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Nonstandard finite difference method for solving singularly perturbed time-fractional delay parabolic reaction-diffusion problems

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Abstract. This work addresses the singularly perturbed time-fractional delay parabolic reaction-diffusion of initial boundary value problems. The temporal derivatives discretization is handled by the Caputo fractional derivative combined with the implicit Euler technique with uniform step size. It also utilizes the nonstandard finite difference approach for the spatial derivative. The scheme has been demonstrated to converge and has an accuracy of $O(h^2 + (\Delta t^{2-\alpha}))$. To assess the suitability of the approach, two model examples are taken into consideration. The findings, which are provided in tables and figures, illustrate that the system has twin layers at the end of space domain and is uniformly convergent.

Keywords: Nonstandard finite difference, time-fractional, delay parabolic reaction-diffusion, convergence analysis.
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1 Introduction

For prior couple decades, the fractional calculus approach has been extensively utilized in a broad range of domains, notably mechanics, electricity, physics, biology, economics, control theory, signal and image processing, transportation, resolve medical images issues, salesman and fuzzy problems [1–14]. A fractional differential equation is a differential equation that can be earned in either the space or time variable, which involves integer order derivatives to an arbitrary degree. For non-conservative structures, time fractional refers to anomalous diffusion processes that are connected with time. Though it is very tricky to obtain analytical solutions for these kinds of problems, numerical approaches are adopted to approximate the solutions. Fractional differential equations are currently enticed by different investigators working on ways to approximate numerical solutions for these types of mathematical models owing to their potential to simulate complex processes [15–17].

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The models of oil reservoir simulation, fluid flow in porous media, global water production, and numerous other organic occurrences have been addressed with time-fractional reaction-diffusion equations [18]. The mere existence of an arbitrary order makes it tough to find exact solutions for such situations. It is therefore becoming more and more crucial to develop dependable and efficient numerical techniques for the numerical solutions of such equations. Time fractional reaction-diffusion has been explored in [19–22].

This study addresses the initial boundary value problems of singularly perturbed time-fractional delay parabolic reaction-diffusion in the domain $D = (0,l) \times (0,1]$:

$$\left(\frac{\partial^{\alpha}}{\partial t^{\alpha}} u - \varepsilon \frac{\partial^{w} u}{\partial w^{2}} + b(w) u \right)(w,t) + p(w,t) u(w,t-\tau) = f(w,t), \quad (w,t) \in D \tag{1}$$

with the vector condition

$$u(w,t) = \varphi_b(w,t), \qquad w \in \bar{D} = [0,1] \tag{2}$$

and the boundary conditions

$$u(0,t) = \varphi_l(t), \quad u(l,t) = \varphi_r(t), \qquad t \in [0,T] \tag{3}$$

where ε is a small perturbation parameter which fulfills $0 < \varepsilon < 1$, τ is delay parameter and $b(w) \geq \vartheta > 0$ is a smooth function. As soon as the functions $b(w), p(w,t), \varphi_b(w,t), \varphi_l(t), \varphi_r(t)$, and $f(w,t))$ meet the required smoothness and compatibility requirements, the initial-boundary value problem admits a unique solution $u(w,t)$. This solution illustrates twin boundary layers with a thickness of $O(\sqrt{\varepsilon})$, that lies near the boundaries $w = 0$ and $w = 1$.

When perturbation parameter is multiplied with the highest order derivative of singularly perturbed parabolic reaction-diffusion equations, large deviation often serve to visualize the solution for the issue [23]. In light of boundary layer behavior, while solving such problems, employing conventional numerical methods upon an even grid, substantial instabilities might emerge whenever the perturbation parameter gets nearer to zero throughout the entire region of interest. To prevent such oscillations, an appropriate numerical method whose accuracy is independent of perturbation parameter will be used. Therefore, an extensive amount of work has gone into establishing numerical techniques to handle the singularly perturbed delay parabolic reaction-diffusion equations of one-dimensional initial boundary value problems [24–32].

Yet nobody has attempted to apply the numerical approach to singularly perturbed time-fractional delay parabolic reaction-diffusion equation, even though there have been several attempts to solve singularly perturbed delay parabolic reaction-diffusion with integer order. Consequently, in this study the nonstandard finite difference approach for solving one-dimensional singularly perturbed time-fractional delay parabolic reaction-diffusion equations is presented.

This study has a distinct structure: Section 2 covers preliminaries and properties of continuous solution. Numerical formulation and convergence analysis are presented in Sections 3 and 4. Sections 5 and 6 provide a discussion of the numerical results and conclusions, respectively.

2 Preliminaries and properties of continuous solution

Definition 1. *Assume that $Re(J) > 0$ for any complex number J. The function specified by*

$$\Gamma(J) = \int_0^\infty e^{-w} w^{J-1} ds$$

is a gamma function.

Definition 2. *When the function $h(t)$ possesses lowest bound of zero, then Caputo fractional derivative is defined as*

$$\frac{\partial^\alpha}{\partial t^\alpha} u(w,t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t h^{(k)}(\zeta)(t-\zeta)^{k-\alpha-1} d\zeta; \quad \alpha \in (k-1,k)$$

Definition 3. *A function $u(w,t)$ and its Caputo fractional differentiation concerning t is described as*

$$\frac{\partial^\alpha}{\partial t^\alpha} u(w,t) = \begin{cases} \frac{1}{\Gamma(k-\alpha)} \int_0^t \frac{\partial^k u(w,\zeta)}{\partial \zeta^k} (t-\zeta)^{k-\alpha-1} d\zeta; & if \quad \alpha \in (k-1,k) \\ \frac{\partial^k u(w,t)}{\partial t^k}; & if \quad \alpha = k \end{cases}$$

Lemma 1. *Let $0 < t_0 < 1$ be the lowest possible value of the function h, where $h \in C^1[0,1]$. Then*

$$\partial_c^\alpha h(t_0) \leq \frac{t_0^\alpha}{\Gamma(1-\alpha)} (h(t_0) - h(0)) \leq 0,$$

where $\alpha \in (0,1)$ and ∂_C^α stands for the Caputo fractional derivative.

Proof. Let an auxiliary function $q(t) = h(t) - h(t_0)$. Then, $p(t) \geq 0$ and $q(t_0) = h(t_0) - h(t_0) = 0$. Now,

$$\partial_C^\alpha q(t_0) = \frac{1}{\Gamma(k-\alpha)} \int_0^{t_0} (t_0 - \zeta)^{-\alpha} q'(\zeta) d\zeta.$$

Applying integration by parts, we obtain

$$\begin{aligned} \partial_C^\alpha q(t_0) &= \frac{1}{\Gamma(k-\alpha)} \left(-t_0^{-\alpha} q(0) - \alpha \int_0^{t_0} (t_0 - \zeta)^{-\alpha-1} q'(\zeta) d\zeta \right) \\ &\leq \frac{1}{\Gamma(k-\alpha)} (-t_0^{-\alpha} q(0)) \\ &\leq \frac{1}{\Gamma(k-\alpha)} (-t_0^{-\alpha} (h(t_0) - h(0))) \\ &\leq 0. \end{aligned}$$

□

Given the data $b(w), p(w,t)$ and $f(w,t)$ are Holder's continuous, and the compatibility criteria s at the corner points $(0,0), (1,0), (0,-\tau)$ and $(0,\tau)$ have been met, it is shown, to ensure the existence of the unique solution for Eqs. (1,2, 3). That is

$$\begin{cases} \varphi_b(0,0) = \varphi_l(0), \\ \varphi_b(1,0) = \varphi_r(0), \end{cases}$$
$$\begin{cases} \frac{\partial^\alpha \varphi_l(0)}{\partial t^\alpha} - \varepsilon \frac{\partial^2 \varphi_b(0,0)}{\partial w^2} + b(0)\varphi_b(0,0) + p(0,0)\varphi_b(0,-\tau) = f(0,0), \\ \frac{\partial^\alpha \varphi_r(0)}{\partial t^\alpha} - \varepsilon \frac{\partial^2 \varphi_b(1,0)}{\partial w^2} + b(1)\varphi_b(1,0) + p(1,0)\varphi_b(1,-\tau) = f(1,0), \end{cases} \quad (4)$$

Lemma 2. *Given $a, b \in C^0(\bar{D})$, let $u \in C^2(D) \bigcap C^0(\bar{D})$. Considering $u \geq 0$ on $\partial D = \bar{D} - D$, it follows that, $\mathscr{L}_\varepsilon u \geq 0$ in D yields $u \geq 0$ in $\bar{D}$.*

Proof. Let (υ, ω) be a point that satisfy $\xi(\omega, \upsilon) = \min_{(w,t)\in\bar{D}} \xi(w,t)$ and $\xi(\omega, \upsilon) < 0$. Then $\xi(\omega, \upsilon) \notin \partial D$ and we have,

$$\mathscr{L}_\varepsilon \xi(\omega, \upsilon) = \varepsilon \xi_{ww}(\omega, \upsilon) - b(\omega, \upsilon)\xi(\omega, \upsilon) - \frac{\partial^\alpha \xi(\omega, \upsilon)}{\partial t^\alpha} \leq 0.$$

Since $\xi_{ww}(\omega, \upsilon) \geq 0$ and $\frac{\partial^\alpha \xi(\omega,\upsilon)}{\partial t^\alpha} = 0$, $\mathscr{L}_\varepsilon \xi(\omega, \upsilon) \leq 0$, which contradicts the initial assumption. Therefore $\xi(\omega, \upsilon) \geq 0 \quad \forall (w,t) \in \bar{D}$. □

Lemma 3. *(Stability Estimate)*
Let $u(w,ţ)$ represent the solution of continuous problem of Eq. (1). Consequently,

$$\|u(w,ţ)\| \leq (1 + \vartheta T) \max\{\|\mathscr{L}_\varepsilon u\|, \|u\|_{\partial D}\}$$

where $\vartheta = \max_{\bar{D}}\{0, 1 - \vartheta\} \leq 1$ and $\|u\| = \max_{\bar{D}} |u(w,ţ)|$ is the greatest that expressed in terms of $\|.\|$.

Proof. One may refer [33] for details. □

Lemma 4. *The solution of problem (1) and its derivative satisfy*

$$\left|\frac{\partial^{k+s} u(w,t)}{\partial w^k \partial t^s}\right| \leq C\left[1 + \varepsilon^{\frac{-i}{2}}\left(exp(\frac{-w}{\sqrt{\varepsilon}} + exp(\frac{-(1-w)}{\sqrt{\varepsilon}})\right)\right]$$

with $0 \leq k + 2s \leq 4$

Proof. The proof of this Lemma can be found in [34]. □

3 Development of numerical schemes

3.1 Temporal discretization

To discretize the time derivative of Eq.(1), we utilize the implicit Euler's technique with uniform mesh size $\Delta ţ$ on the temporal domain $D^M_ţ = \{ţ_j = j\Delta t; j = 0, 1, \ldots, M, ţ_M = T, \Delta ţ = \frac{T}{M}\}$, where M denotes the number of grid points along the time axis. The mesh for $[-\tau, T]$ is defined as $D^M_ţ = \{ţ_j = j\Delta ţ; -m \leq j \leq M\}$. In the Caputo notion, the time-fractional derivative is contemplated. Therefore, the time-fractional derivative term of Eq.(1) at time $ţ_{j+1}$ can be computed with the following quadrature formula utilizing the implicit Euler technique:

$$\begin{aligned}
\partial_ţ^\alpha u(w, ţ_{j+1}) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{ţ_{j+1}} \frac{\partial u(w,s)}{\partial s} (ţ_{j+1} - s)^{-\alpha} ds \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{r=0}^{j} \left(\frac{u(w, ţ_{r+1}) - u(w, ţ_r)}{\Delta ţ}\right) \int_{ţ_r}^{ţ_{r+1}} (ţ_{j+1} - s)^{-\alpha} ds + e^{j+1}_{\Delta ţ} \\
&= \beta \sum_{r=0}^{j} \upsilon_r \left(u(w, ţ_{j-r+1}) - u(w, ţ_{j-r})\right) + e^{j+1}_{\Delta ţ},
\end{aligned}$$

where

$$\beta = \frac{(\Delta\mathfrak{t})^{-\alpha}}{\Gamma(1-\alpha)}, \quad \upsilon_r = \left((r+1)^{1-\alpha} - (r)^{1-\alpha}\right) \text{ and } e_{\Delta\mathfrak{t}}^{j+1} = \frac{(\Delta\mathfrak{t})}{\Gamma(1-\alpha)} \int_{\mathfrak{t}_r}^{\mathfrak{t}_{r+1}} (\mathfrak{t}_{j+1} - s)^{-\alpha} ds.$$

Therefore, the Caputo fractional derivative $\partial_{\mathfrak{t}}^{\alpha} u(w,\mathfrak{t})$ at the point $(w,\mathfrak{t}_{j+1})$ is estimated as

$$\partial_{\mathfrak{t}}^{\alpha} u(w,\mathfrak{t}_{j+1}) = \beta \left(\left(u(w,\mathfrak{t}_{j+1}) - u(w,\mathfrak{t}_j)\right) + \sum_{r=1}^{j} \upsilon_r \left(u(w,\mathfrak{t}_{j-r+1}) - u(w,\mathfrak{t}_{j-r})\right) \right). \quad (5)$$

The time semi-discrete equation is obtained by substituting Eq.(5) into (1)

$$\begin{aligned} \beta\left(\frac{u(w,\mathfrak{t}_{j+1})-u(w,\mathfrak{t}_j)}{2^{1-\alpha}}\right) + \beta\left(\sum_{r=1}^{j} \left(u(w,\mathfrak{t}_{j-r+1}) - u(w,\mathfrak{t}_{j-r})\right)\upsilon_r\right) - \varepsilon u_{ww}^{j+1}(w) + b(w)u^{j+1}(w) \\ = f^{j+1}(w) - \begin{cases} p(w,\mathfrak{t}_{j+1})\varphi_b(w,\mathfrak{t}_{j+1}), & j = 0,1,\ldots,m, \\ p(w,\mathfrak{t}_{j+1})u(w,\mathfrak{t}_{j-m+1}), & j = m+1,\ldots,M. \end{cases} \end{aligned} \quad (6)$$

Adjusting Eq.(5) yields

$$\left(\beta + \mathscr{L}_{\varepsilon}^{\Delta t}\right) u^{j+1}(w) = R_i, \quad (7)$$

where

$$\begin{aligned} &\left(\beta + \mathscr{L}_{\varepsilon}^{\Delta t}\right) u^{j+1}(w) = -\varepsilon u_{ww}^{j+1}(w) + (\beta + b^{j+1}(w))u^{j+1}(w), \\ &R_i = \begin{cases} \left(\beta u^j + f^{j+1} - q^{j+1}\varphi_b(t_{j+1}) - \beta\sum_{r=1}^{j} \upsilon_r\left(u^{j-r+1} - u^{j-r}\right)\right)(w), & j = 0,1,\ldots,m, \\ \left(\beta u^j + f^{j+1} - q^{j+1}u^{j-m+1} - \beta\sum_{r=1}^{j} \upsilon_r\left(u^{j-r+1} - u^{j-r}\right)\right)(w), & j = m+1,\ldots,M, \end{cases} \end{aligned}$$

which is the time semi-discrete of Eq.(1).

Lemma 5. *In (6) an error R is limited as* $\left|\underset{\cdot}{e}^{j+1}\right| \leq C(\Delta t)^{(2-\alpha)}$.

Proof.

$$\begin{aligned} \underset{\cdot}{e}^{j+1} &= \frac{O(\Delta\mathfrak{t})}{\Gamma(1-\alpha)} \sum_{r=0}^{j-1} \int_{\mathfrak{t}_r}^{\mathfrak{t}^{r+1}} \left(\mathfrak{t}_{j+1} - s\right) ds \\ &= \frac{O(\Delta\mathfrak{t})}{\Gamma(1-\alpha)} \sum_{r=1}^{j} \left(\frac{(j-r+1)^{1-\alpha} - (j-r)^{1-\alpha}}{1-\alpha}\right)(\Delta\mathfrak{t})^{1-\alpha} \\ &= \frac{O((\Delta\mathfrak{t})^{2-\alpha})}{\Gamma(2-\alpha)} \left((j-r+1)^{1-\alpha} - (j-r)^{1-\alpha}\right) \\ &= \frac{O((\Delta\mathfrak{t})^{2-\alpha})}{\Gamma(2-\alpha)} \left((j+1)^{1-\alpha}\right) \\ &\leq C(\Delta\mathfrak{t})^{2-\alpha}. \end{aligned}$$

Then $\left|\underset{\cdot}{e}^{j+1}\right| \leq C(\Delta\mathfrak{t})^{(2-\alpha)}$ where C is a constant that remains unaltered by ε or $\Delta\mathfrak{t}$. □

3.2 Spatial discretization

To achieve the discretization of space derivative, we first split the spatial interval evenly using the step size $h = w_{i+1} - w_i = \frac{1}{N}; i = 0,1,\ldots,N-1$. This results in a consistent division of the solution area $0 \leq w \leq 1$. Considering the nonstandard finite difference methods from [35], we can apply to Eq.(7) and get the following results

$$
\begin{aligned}
&-\varepsilon\left(\frac{u_{i-1}^{j+1}-2u_i^{j+1}+u_{i+1}^{j+1}}{\varsigma(i)}\right)+(b(i)+\beta)\,u_i^{j+1}\\
&\quad=\begin{cases}\beta u_i^j+f_i^{j+1}-p_i^{j+1}\varphi_{b_i^{j+1}}-\beta\sum_{r=1}^{j}\upsilon_r\left(u_i^{j-r+1}-u_i^{j-r}\right), & j=0,1,\ldots,m,\\ \beta u_i^j+f_i^{j+1}-p_i^{j+1}u_i^{j-m+1}-\beta\sum_{r=1}^{j}\upsilon_r\left(u_i^{j-r+1}-u_i^{j-r}\right), & j=m+1,\ldots,M,\end{cases}
\end{aligned}
\tag{8}
$$

with $\varsigma(i) = \frac{4\varepsilon}{\mu(i)}\sinh^2\left(\mu\frac{h}{2}\right)$, $\mu = \sqrt{\frac{b(i)}{\varepsilon}}$. Eq.(8) is concisely expressed as

$$
E_i u_{i-1}^{j+1} + F_i u_i^{j+1} + G_i u_{i+1}^{j+1} = H_i \tag{9}
$$

where

$$
\begin{aligned}
&E_i = G_i = -\frac{\varepsilon}{\varsigma(i)},\quad F_i = \frac{2\varepsilon}{\varsigma(i)} + b(i) + \beta,\\
&H_i = \begin{cases}\beta u_i^j+f_i^{j+1}-p_i^{j+1}\varphi_{b_i^{j+1}}-\beta\sum_{r=1}^{j}\upsilon_r\left(u_i^{j-r+1}-u_i^{j-r}\right), & j=0,1,\ldots,m,\\ \beta u_i^j+f_i^{j+1}-p_i^{j+1}u_i^{j-m+1}-\beta\sum_{r=1}^{j}\upsilon_r\left(u_i^{j-r+1}-u_i^{j-r}\right), & j=m+1,\ldots,M.\end{cases}
\end{aligned}
$$

4 Convergence analysis

Lemma 6. *Let u_i^{j+1} denote arbitrary grid function that fulfills both $u_0^{j+1} \geq 0$ and $u_N^{j+1} \geq 0$. Then, $\left(\beta + \mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1} \geq 0$ for $i = 1,2,\ldots,N-1$ implies $u_i^{j+1} \geq 0$ for all $i = 0,1,\ldots,N$.*

Proof. Let $k \in \{0,1,\ldots,N\}$ such that

$$
u_k^{j+1} = \min_{0\leq i\leq N} u_i^{j+1}.
$$

Assume that $u_k^{j+1} < 0$. Then $k \neq \{0,N\}$. Further, we have $u_{k+1} - u_k^{j+1} > 0$ and $u_k - u_{k-1}^{j+1} < 0$. Now,

$$
\left(\beta + \mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1} = \beta u_i^{j+1} + \frac{\varepsilon\sigma}{\varsigma(i)}\left(u_{i-1}^{j+1} - 2u_{i-}^{j+1} + u_{i+1}^{j+1}\right) + b_i u_i^{j+1} \leq 0,
$$

which contradicts the assumption. Hence, $u_i^{j+1} \geq 0, \quad \forall i = 0,1,\ldots,N$. □

Lemma 7. *(Stability) The result u_i^{j+1} of the semi-discrete problem (9) provides the bound*

$$
\left\|u_i^{j+1}\right\|_\infty \leq \frac{\|\left(\beta + \mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1}\|}{\beta + \vartheta} + \max\left\{|\psi_i|, \max\left(\varphi_l, \varphi_r\right)\right\}, \tag{10}
$$

where ϑ is the lower limit of b_i.

Proof. Consider the function $(\Upsilon^{\pm})_i^{j+1}$ defined by

$$(\Upsilon^{\pm})_i^{j+1} = +\frac{\|\left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1}\|}{\beta+\vartheta} + \max\left\{|\Upsilon_i|, \max\left(\varphi_l, \varphi_r\right)\right\} \pm u_i^{j+1}.$$

At the boundaries we have

$$(\Upsilon^{\pm})_0^{j+1} = +\frac{\|\left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1}\|}{\beta+\vartheta} + \max\left\{|\psi_i|, \max\left(\varphi_l, \varphi_r\right)\right\} \pm \varphi_0^{j+1}$$

and

$$(\Upsilon^{\pm})_N^{j+1} = +\frac{\|\left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1}\|}{\beta+\vartheta} + \max\left\{|\psi_i|, \max\left(\varphi_l, \varphi_r\right)\right\} \pm \varphi_N^{j+1}.$$

Now, for all $i = 1, 2, \ldots, N-1$

$$\left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)(\Upsilon^{\pm})_i^{j+1} = (\beta+b_i)\frac{\|\left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1}\|}{\beta+\vartheta} + \max\left\{|\psi_i|, \max\left(\varphi_l, \varphi_r\right)\right\} \pm \left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1}.$$

This yields

$$\begin{aligned}\left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)(\Upsilon^{\pm})_i^{j+1} &= (\beta+\vartheta)\frac{\|\left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1}\|}{\beta+\vartheta}\max\left\{|\psi_i|, \max\left(\varphi_l, \varphi_r\right)\right\} \pm \left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1} \\ &\geq (\beta+\vartheta)\frac{\|\left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1}\|}{\beta+\vartheta} \pm \left(\beta+\mathscr{L}_\varepsilon^{h,\Delta t}\right)u_i^{j+1} \geq 0.\end{aligned}$$

Hence, using Lemma 7, we get $(\Upsilon^{\pm})_i^{j+1}$ for all $i = 0, 1, \ldots, N$. □

Lemma 8. *The bound below corresponds to the value for the denominator $\varsigma(i)$ of Eq. (8)*

$$\varepsilon\left|\frac{h^2}{\varsigma(i)} - 1\right| \leq Kh^2$$

where K is a constant independent of ε

Proof. We have

$$\varepsilon\left|\frac{h^2}{\varsigma(i)} - 1\right| = \left|\frac{\varepsilon\delta}{e^{\sqrt{\delta}} - 2 + e^{-\sqrt{\delta}}} - \varepsilon\right| = b(i)h^2\left|\frac{\delta+2-e^{\sqrt{\delta}}-e^{-\sqrt{\delta}}}{\delta\left(e^{\sqrt{\delta}}-2+e^{-\sqrt{\delta}}\right)}\right|,$$

where $\delta = \frac{b_i h^2}{\varepsilon}$. Let $\Theta(\delta) = \frac{\delta+2-e^{\sqrt{\delta}}-e^{-\sqrt{\delta}}}{\delta\left(e^{\sqrt{\delta}}-2+e^{-\sqrt{\delta}}\right)}$. Now, to evaluate the limit we can apply the L'Hopital's rule repeatedly and we obtain

$$\lim_{\delta\to 0}\Theta(\delta) = -\frac{1}{12} \quad \text{and} \quad \lim_{\delta\to\infty}\Theta(\delta) = 0.$$

Thus $|\Theta(\delta)| \leq K$ and

$$\varepsilon\left|\frac{h^2}{\varsigma(i)} - 1\right| \leq Kh^2 b_i \leq Kh^2.$$

□

Lemma 9. *For every $k \in \mathbb{N}$, and for every fixed mesh, we have*

$$\lim_{\varepsilon \to 0} \max_{1<i<N-1} \left(\varepsilon^{-\frac{k}{2}} exp\left(-\frac{Cw_i}{\sqrt{\varepsilon}}\right)\right) = 0$$

and

$$\lim_{\varepsilon \to 0} \max_{1<i<N-1} \left(\varepsilon^{-\frac{k}{2}} exp\left(-\frac{C(1-w_i)}{\sqrt{\varepsilon}}\right)\right) = 0$$

for $w_i = ih; h = \frac{1}{N}; \forall i = 1,..,N-1$.

Proof. Taking into account the partition of an interval $[0,1]$, where the points for $i = 1(1)N-1$ are $w_i = ih; h = \frac{1}{N}$, we have

$$\max_{1<i<N-1} \left(\varepsilon^{-\frac{k}{2}} \exp\left(-\frac{Cw_i}{\sqrt{\varepsilon}}\right)\right) \leq \left(\varepsilon^{-\frac{k}{2}} \exp\left(-\frac{Cw_1}{\sqrt{\varepsilon}}\right)\right) = \left(\varepsilon^{-\frac{k}{2}} \exp\left(-\frac{Ch}{\sqrt{\varepsilon}}\right)\right)$$

and

$$\max_{1<i<N-1} \left(\varepsilon^{-\frac{k}{2}} \exp\left(-\frac{C(1-w_i)}{\sqrt{\varepsilon}}\right)\right) \leq \left(\varepsilon^{-\frac{k}{2}} \exp\left(-\frac{Cw_{N-1}}{\sqrt{\varepsilon}}\right)\right) = \left(\varepsilon^{-\frac{k}{2}} \exp\left(-\frac{Ch}{\sqrt{\varepsilon}}\right)\right).$$

Then applying L'Hopital's rule repeatedly we get

$$\lim_{\varepsilon \to 0} \left(\varepsilon^{-\frac{k}{2}} \exp\left(-\frac{Ch}{\sqrt{\varepsilon}}\right)\right) = \lim_{r=\frac{1}{\varepsilon} \to \infty} r^k \exp\left(-Chr\right) = \lim_{r \to \infty} k!(Ch)^{-k} \exp\left(-Chr\right) = 0.$$

□

Lemma 10. *The discrete solution u_i^{j+1} in algorithm (8) fulfills the following constraint in terms of its truncation error*

$$\left| u^{j+1}(x_i) - u_i^{j+1} \right| \leq Ch^2.$$

Proof. Truncation error of spatial discretization by using rule in Lemma 8 gives

$$\begin{aligned} \left| \left(\beta + \mathscr{L}_\varepsilon^{h,\Delta t}\right) \left(u^{j+1}(w_i) - u_i^{j+1}\right) \right| &= \left| -\varepsilon \left(\frac{d^2}{dw^2} - \frac{1}{\varsigma(i)} \left(D_w^+ D_w^-\right)\right) u^{j+1}(w_i) \right| \\ &\leq \varepsilon C \left| \left(\frac{d^2}{dw^2} - \frac{1}{\varsigma(i)} \left(D_w^+ D_w^-\right)\right) u^{j+1}(w_i) \right| \\ &\quad + \varepsilon C \left| \left(\frac{h^2}{\varsigma(i)} - 1\right) D_w^+ D_w^- u^{j+1}(w_i) \right| \\ &\leq \varepsilon C h^2 \left| \frac{d^4}{dw^4} u^{j+1}(w_i) \right| + Ch^2 \left| \frac{d^2}{dw^2} u^{j+1}(w_i) \right|. \end{aligned} \tag{11}$$

Again applying Lemma 4 to Eq. (11) become

$$\begin{aligned} \left| \left(\beta + \mathscr{L}_\varepsilon^{h,\Delta t}\right) \left(u^{j+1}(w_i) - u_i^{j+1}\right) \right| \leq \varepsilon C^* h^2 &\left| \left(1 + \varepsilon^{-2} \left(\exp(\tfrac{-w_i}{\sqrt{\varepsilon}} + \exp(\tfrac{-(1-w_i)}{\sqrt{\varepsilon}})\right)\right) \right| \\ + C^* h^2 &\left| \left(1 + \varepsilon^{-1} \left(\exp(\tfrac{-w_i}{\sqrt{\varepsilon}} + \exp(\tfrac{-(1-w_i)}{\sqrt{\varepsilon}})\right)\right) \right|, \end{aligned} \tag{12}$$

$$\begin{aligned} \left| \left(\beta + \mathscr{L}_\varepsilon^{h,\Delta t}\right) \left(u^{j+1}(w_i) - u_i^{j+1}\right) \right| \leq C^* h^2 &\left| \left(\varepsilon + \varepsilon^{-1} \left(exp(\tfrac{-w_i}{\sqrt{\varepsilon}} + exp(\tfrac{-(1-_i)}{\sqrt{\varepsilon}})\right)\right) \right| \\ + C^* h^2 &\left| \left(1 + \varepsilon^{-1} \left(\exp(\tfrac{-w_i}{\sqrt{\varepsilon}} + \exp(\tfrac{-(1-w_i)}{\sqrt{\varepsilon}})\right)\right) \right|. \end{aligned} \tag{13}$$

Results from Eq.(13) and Lemma 9, gives

$$\left|\left(\beta+\mathscr{L}_{\varepsilon}^{h,\Delta t}\right)\left(u^{j+1}(w_i)-u_i^{j+1}\right)\right| \leq C^* h^2 \,|1+\varepsilon| \tag{14}$$

Hence, an error bound becomes $\left|u^{j+1}(w_i)-u_i^{j+1}\right| \leq Ch^2$. □

Theorem 1. *Let $u(w,t)$ be the solution of Eq.(1) and $u(w_i,t_{j+1})$ be the solution of the total discretized equation. Under the hypothesis of Lemma 5 and Lemma 10, the $\varepsilon-$uniform estimate holds*

$$\sup_{1\leq i\leq N-1} = \max_{1\leq i\leq N-1, 0<j<M} \left|u(w_i,t_{j+1})-U(w_i,t_{j+1})\right| \leq C\left(h^2+(\Delta t)^{2-\alpha}\right), \tag{15}$$

where C is a constant that is not altered by ε, h or (Δt).

Proof. The proof followed by applying triangle inequality. □

5 Numerical results

Model examples are offered to demonstrate the validity of the proposed data plan. The provided example are calculated as follows: Double mesh formula used to determine maximum point-wise error is:

$$E_{\varepsilon}^{N,M} = \max_{1\leq i\leq N-1} \left|u_i^{N,M}-u_i^{2N,2M}\right|,$$

where $u_i^{N,M}$ is the numerical solution obtained on the mesh $D^{N,M}=D_x^N\times D_t^M$. To determine the $\varepsilon-$uniform errors for any value of N and M, employ $E^{N,M}=\max_{\varepsilon} E_{\varepsilon}^{N,M}$. The formula for estimating the ε-uniform rate of convergence of the scheme is:

$$r_{\varepsilon}^{N,M} = \frac{\log(E_{\varepsilon}^{N,M})-\log(E_{\varepsilon}^{2N,2M})}{\log(2)}.$$

Example 1.

$$\frac{\partial^{\alpha}u(w,t)}{\partial t^{\alpha}} - \varepsilon\frac{\partial^2 u(w,t)}{\partial w^2} + (1.1+w^2)u(w,t)+u(w,t-\tau) = t^3$$

with initial and boundary conditions

$$u(w,0)=0, \qquad (w,t)\in[0,1]\times[-1,0],$$

and

$$u(0,t)=0=u(l,t)=0; \qquad t\in[0,2].$$

Example 2.

$$\frac{\partial^{\alpha}u(w,t)}{\partial t^{\alpha}} - \varepsilon\frac{\partial^2 u(w,t)}{\partial w^2} + \frac{(1+w)^2}{2}u(w,t)+u(w,t-\tau) = t^3$$

with constraints of

$$u(w,0)=0, \qquad (w,t)\in[0,1]\times[-1,0],$$

and

$$u(0,t)=0=u(l,t)=0; \qquad t\in[0,2].$$

Table 1: Maximum absolute errors for different α values and fixed $\varepsilon = 10^{-10}$ of Example 1 with (N,M)

α	(32,4)	(64,8)	(128,16)	(256,32)
0.25	1.3968e-02	1.0085e-02	1.6262e-02	2.0043e-02
0.50	4.7734e-02	2.0990e-02	1.8678e-02	2.4170e-02
0.75	1.0658e-01	5.9050e-02	2.8035e-02	1.7859e-02
0.90	1.5295e-01	1.0331e-01	5.4487e-02	2.6586e-02

Table 2: Maximum absolute error of Example 1 for $\alpha = 0.8$ and (N,M)

ε	(32,4)	(64,8)	(128,16)	(256,32)
10^{0}	4.7900e-03	2.3719e-03	1.0967e-03	6.9576e-04
10^{-2}	1.0788e-01	6.3516e-02	3.0331e-02	1.6603e-02
10^{-4}	1.2061e-01	7.1515e-02	3.6598e-02	1.8552e-02
10^{-6}	1.2120e-01	7.1866e-02	3.6754e-02	1.8625e-02
10^{-8}	1.2120e-01	7.1866e-02	3.6760e-02	1.8626e-02
10^{-10}	1.2120e-01	7.1866e-02	3.6760e-02	1.8626e-02
10^{-12}	1.2120e-01	7.1866e-02	3.6760e-02	1.8626e-02
10^{-14}	1.2120e-01	7.1866e-02	3.6760e-02	1.8626e-02
10^{-16}	1.2120e-01	7.1866e-02	3.6760e-02	1.8626e-02
$E^{N,M}$	1.2120e-01	7.1866e-02	3.6760e-02	1.8626e-02
$r^{N,M}$	0.75401	0.96717	0.98082	

Table 3: Maximum absolute errors for different α values and fixed $\varepsilon = 10^{-10}$ of Example 2 with (N,M)

α	(32,4)	(64,8)	(128,16)	(256,32)
0.25	1.5356e-02	1.2000e-02	2.6519e-02	3.1007e-02
0.50	6.9433e-02	2.4254e-02	2.6583e-02	3.7466e-02
0.75	1.6207e-01	8.3164e-02	3.7525e-02	2.4763e-02
0.90	2.3213e-01	1.5015e-01	7.7367e-02	3.7252e-02

Table 4: Maximum absolute error of Example 2 for $\alpha = 0.8$ and (N, M)

ε	(32,4)	(64,8)	(128,16)	(256,32)
10^{0}	4.7122e-03	2.1731e-03	9.9120e-04	6.5133e-04
10^{-2}	1.2994e-01	6.9327e-02	3.1071e-02	1.7352e-02
10^{-4}	1.7689e-01	9.7089e-02	4.7711e-02	2.4097e-02
10^{-6}	1.8435e-01	1.0291e-01	5.0301e-02	2.5351e-02
10^{-8}	1.8435e-01	1.0291e-01	5.0760e-02	2.5578e-02
10^{-10}	1.8435e-01	1.0291e-01	5.0760e-02	2.5578e-02
10^{-12}	1.8435e-01	1.0291e-01	5.0760e-02	2.5578e-02
10^{-14}	1.8435e-01	1.0291e-01	5.0760e-02	2.5578e-02
10^{-16}	1.8435e-01	1.0291e-01	5.0760e-02	2.5578e-02
$E^{N,M}$	1.8435e-01	1.0291e-01	5.0760e-02	2.5578e-02
$r^{N,M}$	0.84106	1.0196	0.98879	

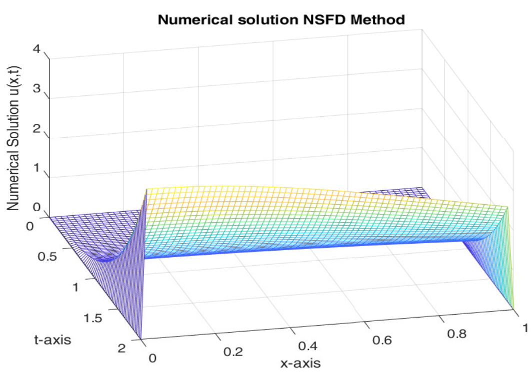

Figure 1: Numerical solution of Example 1 for $N = M = 64$ and $\varepsilon = 10^{-10}$.

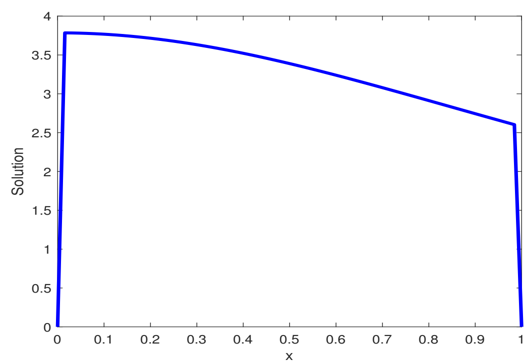

Figure 2: Boundary layer formation of Example 1.

The effectiveness of the nonstandard finite difference approach is verified on the problems 1 and 2 using different spatial and temporal sizes. The maximum absolute errors for different options of α and fixed ε are reported in Tables 1 and 3. The statistics show that when α drops, the maximum absolute errors likewise drops, suggesting that the fractional order model does an improved task of representing real-world problems than the integer order model. Tables 2 and 4 demonstrate the maximum absolute value that was attained. As the step size is reduced and $\varepsilon \to 0$, the numerical results converge uniformly and the rate of convergence becomes order one as noticed in the tables. Figures 1 and 3 also display the behavior of the solution as determined by the numerical approach. The boundary layer behavior of the problem, which features parabolic boundary layers at $w = 0$ and $w = 1$ is shown in Figures 2 and 4. In conclusion, Figures 5 and 6 exhibit the log-log scale of the maximum absolute error. This scale indicates a monotonic decline in the maximum absolute error and perturbation value, so verifying a correspondence with theoretical results.

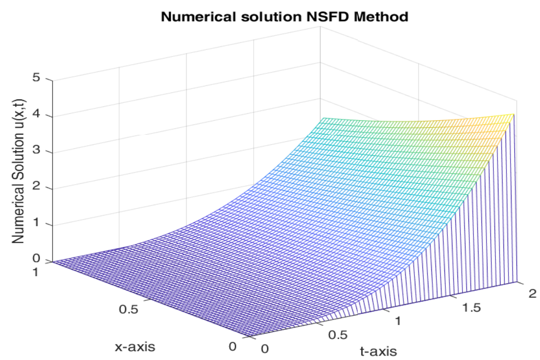

Figure 3: Numerical solution of Example 2 for $N = M = 64$ and $\varepsilon = 10^{-10}$.

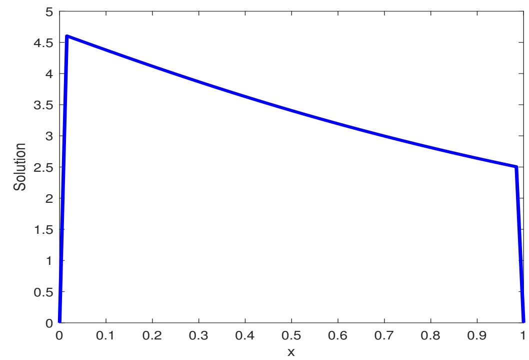

Figure 4: Boundary layer formation of Example 2.

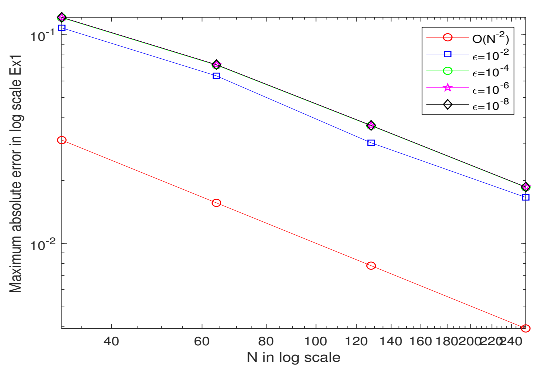

Figure 5: Log-log scale plot for Example 1.

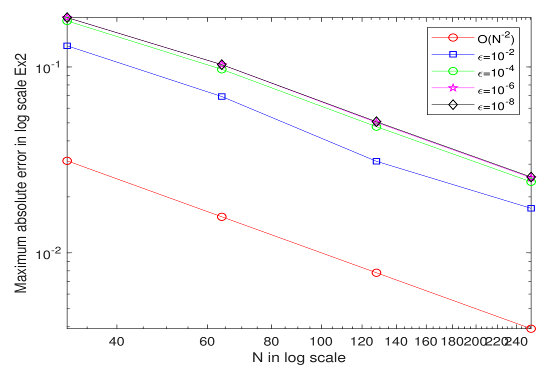

Figure 6: Log-log scale plot for Example 2.

6 Conclusions

For the one-dimensional initial boundary value problem of singularly perturbed time fractional delay parabolic reaction-diffusion equations, a nonstandard finite difference approach is implemented. Caputo fractional derivative combined with the implicit Euler method are employed to discretize the time derivative. Through the use of the nonstandard finite difference method, the spatial derivative is discretized. The convergence analysis of the scheme is proven to be accurate of order $O(h^2 + (\Delta t)^{2-\alpha})$. The scheme has twin layers at $w = 0$ and $w = 1$, and the findings from numerical examples verified an agreement. The scheme is uniformly convergent.

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