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On the preconditioning of the Schur complement matrix of a class of two-by-two block matrices

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Abstract. We consider a class of two-by-two block complex system of linear equations obtained from finite element discretization of the distributed optimal control with time-periodic parabolic equations. Using the Schur complement technique we transform the obtained system to two subsystems. We propose a preconditioner to the subsystem with the Schur complement matrix. Spectral properties of the preconditioned matrix are analyzed. Some numerical results are presented to show the effectiveness of the preconditioner.

Keywords: Preconditioner, GMRES, finite element, PDE-constrained, optimization, Schur complement.
AMS Subject Classification 2010: 49M25, 49K20, 65F10, 65F50.

1 Introduction

We consider the problem of computing the state function $y(x,t)$ and the control function $u(x,t)$ which minimize the functional [12, 14]

$$\mathscr{J}(y,u) = \frac{1}{2}\int_0^T \int_\Omega |y(x,t) - y_d(x,t)|^2 dxdt + \frac{\nu}{2}\int_0^T \int_\Omega |u(x,t)|^2 dxdt,$$

subject to the time-dependent parabolic problem

$$\begin{aligned}
\frac{\partial}{\partial t}y(x,t) - \Delta y(x,t) &= u(x,t) \;\text{ in }\; Q_T,\\
y(x,t) &= 0 \qquad\;\; \text{on }\; \Sigma_T,\\
y(x,0) &= y(x,T) \;\text{ on }\; \partial\Omega,\\
u(x,0) &= u(x,T) \;\text{ in }\; \Omega,
\end{aligned}$$

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with $T > 0$, where Ω is an open and bounded domain in $\mathbb{R}^d$ $(d = 1,2,3)$ and the boundary of Ω, $\partial\Omega$, is Lipschitz-continuous. Let $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. Here, $y_d(x,t)$ is a desired state and ν is the regularization parameter. It can be assumed that the function $y_d(x,t)$ is time-harmonic [11, 13], which means that the function $y_d(x,t)$ can be written as

$$y_d(x,t) = y_d(x)e^{i\omega t},$$

with $\omega = \frac{2\pi m}{T}$ for some $m \in \mathbb{Z}$ and $i = \sqrt{-1}$. With this assumption, both the solution and the control functions are also time-harmonic, i.e.,

$$y(x,t) = y(x)e^{i\omega t} \text{ and } u(x,t) = u(x)e^{i\omega t}.$$

Substituting the functions $y(x,t)$, $y_d(x,t)$ and $u(x,t)$ in the problem results in the following time-independent problem

$$\begin{aligned} \min_{y,u} \quad & \frac{1}{2}\int_\Omega |y(x) - y_d(x)|^2 dx + \frac{\nu}{2}\int_\Omega |u(x)|^2 dx, \\ s.t: \quad & i\omega y(x) - \Delta y(x) = u(x) \ \text{ in } \ \Omega, \\ & \qquad\qquad u(x) = 0, \quad \text{on } \ \partial\Omega. \end{aligned}$$

Using an approximate finite element V_h for computing both y and u and the idea of discretize-then-optimization approach, the above problem can be written

$$\begin{aligned} \min_{y,u} \quad & \frac{1}{2}(\bar{y} - \bar{y}_d)^* M (\bar{y} - \bar{y}_d) + \frac{\nu}{2}\bar{u}^* M \bar{u}, \\ s.t: \quad & i\omega M\bar{y} + K\bar{y} = M\bar{u}, \end{aligned}$$

where the real matrix M is the mass matrix and K is the discretized negative Laplacian. It is noted that the matrices M and K are symmetric positive definite (SPD). Also, $\bar{y}$, $\bar{y}_d$, and $\bar{u}$ denote the coefficient vectors of the function y, y_d and u in V_h. If we define the Lagrangian functional for the above problem as

$$\mathscr{L}(\bar{y},\bar{u},\bar{p}) = \frac{1}{2}(\bar{y} - \bar{y}_d)^* M (\bar{y} - \bar{y}_d) + \frac{\nu}{2}\bar{u}^* M \bar{u} + \bar{p}^*(i\omega M\bar{y} + K\bar{y} - M\bar{u}),$$

where $\bar{p}$ is the Lagrange multiplier associated with the constraint, then the first order necessary conditions which are also sufficient for the existence of a solution is $\nabla\mathscr{L}(\bar{y},\bar{u},\bar{p}) = 0$, can be written as

$$\begin{pmatrix} M & 0 & K - i\omega M \\ 0 & \nu M & -M \\ K + i\omega M & -M & 0 \end{pmatrix} \begin{pmatrix} \bar{y} \\ \bar{u} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} M\bar{y}_d \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

If we compute $\bar{u}$ from the the second equation in (1), we get $\bar{u} = \frac{1}{\nu}\bar{p}$. Substituting $\bar{u}$ in the third equation and an scaling, gives the following system

$$\begin{pmatrix} M & \sqrt{\nu}(K - i\omega M) \\ \sqrt{\nu}(K + i\omega M) & -M \end{pmatrix} \begin{pmatrix} \bar{y} \\ \bar{q} \end{pmatrix} = \begin{pmatrix} \hat{y}_d \\ 0 \end{pmatrix}, \tag{2}$$

where $\bar{q} = \frac{1}{\sqrt{\nu}}\bar{p}$ and $\hat{y}_d = M\bar{y}_d$.

To solve the linear system (2), we can use the GMRES method [17], HSS method [7–9], MHSS method [4], PMHSS method [5,6] and so on. Krendl [14] proposed the real block diagonal preconditioner and the alternative indefinite preconditioner. Zheng et al. in [19] proposed the block alternating splitting (BAS) iteration method. Recently, Zheng in [18] writes the following real form of the system (2)

$$\begin{pmatrix} -D_1 & B_1 \\ B_1 & D_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}, \tag{3}$$

where

$$D_1 = \begin{pmatrix} \sqrt{\nu}K & 0 \\ 0 & -\sqrt{\nu}K \end{pmatrix}, \quad B_1 = \begin{pmatrix} M & \omega\sqrt{\nu}M \\ -\omega\sqrt{\nu}M & M \end{pmatrix},$$

where $\tilde{p} = (0; \Im(\hat{y}_d))$, $\tilde{q} = (\Re(\hat{y}_d); 0)$, $x = (\Re(\bar{y}), \Im(\bar{q}))$ and $y = (\Re(\bar{q}); \Im(\bar{y}))$. Then, the author splits the above system into the following subsystems

$$(D_1 + B_1 D_1^{-1} B_1)y = B_1 D_1^{-1}\tilde{p} + \tilde{q}, \tag{4}$$

$$D_1 y = B_1 y - \tilde{p}. \tag{5}$$

For solving the system (5), we need to solve two subsystems with the coefficient matrix K which can be solved exactly using the Cholesky factorization or inexactly using the conjugate gradient method. However, solving the system (4) is a more changing problem. In [18], Zheng proposed solving the system (4) by a Krylov subspace method in conjunction with the preconditioner $P_K = D_1$. The author proved that the eigenvalues of the preconditioned matrix

$$P_K^{-1}(D_1 + B_1 D_1^{-1} B_1) = I + \left(D_1^{-1} B_1\right)^2,$$

are of the form

$$\lambda = 1 + \frac{1 + \omega^2 \nu}{\nu \eta^2},$$

where η is an eigenvalue of the matrix $M^{-1}K$. The relation shows that when ν is sufficiently small then the preconditioner would not be efficient, since the eigenvalues of the preconditioner are far from 1 [16]. In this paper, we consider another real form of the system (2) and propose a method for solving it that is more efficient.

The following notations are used throughout the paper. The real and imaginary parts of a complex number z are denoted by $\Re(z)$ and $\Im(z)$, respectively. The conjugate transpose of a the matrix A shown by A^*. i stands for imaginary unit ($i = \sqrt{-1}$). For a square matrix W the block diagonal matrix

$$A = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix},$$

is denoted by $A = \text{blkdiag}(W, W)$. For the vector $z = (x^T, y^T)^T$, we use the MATLAB notation $z = (x; y)$. The spectrum of an square matrix A is denoted $\sigma(A)$.

The rest of the paper is organized as follows. In Section 2 we present the proposed preconditioner. Section 3 is devoted to implementation issues of the proposed preconditioner. In Section 4 we present some numerical results. Concluding remarks are presented in Section 5.

2 The proposed precondtioner

We write down the system (2) in the real form

$$\begin{pmatrix} D & B \\ -B^T & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}, \tag{6}$$

where

$$D = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}, \quad B = \begin{pmatrix} \sqrt{\nu}K & \omega\sqrt{\nu}M \\ -\omega\sqrt{\nu}M & \sqrt{\nu}K \end{pmatrix},$$

with $p = (\Re(\hat{y}_d); \Im(\hat{y}_d))$, $x = (\Re(\bar{y}); \Im(\bar{y}))$ and $y = (\Re(\bar{q}); \Im(\bar{q}))$. It is easy to see that solving the system (6) is equivalent to solving

$$Sy = B^T D^{-1} p, \tag{7}$$
$$Dx = p - By, \tag{8}$$

where $S = D + B^T D^{-1} B$ is the Schur complement matrix of the system (6). The system (8) can be split into two subsystems with the coefficient matrix M, and since M is SPD, the corresponding systems can be solved exactly using the Cholesky factorization or inexactly using the CG method. However, in general, the matrix S is often ill-conditioned and there is no efficient direct method to solve the corresponding system. So to solve the system (7) by an iterative method, we need an efficient preconditioner.

Using the idea of [1,2] we present the preconditioner

$$P_S = (D + B^T) D^{-1} (D + B),$$

to the system (7). The next theorem analyses the eigenvalue distribution of the preconditioned matrix $P_S^{-1} S$.

Theorem 1. *The eigenvalues of matrix $P_S^{-1} S$ are of the form*

$$\lambda = \frac{1 + \nu(\omega^2 + \mu^2)}{\nu\omega^2 + (1 + \sqrt{\nu}\mu)^2}, \tag{9}$$

Proof. Since the matrix M is SPD, there is an SPD matrix $M^{\frac{1}{2}}$ such that $M = M^{\frac{1}{2}} M^{\frac{1}{2}}$. Now, by setting $D^{\frac{1}{2}} = \mathrm{blkdiag}(M^{\frac{1}{2}}, M^{\frac{1}{2}})$, we get

$$S = D^{\frac{1}{2}} \left(I + \tilde{B}^T \tilde{B}\right) D^{\frac{1}{2}}, \quad P = D^{\frac{1}{2}} (I + \tilde{B}^T)(I + \tilde{B}) D^{\frac{1}{2}},$$

where $\tilde{B} = D^{-\frac{1}{2}} B D^{-\frac{1}{2}}$. Hence,

$$P_S^{-1} S = D^{-\frac{1}{2}} \left(\left(I + \tilde{B}^T\right)\left(I + \tilde{B}\right)\right)^{-1} \left(I + \tilde{B}^T \tilde{B}\right) D^{\frac{1}{2}},$$

which is similar to the matrix

$$W = \left((I + \tilde{B}^T)(I + \tilde{B})\right)^{-1} \left(I + \tilde{B}^T \tilde{B}\right).$$

Let (λ, x) be an eigenpair of W, i.e., $Wx = \lambda x$ which is equivalent to

$$\left(I+\tilde{B}^T\tilde{B}\right)^{-1}\left(I+\tilde{B}^T\tilde{B}+\tilde{B}+\tilde{B}^T\right) = \frac{1}{\lambda}x$$

which is itself equivalent to

$$(I+\tilde{B}^T\tilde{B})^{-1}(\tilde{B}+\tilde{B}^T)x = (\frac{1}{\lambda}-1)x. \tag{10}$$

On the other hand, we have

$$\tilde{B} = D^{-\frac{1}{2}}BD^{-\frac{1}{2}} = \begin{pmatrix} \sqrt{\nu}\hat{K} & \omega\sqrt{\nu}I \\ -\omega\sqrt{\nu}I & \sqrt{\nu}\hat{K} \end{pmatrix},$$

where $\hat{K} = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$. It is straightforward to see that

$$\tilde{B}^T\tilde{B} = \begin{pmatrix} \nu(\omega^2 I+\hat{K}^2) & 0 \\ 0 & \nu(\omega^2 I+\hat{K}^2) \end{pmatrix},$$

$$\tilde{B}^T+\tilde{B} = \begin{pmatrix} 2\sqrt{\nu}\hat{K} & 0 \\ 0 & 2\sqrt{\nu}\hat{K} \end{pmatrix}.$$

So, we have

$$(I+\tilde{B}^T\tilde{B})^{-1}(\tilde{B}+\tilde{B}^T) = \begin{pmatrix} 2\sqrt{\nu}\left(I+\nu\left(\omega^2 I+\hat{K}^2\right)\right)^{-1}\hat{K} & 0 \\ 0 & 2\sqrt{\nu}\left(I+\nu\left(\omega^2 I+\hat{K}^2\right)\right)^{-1}\hat{K} \end{pmatrix}. \tag{11}$$

From Eqs. (10) and (11), we deduce that $\frac{1}{\lambda}-1$ is an eigenvalue of

$$T = 2\sqrt{\nu}\left(I+\nu\left(\omega^2 I+\hat{K}^2\right)\right)^{-1}\hat{K}.$$

Hence,

$$\frac{1}{\lambda}-1 = 2\sqrt{\nu}\left(1+\nu\left(\omega^2+\mu^2\right)\right)^{-1}\mu, \tag{12}$$

where $\mu \in \sigma(\hat{K})$. It is noted that $\mu > 0$. Straightforward computations reveal that

$$\lambda = \frac{1+\nu(\omega^2+\mu^2)}{\nu\omega^2+(1+\sqrt{\nu}\mu)^2},$$

which completes the proof. □

Theorem 2. *(I) The eigenvalues of the matrix $P_S^{-1}S$ are included in the interval $(\frac{1}{2},1)$.*
(II) If $\nu \to 0^+$ or $w \to +\infty$, then the eigenvalues of the preconditioned matrix $P_S^{-1}S$ tend to 1.

Proof. To prove (I), it follows from (12) that $\frac{1}{\lambda}-1>0$, which results in $\lambda<1$. To complete the proof, we write down Eq. (9) as

$$\lambda = 1-\frac{2\sqrt{\nu}\mu}{\nu\omega^2+\left(1+\sqrt{\nu}\mu\right)^2}. \tag{13}$$

Using the Arithmetic Mean - Geometric Mean (AM-GM) inequality we have

$$\left(1+\sqrt{\nu}\mu\right)^2 \geq 4\sqrt{\nu}\mu.$$

So, from the latter equation we arrive at

$$\begin{aligned}\lambda &= 1-\frac{2\sqrt{\nu}\mu}{\nu\omega^2+\left(1+\sqrt{\nu}\mu\right)^2}\\ &\geq 1-\frac{2\sqrt{\nu}\mu}{\nu\omega^2+4\sqrt{\nu}\mu}\\ &> 1-\frac{2\sqrt{\nu}\mu}{4\sqrt{\nu}\mu}=\frac{1}{2}.\end{aligned}$$

Part (II) of the theorem follows from Eq. (13). □

The above theorem shows that the eigenvalues of the preconditioned matrix $P_S^{-1}S$ are clustered in $(\frac{1}{2},1)$. Especially, for sufficiently large values of ω or small values of ν, the eigenvalues of $P_S^{-1}S$ are clustered around 1. In this case, the convergence of a Krylov subspace method, like GMRES, would be high for solving the corresponding preconditioned system [10, 16].

3 Implementation of the preconditioner P_S

In the implementation of the preconditioner P_S in a Krylov subspace method, like GMRES, for solving the system (7), two subsystems with coefficient matrices $D+B$ and $D+B^T$ should be solved. We first we propose an efficient method for solving the system with the coefficient matrix $D+B$.

We have

$$D+B=\begin{pmatrix} M+\sqrt{\nu}K & \omega\sqrt{\nu}M \\ -\omega\sqrt{\nu}M & M+\sqrt{\nu}K \end{pmatrix}.$$

For solving the system with the coefficient matrix $D+B$ we apply the PRESB preconditioner (see [3])

$$P_1=\begin{pmatrix} (1+2w\sqrt{\nu})M+\sqrt{\nu}K & \omega\sqrt{\nu}M \\ -\omega\sqrt{\nu}M & M+\sqrt{\nu}K \end{pmatrix}.$$

It is known that the eigenvalues of the matrix $P_1^{-1}(D+B)$ are included in the interval $[\frac{1}{2},1]$. So the corresponding system can be efficiently solved using the GMRES method. In each iteration of the GMRES method with the preconditioner P_1 we should solve a system of the form

$$\left((1+2\omega\sqrt{\nu})M+\sqrt{\nu}K\right)r+\omega\sqrt{\nu}Ms = e, \tag{14}$$

$$-\omega\sqrt{\nu}Mr+\left(M+\sqrt{\nu}K\right)s = f, \tag{15}$$

Summing up Eqs. (14) and (15) results in the system

$$\left((1+\omega\sqrt{\nu})M+\sqrt{\nu}K\right)z=e+f,$$

where $z=r+s$. On the other hand, from Eq. (14) we get

$$\left((1+\omega\sqrt{\nu})M+\sqrt{\nu}K\right)r=e-\omega\sqrt{\nu}Mz.$$

Finally, we have $s = z - r$. By summarizing the above results we present the following algorithm to solve the system $P_1(r;s) = (e;f)$.

Algorithm 1 Solution of $P_1(r;s) = (e;f)$.

1: Solve $\left((1+\omega\sqrt{\nu})M + \sqrt{\nu}K\right)z = e + f$ for z;
2: Solve $\left((1+\omega\sqrt{\nu})M + \sqrt{\nu}K\right)r = e - \omega\sqrt{\nu}Mz$ for r;
3: $s = z - r$.

As Algorithm 1 shows two subsystems with the coefficient matrix $(1+\omega\sqrt{\nu})M + \sqrt{\nu}K$ should be solved. Since, this matrix is SPD, the corresponding system can be solved exactly using the Cholesky method or inexactly using the CG method.

Similarly, we have

$$D + B^T = \begin{pmatrix} M + \sqrt{\nu}K & -\omega\sqrt{\nu}M \\ \omega\sqrt{\nu}M & M + \sqrt{\nu}K \end{pmatrix}.$$

To solve the system with the coefficient matrix $D + B^T$ we can employ the PRESB preconditioner

$$P_2 = \begin{pmatrix} M + \sqrt{\nu}K & -\omega\sqrt{\nu}M \\ \omega\sqrt{\nu}M & (1+2w\sqrt{\nu})M + \sqrt{\nu}K \end{pmatrix}.$$

Similar to Algorithm 1 we can state the following algorithm for solving $P_2(r;s) = (e;f)$.

Algorithm 2 Solution of $P_2(r;s) = (e;f)$.

1: Solve $\left((1+\omega\sqrt{\nu})M + \sqrt{\nu}K\right)z = e + f$ for z;
2: Solve $\left((1+\omega\sqrt{\nu})M + \sqrt{\nu}K\right)s = f - \omega\sqrt{\nu}Mz$ for r;
3: $r = z - s$.

Solution of the subsystems of this algorithm can be treated similar to Algorithm 1. As we see four subsystems with the same coefficient matrix should be solved in Algorithms 1 and 2. So to solve these subsystems exactly we need only a Cholesky factorization of the matrix $(1+\omega\sqrt{\nu})M + \sqrt{\nu}K$.

4 Numerical experiments

For our numerical tests we consider the distributed control problem in 2-dimensional case in the domain $\Omega = (0,1) \times (0,1) \in \mathbb{R}^2$. The target state is chosen as

$$y_d(x,y) = \begin{cases} (2x-1)^2(2y-1)^2, & \text{if } (x,y) \in (0,\frac{1}{2}) \times (0,\frac{1}{2}), \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

To produce the system (1) we have used the codes of the paper [15] which is available at https://www.numerical.rl.ac.uk/people/tyrone-rees/. All runs are implemented in MATLAB R2018, equipped with a Laptop with 2.60 GHz central processing unit (Intel(R) Core(TM) i7-4510), 8 GB memory and Windows 10 operating system.

We compare the numerical results of solving the system (4) with the preconditioner P_K and those of solving the system (7) with the preconditioner P_S. We use the complete version of the GMRES method

Table 1: Numerical results for $h = 2^{-8}$.

	ω	0.01	0.1	1	10	100
Preconditioner	ν	Its (CPU)	Its (CPU)	Its (CPU)	Its (CPU)	Its (CPU)
P_S	10^{-2}	3(1.84)	3(2.13)	3(2.53)	3(3.40)	3(3.84)
	10^{-4}	5(2.40)	5(2.37)	5(2.99)	5(4.05)	5 (5.52)
	10^{-6}	6(2.76)	6(2.70)	6(2.93)	6 (3.41)	6 (4.53)
	10^{-8}	7(2.61)	7(2.98)	7(2.99)	7(3.36)	7(3.65)
	10^{-10}	7(2.62)	7(2.66)	7(2.98)	7(3.30)	7(3.35)
P_K	10^{-2}	4(0.82)	4(0.77)	4(0.77)	4(0.81)	9(1.19)
	10^{-4}	9(1.19)	9(1.20)	9(1.18)	9(1.18)	10(1.24)
	10^{-6}	33(3.35)	33(3.44)	33(3.84)	33(3.36)	34(3.50)
	10^{-8}	137(16.45)	136(15.89)	136(16.15)	137(16.78)	137(16.30)
	10^{-10}	457(92.65)	457(93.24)	457(93.16)	457(95.97)	457(93.77)

in conjunction with the preconditioners P_K and P_S for solving systems (4) and (7), respectively. We always use right preconditioning. We employ the implementation method presented in Section 3 for the preconditioner P_S. We also solve the systems with the coefficient matrices $D+B$ and $D+B^T$ with the preconditioners P_1 and P_2, using the complete version of the GMRES. For the GMRES method (for the outer and inner systems) a zero vector is used as an initial guess and the iteration is stopped as soon as the residual norm of the original system is reduced by a factor of 10^5. The maximum number of iterations is set to be 500. All the subsystems (with SPD coefficient matrices) are solved using the Cholesky factorization of the coefficient matrix incorporated with the symmetric approximate minimum degree reordering. To do so, the "`symamd`" command of MATLAB is used.

Numerical results for $h = 2^{-8}, 2^{-9}$ for different values ω and ν have been presented in Tables 1-2. It is noted that for $h = 2^{-8}$ the size of the matrix S is 130050 and that of $h = 2^{-9}$ is 522242. In the tables, "Its" and "CPU" stand for the number of iterations and the CPU time (in seconds), respectively. As we see for all the tested problems the number of iterations of our method is less than that of the preconditioner P_K. However, for small values of ν ($\nu \leq 10^{-6}$), the preconditioner P_K can not compete with preconditioner P_S. As we observe, the number of iterations for the preconditioner P_K drastically increases as the value of ν decreases, however this is not the case for the preconditioner P_S. The last point which is mentioned here is that the GMRES method without preconditioning, almost for all the problems, could not compute the solution of the system (6) in 500 iterations.

5 Conclusion

By using the idea of [15] we have presented a preconditioner for solving the linear systems arising from finite element discretization of the time-harmonic parabolic optimal control problem. The eigenvalue distribution of the preconditioned matrix have been analyzed. Numerical results of the proposed preconditioner have been compared with those of the recently proposed preconditioner in [18]. Numerical results show that our preconditioner is superior to the preconditioner of [18].

Table 2: Numerical results for $h = 2^{-9}$.

	ω	0.01	0.1	1	10	100
Preconditioner	ν	Its (CPU)	Its (CPU)	Its (CPU)	Its (CPU)	Its (CPU)
P_S	10^{-2}	3(10.46)	3(11.70)	3(14.00)	3(18.31)	2(17.10)
	10^{-4}	5(13.89)	5(13.29)	5(16.59)	5(22.37)	4 (27.63)
	10^{-6}	6(15.22)	6(15.25)	6(16.81)	6 (18.81)	6 (25.77)
	10^{-8}	7(15.05)	7(16.78)	7(16.77)	7(18.66)	7(20.62)
	10^{-10}	7(14.88)	7(16.91)	7(16.45)	7(17.50)	7(18.34)
P_K	10^{-2}	4(4.60)	4(4.86)	4(4.90)	4(4.37)	9(6.67)
	10^{-4}	9(6.82)	9(6.95)	9(6.76)	9(6.77)	10(7.33)
	10^{-6}	33(19.78)	33(19.81)	33(20.03)	33(20.00)	34(20.50)
	10^{-8}	137(108.53)	137(108.46)	136(107.32)	137(108.18)	137(108.15)
	10^{-10}	470(683.60)	470(685.43)	470(687.32)	470(686.41)	470(688.79)

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