unique solution identical to the zero function. □

7 Conclusions

We obtain the conditions under which the boundary problem $u|_{\partial\Omega} = \psi|_{\partial\Omega}$ for partial differential equation $A(u) = 0$ with the uniformly elliptic matrix has a smooth solution in $C_{2,\alpha}(clos(\Omega))$. We use the fixed-point method in the Leray-Schauder form, and to justify the limiting process we employ certain a priori estimations. Our results can be further generalized to include wide classes of elliptic partial differential equations.

References

[1] M. Agueh, *Gagliardo-Nirenberg inequalities involving the gradient L^2-norm*, C. R. Acad. Sci. Paris, Ser. I **346** (2008) 757–762.

[2] H. Amann, *Invariant sets and existence theorems for semilinear parabolic and elliptic systems*, J. Math. Anal. Appl. **65** (1978) 432–467.

[3] C. Bao, B. Cui, X. Lou, W. Wu, and B. Zhuang, *Fixed-time stabilization of boundary controlled linear parabolic distributed parameter systems with space-dependent reactivity*, IET Control Theory Appl. **15** (2020) 652–667.

[4] C. Chen, R.M. Strain, H. Yau, T. Tsai, *Lower bounds on the blow-up rate of the axisymmetric Navier-Stokes equations II*, Commun. Partial Differ. Equ. **34** (2009) 203–232.

[5] E. DeGiorgi, *Sulla differenziabilita e lanaliticita delle estremali degli integrali multipli regolari*, Mem. Accad Sc. Torino, C. Sc. Fis. Mat. Natur. **3** (1957) 25-43.

[6] E. DeGiorgi, *Un esempio di estremali discontinue per un problema variazionale di tipo ellitico*, Bull. UMI **4** (1968) 135-137.

[7] H. Dong, S. Kim, S. Lee, *Estimates for fundamental solutions of parabolic equations in non-divergence form*, J. Differ. Equ. **340** (2022) 557–591.

[8] J.A. Goldstein, Q.S. Zhang, *Linear parabolic equations with strongly singular potentials*, Trans. Amer. Math. Soc. **355** (2002) 197–211.

[9] D. Hongjie, L. Escauriaza, S. Kim, *On C1/2,1, C1,2, and C0,0 estimates for linear parabolic operators*, J. Evol. Equ. **21** (2021) 4641–4702.

[10] S. Kim, S. Lee, *Estimates for Green's functions of elliptic equations in non-divergence form with continuous coefficients*, Ann. Appl. Math. **37** (2021) 111–130.

[11] D. Kinzebulatov, Y. A. Semenov, *Heat kernel bounds for parabolic equations with singular (form-bounded) vector fields*, Preprint, 2021, arXiv:2103.11482.

[12] M. Kassmann, M. Weidner, *The parabolic Harnack inequality for nonlocal equations*, 2023, arXiv:2303.05975.

[13] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs*, Amer. Math. Soc. **23** (1968).

[14] M. Li, W. Mao, *Finite-time bounded control for coupled parabolic PDE-ODE systems subject to boundary disturbances*, Math. Probl. Eng. **2020** (2020) Article ID 8882382.

[15] Z. Qian, G. Xi, *Parabolic equations with singular divergence-free drift vector fields*, J. London Math. Soc. **100** (2019) 17–40.

[16] Y. Tao, M. Winkler, *Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with the remodeling of non-diffusible attractant*, J. Differ. Equ. **257** (2014) 784–815.

[17] Y. Tao, *Global existence of classical solutions to a predator-prey model with nonlinear prey-taxis*, Nonl. Aanl. RWA **11** (2010) 2056–2064.

[18] Y. Tao, M. Winkler, *Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with the remodeling of non-diffusible attractant*, J. Differ. Equ. **257** (2014) 784–815.

[19] A. Veretennikov, *On weak existence of solutions of degenerate McKean-Vlasov equations*, Preprint, 2023, ArXiv 2301.01532.

[20] F.B. Weissler, *Single-point blow-up for a semilinear initial value problem*, J. Diff. Equ. **5** (1984) 204–224.

[21] S. Wang, Y. Teng, P. Perdikaris, *Understanding and mitigating gradient flow pathologies in physics-informed neural networks*, SIAM J. Sci. Comput. **43** (2021) A3055–A3081.

[22] W.L.J. Wang, W. Guo, *A backstepping approach to adaptive error feedback regulator design for one-dimensional linear parabolic PIDEs*, J. Math. Anal. Appl. **503** (2021) Article ID 125310.

[23] M.I. Yaremenko, *The existence of a solution of evolution and elliptic equations with singular coefficients,* Asian Res. J. Math. **15** (2017) 172–204.

[24] M.I. Yaremenko, *Quasi-linear evolution, and elliptic equations*, J. Progressive Res. Math. **11** (2017) 1645–1669.

[25] M.I. Yaremenko, *The sequence of semigroups of nonlinear operators and their applications to study the Cauchy problem for parabolic equations*, Sci. J. Ternopil Nat. Tech. Univ. **4** (2016) 149–160.

[26] D. Zhang, L. Guo, G. E. Karniadakis, *Learning in modal space: Solving time-dependent stochastic PDEs using physics-informed neural networks*, SIAM J. Sci. Comput. **42** (2020) A639–A665.

[27] Z.-Q. Chen, T. Kumagai, J. Wang, *Stability of heat kernel estimates for symmetric non-local Dirichlet forms*, Mem. Amer. Math. Soc. **271** 2021 1–100.

[28] S. Zhang, Z.-Q. Chen, *Stochastic maximum principle for fully coupled forward-backward stochastic differential equations driven by sub-diffusion*, Preprint, 2023.