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Evaluating cost efficiency of decision-making units in an uncertain environment

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Abstract. The efficiency evaluation of organizational units provides managers with a perspective on the current state of the organization and solutions for their improvement. One of the methods of organizational evaluation is to determine the organization's minimum cost or cost efficiency. Cost efficiency in practice can be calculated when the input prices are available. In traditional models of cost efficiency, input and output data are crisp. However, there are situations where input and/or output may be imprecise. For such cases, experts are invited to model their opinion. Then uncertainty theory can be applied which is introduced by Liu as a mathematical branch rationally dealing with belief degrees. In this paper, a model is proposed to estimate the cost of decision-making units in the uncertain environment, where inputs and outputs are uncertain but the input prices are crisp. Several theorems are presented to discuss some features of the introduced model. When the data has a linear distribution, the cost efficiencies of the decision-making units are calculated. Also, the model is implemented on two numerical examples. The obtained results are compared with previous results. Finally, in the presence of input prices, a different cost efficiency score for the decision-making units is obtained. The proposed model helps decision-makers to improve their performance by using experts' opinions.

Keywords: Data envelopment analysis, cost efficiency, uncertainty, evaluating, decision-making units.
AMS Subject Classification 2010: 34A34, 65L05.

1 Introduction

Managers of organizations, with the development of technology, need to evaluate their organizational units or decision-making units (DMUs) to identify the strengths and weaknesses of each sub-unit in order to improve them. The efficiency evaluation of DMUs provides managers with a perspective on the current state of the organization and solutions for their improvement. One of the methods of DMUs evaluation is to determine the organization's minimum cost or cost efficiency. Cost efficiency in practice

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can be calculated when the input prices are available. Farrell first proposed the idea of cost efficiency of DMUs in the presence of input prices [7]. The concept of cost efficiency was developed by Färe and Grosskopf [5]. They measured the cost efficiency of DMUs as the ratio of the least possible cost of producing the unit's current output using the prices paid by the same unit to the actual cost incurred.

Many researchers have conducted studies on cost efficiency. Among the recent studies, the following can be mentioned. Soleimani-Chamkhorami and Ghobadi examined the cost efficiency of bank branches [24], and Piran et al. explored the cost efficiency of incubators over six years [16]. Fakharzadehet al, who investigated the gap between parallel systems and series of optimal system designed and presented a model for parallel systems [4]. Also, Soofizadeh and Fallahnejhad develop a method based on bargaining for evaluation in network DEA considering shared inputs and undesirable outputs [25]. Cost efficiency is also studied in other researchers papers [2, 3, 5, 23]. Most cost-efficiency studies have assumed that all data are certain and accurate.

In many real-world problems, however, accurate data are not available, and that is why stochastic, fuzzy, interval, and uncertainty approaches have been presented to solve the challenge of data certainty. The first attempt to overcome the challenge of uncertain data was the use of a stochastic theory which was done by researchers such as Sengupta [22]. They considered the data as random or stochastic, with each proposing different methods to solve the problems.

Fuzzy set theory is another method used to deal with imprecise data in Data Envelopment Analysis (DEA). Sengupta who first introduced fuzzy set theory into DEA used the sizes of the fuzzy set in the field of DEA [21]. Many researchers, including Kao and Liu, developed fuzzy set theory when some inputs and outputs are fuzzy numbers [10]. Jahanshahloo et al. applied the concept of fuzzy set theory to cost efficiency [9]. They proposed a minimum cost calculation model for cases when the input prices are certain, but the inputs and outputs are fuzzy. Saati et al, proposed a new fuzzy DEA method for clustering operating units in a fuzzy environment by considering the priority between the clusters and the priority between the operating units in each cluster simultaneously [20]. There have been many studies on cost efficiency with fuzzy data. Among them, Pourmahmood and Bafekr considered a case where both input prices as well as inputs and outputs are fuzzy-data types [17]. Another approach used when dealing with uncertain data is interval data. Cost efficiency in the interval data environment was introduced by Comonho and Dayson [1]. In this method, since commodity price data were in a state where the highest and lowest prices were certain, the prices would fluctuate between these two numbers. Interval DEA has also been used by researchers such as Mombini et al. [14].

The organization of the paper is as follows. In Section 2 we provide a literature review of the related studies. In Section 3, we describe basic concepts including cost efficiency and Liu's uncertainty theory. In Section 4, we present an uncertain cost efficiency evaluation model to evaluate DMUs with uncertain inputs and outputs, and will prove some theorems. The proposed model is applied and analyzed on two numerical examples in Section 5. Finally in section 6 we give of conclusions and possible future research directions.

2 Literature review and research gap

The following two subsections provide literature review of the related studies and research gap.

2.1 Literature on the background

Despite the three approaches presented in many practical everyday problems, it is not possible to predict the type or amount of data. This and other similar cases further highlight the significance of having data that do not have a predictable distribution or frequency. To overcome the challenge of uncertainty of the aforementioned data, Liu proposed the uncertainty theory, indicating that experts are preferred to it [12]. This theory works well in dealing with the problems that a specialist and his view transform the uncertain situation into quantitative data. Also in some very unpredictable phenomena, the expert is the only option relying on his/her opinion about the potential outcome [12].

There are fundamental differences between the assumed data and the fuzzy, stochastic, and interval data in Liu's proposed approach so there is no sample for estimating the distribution function [12]. According to the uncertainty theory, the opinion of relevant experts can be used in unexpected events such as floods, wars, earthquakes, accidents, or even rumors. Uncertainty theory has been used by many researchers in different fields. For example, Wen et al. presented the DEA model in an uncertain environment [27]. Due to the complexity of the uncertain model, they presented an equivalent certain model. Mohammadnejad and Ghaffari-Hadigheh introduced another uncertaint model to obtain the highest degree of belief, which provided a relatively more optimistic view for DMUs [13]. Lio and Liu presented another model to evaluate the efficiency of DMUs in which they used the expected value for the objective function and constraints [11]. Ghaffari-Hadigheh and Lio applied uncertainty theory to network DEA [8]. In their model, input and output data were provided by experts and were used to evaluate DMUs in an uncertain environment. Pourmahmood and Bagheri evaluated the efficiency of DMUs in a case where the units had a two-stage network structure [18]. The inputs, outputs, and intermediate products in their proposed model were uncertain. Peykani et al. presented a new method for ranking efficient units in uncertaint environment [15]. Pourmahmoud and Bagheri applied the Malmquist Productivity Index (MPI) concept in the nonparametric approach of DEA to calculate the efficiency of systems over different periods of time under uncertain conditions. They considered MPI when inputs and outputs are belief degrees of experts [19].

2.2 Research gap

Take the stock price of a company on the stock exchange as an example where it is not possible to predict inputs and outputs. Also, as an observed case, in April 2018, Khouzestan, Lorestan, and Färs provinces as well as other parts of Iran were hit by a large flood whose return period was predicted to be 1000 years which could not be predicted by uncertain (stochastic, fuzzy, or interval) data. However, several meteorologists had already warned that a devastating flood would occur that year.

Consider a DMU that its inputs are the fuzzy trapezoid number. The membership function of an input is as follows:

$$\mu(x) = \begin{cases} 0, & x \leq 2.5, \\ x-2.5, & 2.5 \leq x \leq 3.5, \\ 1, & 3.5 \leq x \leq 4.5, \\ 5.5-x, & 4.5 \leq x \leq 5.5, \\ 0, & x \geq 5.5. \end{cases} \tag{1}$$

According to the membership function (1), the possibility that the system receives score 3.5 is equal to when it does not receive this score and this is 1 for both of them. Obviously, this is a contradiction and fuzzy concept is incapable of dealing with such challenges.

Also, consider data collection of 20 DMUs for two periods of time at which one input is equipment. If the expert regards it as a fuzzy concept with the following membership function, the possibility that ”the amount of equipment is 200 is equal to one

$$\mu(x) = \begin{cases} (x-150)/50, & 150 \leq x \leq 200, \\ (250-x)/50, & 200 \leq x \leq 250. \end{cases} \tag{2}$$

However, the degree of belief in the ”exact amount of 200 for equipment” is near zero, and no one can believe that the ”exact amount of 200 for equipment” is correct. On the other hand, the possibility that the system receives the score of 200 is the same as when it does not, which is a contradiction. Therefore, fuzzy logic is not appropriate to model belief degrees. Furthermore, in the lack of historical data, probability theory would be unable to generate practical results.

The main goal of assessing performance is to recognize inefficient systems and improve them so that the system is able to perform better in future. One of the main issues in measuring the performance of systems is cost efficiencies. It is not possible to evaluate cost efficiency of the DMUs whose data are uncertain with the abovementioned methods. Thus, a challenge can be raised in cost efficiency evaluation of the DMUs which is the subject of this study. Liu’s uncertainty theory can be used to deal with this challenge. The next section studies this concept.

3 Basic concepts

In this section, first some basic concepts of cost efficiency are presented. then uncertainty theory and Lio and Liu’s models are presented in Section 3.2.

3.1 Liu’s Cost efficiency

In different cost efficiency models, the possibility of producing the company’s current output with the least possible cost is evaluated. In other words, in cost efficiency, DMUs efficiency is examined in a situation where, in addition to inputs and outputs, prices for inputs are also considered.

Tone showed that in Färe and Grosskopf’s model, if two units have identical input and output values but different input prices, then identical cost efficiency values are obtained for these two units, which is a challenge [26]. Thus, in order to overcome this challenge, he expanded the production possibility set space to the cost space. Tone proposed model (3) to evaluate cost efficiency of DMUs [26]:

$$\begin{aligned} C_k^* = \min & \sum_{i=1}^{m} \bar{x}_i \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j \bar{x}_{ij} \leq \bar{x}_i, \quad i = 1,2,\ldots,m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}, \quad r = 1,2,\ldots,s, \\ & \lambda_j \geq 0, \quad j = 1,2,\ldots,n, \\ & \bar{x}_i \geq 0, \quad i = 1,2,\ldots,m, \end{aligned} \tag{3}$$

where $\bar{x}_{ij}$ is the cost of the i-th ($i = 1,2,,m$) input of the DMU$_j$ ($j = 1,2,,n$) and $\bar{x}_i$ is the minimum value of the i-th ($i = 1,2,,m$) input cost of the unit under the evaluation DMU$_k$.

Definition 1. *The cost efficiency of DMU$_k$ is defined as the ratio of the minimum cost of $\sum_{i=1}^{m}\bar{x}_i^*$ to the actual observation cost of $\sum_{i=1}^{m}\bar{x}_{ik}$ [26],i.e.,*

$$CE_k = \frac{\sum_{i=1}^{m}\bar{x}_i^*}{\sum_{i=1}^{m}\bar{x}_{ik}},$$

where $(\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_m^*)$ is the optimal solution obtained from model (1) for DMU$_k$.

3.2 Liu's uncertainty

In 2007, Liu introduced the uncertainty theory based on the measurable space $(\Gamma, \mathscr{L})$ [12]. He claimed that uncertainty theory is an alternative to measure theory, so he named each measurable set as an event and defined the uncertainty measure $\mathscr{M}$ on the σ-algebra L. The degree of belief of a person about the occurrence or non-occurrence of the event Λ was shown by the degree of belief $\mathscr{M}\{\Lambda\}$.

Liu defined the uncertain variable as a measurable function from the uncertain space to the set of real numbers as follows [12]. The variable ξ is a function ξ from the uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ to the set of real numbers such that $\xi \in \mathscr{B}$ is an event for every Borel set $\mathscr{B}$. The uncertainty distribution Φ of an uncertain variable ξ is defined by Liu in the following way [12]:

$$\Phi(x) = \mathscr{M}\{\xi \leq x\}, \quad \forall x \in \mathbb{R}. \tag{4}$$

In practice, uncertain variables use different distribution functions. Among the distribution functions that are used in uncertainty theory are linear, zigzag, empirical distribution functions, etc. Since in this study two linear and zigzag uncertain distribution functions are used, they are introduced as follows.

Linear uncertainty disttibution. The linear uncertainty distribution function for an uncertain variable is as follows:

$$\Phi(x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x \geq b, \end{cases}$$

which is denoted by $\mathscr{L}(a,b)$ where a and b are real numbers and $a \leq b$.

Zigzag uncertainty disttibution. The zigzag uncertainty distribution function is as follows:

$$\Phi(x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{2(b-a)}, & a \leq x \leq b, \\ \dfrac{x+c-2b}{2(c-a)}, & b \leq x \leq c, \\ 1, & x \geq c. \end{cases}$$

This distribution is represented by $\mathscr{Z}(a,b,c)$, where a, b, and c are real numbers and $a < b < c$.

The expectation of the uncertain variable ξ which has the uncertain distribution Φ is calculated as follows:

$$E[\xi] = \int_0^{+\infty} (1-\Phi(x))\mathrm{d}x - \int_{-\infty}^{0} \Phi(x)\mathrm{d}x.$$

When is regular, it will be

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha))\mathrm{d}\alpha,$$

provided that, at least, one of the above integrals exists.

Uncertain programming is a type of mathematical programming in which the decision variables are uncertain. Thus, based on uncertainty theory, it is not possible to directly minimize the objective uncertain function, so the minimization can be done on its expected value [12]:

$$\min_x E[f(x,\xi)].$$

On the other hand, the uncertain constrains in uncertain programming are expressed as the level of belief in establishing inequality, that is:

$$\mathscr{M}\left\{g_j(x,\xi) \le 0\right\} \ge \alpha_j, \quad j = 1,2,\ldots,p. \tag{5}$$

Hence, the constrained uncertain programming is written as follows:

$$\begin{aligned} &\min_x E[f(x,\xi)] \\ &\quad \text{s.t. } \mathscr{M}\left\{g_j(x,\xi) \le 0\right\} \ge \alpha_j, \quad j = 1,2,\ldots,p. \end{aligned} \tag{6}$$

Many researchers used uncertain programming in various fields. Lio and Liu are among them who used the expected value form by changing the programming constraints [11]. The proposed model of Lio and Liu is as follows:

$$\begin{aligned} \min_{u,v}\ & \theta = E\left[\frac{v^T\tilde{y}_k}{u^T\tilde{x}_k}\right] \\ \text{s.t. } & E\left[\frac{v^T\tilde{y}_j}{u^T\tilde{x}_j}\right] \le 1, \quad j = 1,2,\ldots,n, \\ & u \ge 0, \qquad v \ge 0. \end{aligned}$$

The above model evaluates the efficiency of DMUs in situations where the inputs and outputs are uncertain. In the seqvel, a theorem is proved which underlies our work.

Theorem 1. *Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. If f is strictly increasing with respect to $\xi_1,\xi_2,\ldots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\ldots,\xi_n$, then*

1. *$\xi = f(\xi_1,\xi_2,\ldots,\xi_n)$ is an uncertain variable and its inverse uncertainty distribution is as follow:*

$$\Phi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha),\ldots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha)\right).$$

2. *the uncertain variable* $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ *has an expected value as follows:*

$$E[f] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)\right) \mathrm{d}\alpha.$$

There are situations where the information of data should be determined by expert'opinion. Uncertainty theory can be used to evaluate cost efficiency of a system in these circumstances. Accordingly, in the next section, we will utilize the uncertainty theory to evaluate cost efficiency of systems with uncertain variables and crisp prices of inputs.

4 Uncertain cost efficiency

Many studies have been performed to evaluate the cost efficiency of DMUs with certain data [2, 3, 5–7] and [23]. Among these studies, Tone's model examined cost efficiency evaluation of DMUs in certain mode. On the other hand, Lio and Liu used expected values to calculate the efficiency of DMUs in the case where the input and output data are uncertain [23]. The methods presented in the studies are not suitable for evaluating the cost efficiency of DMUs with uncertain data. Next, a model is presented based on Tone's model and Lio and Liu's model that deals successfully with the uncertainty data challenges raised in [11] and [26]. This paper studies a state of cost efficiency where the inputs and outputs are uncertain while the input prices are certain.

4.1 Proposed model

Suppose DMU_j $(j = 1, 2, \dots, n)$ are evaluated, so that each unit consumed m nonnegative uncertain inputs $\tilde{X}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \dots, \tilde{x}_{mj})$ with certain prices $P_j = (p_{1j}, p_{2j}, , p_{mj})$ to produce s nonnegative uncertain output $\tilde{Y}_j = (\tilde{y}_{1j}, \tilde{y}_{2j}, \dots, \tilde{y}_{sj})$. It is also assumed that the return to scale of the production technology is constant. In order to use Lio and Liu's model in cost efficiency evaluation of units, it is necessary to transform the problem space into the space of uncertain prices [11]. In the transformation of the problem space, the multiplication of the i-th uncertain input of the j-th unit $\tilde{x}_{ij}$ in the i-th certain price of the j-th input p_{ij} is used, which is

$$\tilde{\bar{x}}_{ij} = p_{ij}\tilde{x}_{ij}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.$$

Symbol $\sim$ indicates the uncertain variables.

To evaluate DMU_k in this case, Tone's model can be rewritten as below [26]:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} \bar{x}_i \\ \text{s.t.} \quad & E\Big(\sum_{j=1}^{n} \lambda_j \tilde{\bar{x}}_{ij}\Big) - \bar{x}_i \le 0, \quad i = 1, 2, \dots, m, \\ & E\Big(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}\Big) - \tilde{y}_{rk} \ge 0, \quad j = 1, 2, \dots, s, \\ & \lambda_j \ge 0, \quad j = 1, 2, \dots, n, \\ & \bar{x}_i \ge 0, \quad i = 1, 2, \dots, m. \end{aligned} \tag{7}$$

Let $\bar{x}^* = (\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_m^*)$ be the optimal solution of model (7) and $\sum_{i=1}^{m} \bar{x}_i^*$ be the optimal objective function value. The cost efficiency of DMU$_k$ in this case is represented by $\widetilde{CE}_k$ and is defined as follows.

Definition 2. *The efficiency of the uncertain cost DMU$_k$ is defined as the ratio of the minimum cost $\sum_{i=1}^{m} \bar{x}_i^*$ to the observed uncertain the cost $\sum_{i=1}^{m} \tilde{\bar{x}}_{ik}^*$, i.e.,*

$$\widetilde{CE}_k = \frac{\sum_{i=1}^{m} \bar{x}_i^*}{E(\sum_{i=1}^{m} \tilde{\bar{x}}_{ik})}. \tag{8}$$

Cost efficiency by the above definition is possible when the third proposed model is feasible and bounded. This will be reviewed later.

Theorem 2. *Uncertain cost efficiency model (7) is feasible and has an optimal boundary solution.*

Proof. The following is a feasible solution for model (7):

$$\begin{cases} \lambda_k = 1, \\ \lambda_j = 0, \quad j = 1, 2, \ldots, n, j \neq k, \\ \tilde{x}_i = \tilde{\bar{x}}_{ik}, \quad i = 1, 2, \ldots, m. \end{cases}$$

Note that according to Liu, any certain variable can be written as an uncertain variable [12]. By placing the above solution in the constraints, we will have:

$$\begin{aligned} E\Big(\sum_{j=1}^{n} \lambda_j \tilde{\bar{x}}_{ij}\Big) - \tilde{x}_i &= E(\tilde{\bar{x}}_{ik}) - \tilde{x}_i = 0, \\ E\Big(\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \tilde{y}_{rk}\Big) &= E(\tilde{y}_{rk} - \tilde{y}_{rk}) = 0. \end{aligned}$$

On the other hand, the nonnegativity of the inputs and their prices results in the nonnegativity of $\tilde{\bar{x}}_{ik}$ and finally the non-negativity of $\bar{x}_i = \tilde{\bar{x}}_{ik}$.

As a result, the introduced solution is applied within the constraints and is a feasible solution for model (7). Hence, the first part of the theorem is established.

Due to the nonnegativity of $\bar{x}_i, i = 1, 2, ..., m$ and at least one of them being nonzero, it follows that the value of the objective function is always positive. According to the type of objective function, the problem is bounded and the theorem is established. □

Since the data in model (7) are uncertain, it is not possible to calculate the model using common methods. In the following, we try to provide of model under uncertain conditions with the certain form. In this case, the expected value of each of the uncertain values is replaced by their uncertain values, and in this way, the expected value is used as a certain value in modeling.

Theorem 3. *Suppose DMU$_j$; $j = 1, 2, \ldots, n$, $\tilde{\bar{x}}_{ij}$; $i = 1, 2, \ldots, m$ are the uncertain cost inputs, $\tilde{y}_{rj}$, $r = 1, 2, \ldots, s$, are independent uncertain outputs that have regular distributions Φ_{ij} and Ψ_{rj}, respectively.*

The certain form of model (7) *can be written as follows:*

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} \bar{x}_i \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j \Big(\int_0^1 \Phi_{ij}^{-1}(\alpha)\mathrm{d}(\alpha) \Big) - \tilde{x}_i \leq 0, \quad i = 1,2,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j \Big(\int_0^1 \Psi_{rj}^{-1}(\alpha)\mathrm{d}(\alpha) \Big) - \Big(\int_0^1 \Psi_{rk}^{-1}(1-\alpha)\mathrm{d}(\alpha) \Big) \geq 0, \quad r = 1,2,\ldots,s, \\
& \lambda_j \geq 0, \quad j = 1,2,\ldots,n, \\
& \tilde{x}_i \geq 0, \quad i = 1,2,\ldots,m.
\end{aligned} \tag{9}
$$

Proof. According to the assumption that the inputs $\tilde{\tilde{x}}_{ik}$, and outputs $\tilde{y}_{rj}$ are uncertain, consider the following equations.

$$\bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)^t, \tag{10}$$

$$\tilde{\tilde{X}}_i = (\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{in})^t, \quad i = 1,2,\ldots,m, \tag{11}$$

$$\tilde{Y}_r = (\tilde{y}_{r1}, \tilde{y}_{r2}, \ldots, \tilde{y}_{rn})^t, \quad r = 1,2,\ldots,s, \tag{12}$$

$$\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^t. \tag{13}$$

The first parts of each constraint of model (7) are functions of uncertain variables. On the other hand, functions of independent variables are uncertain variables for which inverse distribution functions can be considered [12]. Hence, suppose

$$
\begin{aligned}
f_i^1(\Lambda, \tilde{\tilde{X}}_i) &= \sum_{j=1}^{n} \lambda_j \tilde{\tilde{x}}_{ij}, \quad i = 1,2,\ldots,m, \\
f_r^2(\Lambda, \tilde{Y}_r, \tilde{y}_{rk}) &= \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \tilde{y}_{rk}, \quad r = 1,2,\ldots,s.
\end{aligned}
$$

The function f_i^1 is increasing with respect to Λ and $\tilde{\tilde{X}}_i$. Also, f_r^2 function is increasing with respect to $\tilde{Y}_r$ and Λ, but is decreasing in relation to $\tilde{y}_{rk}$. Therefore, based on the Theorem 1 [12], the inverse uncertain distribution of these functions is as follows:

$$
\begin{aligned}
[F_i^1]^{-1}(\alpha) &= \sum_{j=1}^{n} \lambda_j \Phi_{ij}^{-1}(\alpha), \quad i = 1,2,\ldots,m, \\
[F_r{}^2]^{-1}(\alpha) &= \sum_{j=1}^{n} \lambda_j \Psi_{rj}^{-1}(\alpha) - \Psi_{rk}^{-1}(1-\alpha), \quad r = 1,2,\ldots,s.
\end{aligned}
$$

By applying the expected value function on the above functions (Theorem 1) we will have:

$$
\begin{aligned}
E\big(f_i^1(\Lambda, \tilde{\tilde{X}}_i)\big) &= \int_0^1 \Big(\sum_{j=1}^{n} \lambda_j \Phi_{ij}^{-1}(\alpha) \Big) \mathrm{d}(\alpha), \quad i = 1,2,\ldots,m, \\
E\big(f_r^2(\Lambda, \tilde{Y}_r, \tilde{y}_{rk})\big) &= \int_0^1 \Big(\sum_{j=1}^{n} \lambda_j \Psi_{rj}^{-1}(\alpha) - \Psi_{rk}^{-1}(1-\alpha) \Big) \mathrm{d}(\alpha), \quad r = 1,2,\ldots,s.
\end{aligned}
$$

Since the integral has linear property, the above relations can be written as follows:

$$E\left(f_i^1(\Lambda,\tilde{\tilde{X}}_i)\right) = \sum_{j=1}^{n}\lambda_j\Big(\int_0^1 \Phi_{ij}^{-1}(\alpha)\mathrm{d}(\alpha)\Big), \quad i=1,2,\ldots,m,$$
$$E\left(f_r^2(\Lambda,\tilde{Y}_r,\tilde{y}_{rk})\right) = \sum_{j=1}^{n}\lambda_j\Big(\int_0^1 \Psi_{rj}^{-1}(\alpha)\mathrm{d}(\alpha)\Big) - \Big(\int_0^1 \Psi_{rk}^{-1}(1-\alpha)\mathrm{d}(\alpha)\Big), \quad r=1,2,\ldots,s.$$

Therefore, considering the above relationships and according to Theorem 1, model (7) is balanced to model (9). So, the proof is complete. □

Model (9) is a general model for an uncertain state where no specific distribution is considered for the variables, but, in the real world, variables follow different distributions. Some of these distributions are presented in the book by Liu [12]. In this part, model (9) is rewritten by considering linear distribution for input variables and zigzag distribution for output variables.

4.2 Special case

The expert's opinion about the input variables follows the linear uncertain distribution and about the output variables follows the zigzag distribution. In other words, the inputs should be linear uncertain $\mathscr{L}(a_{ij},b_{ij})$ and the outputs should be zigzag uncertain $\mathscr{Z}(a_{rj},b_{rj},c_{rj})$. Considering distribution functions for inputs and outputs data, model (9) is written as follows:

$$\begin{aligned}
\min\quad & \sum_{i=1}^{m}\bar{x}_i \\
\text{s.t.}\quad & \sum_{j=1}^{n}\frac{1}{2}\lambda_j(a_{ij}+b_{ij})-\tilde{x}_i \le 0, \quad i=1,2,\ldots,m, \\
& \sum_{j=1}^{n}\frac{1}{4}\lambda_j(a_{rj}+b_{rj}+c_{rj})-\frac{1}{4}(2b_{rk}+a_{rk}+c_{rk}) \ge 0, \quad r=1,2,\ldots,s, \\
& \lambda_j \ge \varepsilon, \quad j=1,2,\ldots,n, \\
& \bar{x}_i \ge 0, \quad i=1,2,\ldots,m.
\end{aligned} \tag{14}$$

The above model is a certain linear one that can be easily solved via linear program solvers. After running model (14) on the data of the DMUs, the optimal values $\bar{x}_i^*$ and λ_j^* are obtained. By having the optimal values as well as the expected value of the costs incurred, the efficiency of each DMU is obtained with the help of Definition 1.

5 Numerical examples

In this paper, a model is presented to evaluate the cost efficiency of units that have uncertain inputs and outputs. To show the power and applicability of the model, two numerical examples in different situations are examined and compared.

Example 1. Consider five DMUs of Lio and Liu's paper [11]. The information related to three linear uncertain inputs and three zigzag uncertain outputs of these DMUs is reported in Table 1.

Table 1: Data for lio & liu article.

#DMU$_j$	$\tilde{x}_{1j}$	$\tilde{x}_{2j}$	$\tilde{x}_{3j}$	$\tilde{y}_{1j}$	$\tilde{y}_{2j}$	$\tilde{y}_{3j}$
1	$\mathscr{L}(4,7)$	$\mathscr{L}(5,8)$	$\mathscr{L}(4,8)$	$\mathscr{Z}(1,2,3)$	$\mathscr{Z}(1,2,4)$	$\mathscr{Z}(1,3,4)$
2	$\mathscr{L}(2,5)$	$\mathscr{L}(3,6)$	$\mathscr{L}(1,3)$	$\mathscr{Z}(4,7,10)$	$\mathscr{Z}(8,10,12)$	$\mathscr{Z}(9,11,13)$
3	$\mathscr{L}(3,4)$	$\mathscr{L}(4,6)$	$\mathscr{L}(3,7)$	$\mathscr{Z}(2,3,4)$	$\mathscr{Z}(2,6,8)$	$\mathscr{Z}(3,5,7)$
4	$\mathscr{L}(2,5)$	$\mathscr{L}(1,6)$	$\mathscr{L}(1,3)$	$\mathscr{Z}(4,6,8)$	$\mathscr{Z}(5,7,8)$	$\mathscr{Z}(5,7,9)$
5	$\mathscr{L}(1,6)$	$\mathscr{L}(1,3)$	$\mathscr{L}(1,3)$	$\mathscr{Z}(10,12,14)$	$\mathscr{Z}(5,6,7)$	$\mathscr{Z}(8,9,10)$

To show the effect of input prices in the evaluation of units, two different cases are considered, one of which is the identical input prices for all units and the second case is where units cannot obtain their inputs at the same prices as other units.

The first case: Cost efficiency evaluation under identical input prices

Consider a case where all DMUs receive their i-th input from a given source. In this case, prices for each input are assumed to be identical across all DMUs. The price information is listed below in Table 2.

Table 2: Cost of Uncertain inputs.

DMU$_j$	p_{1j}	p_{2j}	p_{3j}
$j = 1,2,\dots,n$	3	12	4

In order to provide information related to the proposed model, it is necessary to multiply the input prices by the input values of the DMUs. Afterward, the corresponding values are listed in columns 2 to 4 of Table 3.

Table 3: Cost efficiency with same prices.

#DMU$_j$	$\tilde{x}_{1j}$	$\tilde{x}_{2j}$	$\tilde{x}_{3j}$	Proposed cost efficiency	Lio & Liu's efficiency
1	$\mathscr{L}(12,21)$	$\mathscr{L}(60,96)$	$\mathscr{L}(16,32)$	0.1345	0.1465
2	$\mathscr{L}(6,15)$	$\mathscr{L}(36,72)$	$\mathscr{L}(4,12)$	0.9770	1.0000
3	$\mathscr{L}(9,12)$	$\mathscr{L}(48,72)$	$\mathscr{L}(12,28)$	0.4305	0.5028
4	$\mathscr{L}(6,15)$	$\mathscr{L}(12,72)$	$\mathscr{L}(4,12)$	0.7903	0.9407
5	$\mathscr{L}(3,18)$	$\mathscr{L}(12,36)$	$\mathscr{L}(4,12)$	1.0000	1.0000

The fifth column of Table 3 shows the results of the implementation of the proposed model (unit cost efficiency) on the data of Table 1 using input prices in Table 2 and the sixth column shows the results of Lio and Liu [11] model implementation (unit efficiency) regardless of the input price.

This table shows that the number of efficient DMUs is not the same in both models and two DMUs, namely numbers 2 and 5, are efficient in Lio and Liu's model [11], while DMU_2 in the proposed model is not efficient and only DMU_5 is efficient. In other words, while DMU_2 is a technically efficient unit, it has lost its efficiency in the presence of prices. Also, despite allocating the identical prices to inputs, the efficiencies obtained for DMUs 1, 3, and 4 in the two models are different. Therefore, efficiency scores are affected by prices. In other words, considering and ignoring input prices would alter the efficiency of the units in two cases.

Second case: Cost efficiency evaluation in the conditions of different input prices

In this case, it is assumed that each DMU supplies its inputs from different sources. Hence, there are different prices for each input in each DMU. Considering the prices of inputs in different DMUs, columns 4 to 6 of Table 4 are obtained. The result of applying the proposed model and Lio and Liu's model on the data of columns 4 to 6 of the same table are listed in columns 7 and 8, respectively [11].

Table 4: Cost efficiency with same prices.

#DMU_j	p_{1j}	p_{2j}	p_{3j}	$\tilde{\tilde{x}}_{1j}$	$\tilde{\tilde{x}}_{2j}$	$\tilde{\tilde{x}}_{3j}$	Proposed cost efficiency	Lio & Liu's efficiency
1	6	5	15	$\mathscr{L}(24,42)$	$\mathscr{L}(25,40)$	$\mathscr{L}(60,120)$	0.0793	0.1465
2	6	5	15	$\mathscr{L}(12,30)$	$\mathscr{L}(15,30)$	$\mathscr{L}(14,45)$	0.5873	1.0000
3	1	1	2	$\mathscr{L}(33,44)$	$\mathscr{L}(28,42)$	$\mathscr{L}(48,112)$	1.0000	0.5028
4	11	6	17	$\mathscr{L}(6,15)$	$\mathscr{L}(12,72)$	$\mathscr{L}(4,12)$	0.7903	0.9407
5	13	10	18	$\mathscr{L}(13,96)$	$\mathscr{L}(10,30)$	$\mathscr{L}(18,54)$	0.7291	1.0000

Observing Table 4, in the comparison between the two presented models by Liu and the proposed model, it can be seen that the introduction of different prices for the inputs in the proposed model has caused a difference in the number of efficient DMUs. That is, in Lio and Liu's [11] model, DMU_2 and DMU_5 were efficient, while in the proposed cost efficiency model, only DMU_3 is efficient and the rest are inefficient. In other words, while DMU_3 had a technically low efficiency score, it could be efficient by supplying its inputs at lower prices from sources different from other units which indicates the further effect of prices. This suggests that an inefficient DMU can be made efficient by providing its inputs at lower prices than from a cost efficiency point of view. Also, the efficiency score is different in two models for the same DMUs. Thus, it can be concluded that in the presence of different prices for inputs, the number of efficient units, the efficiency number, as well as the efficiency of a unit are different from the model with no prices.

Example 2. Consider 20 wastewater treatment systems as DMUs for evaluation [18]. In order to show the effect of input prices in the evaluation, information about prices has been added to them in two cases. The information related to the three linear uncertain inputs and the three zigzag uncertain outputs of these DMUs is outlined in Table 5.

Similar to the previous example, two different cases are considered in the evaluation of the units: the first case is the identical input prices and the second is different input prices in each of the units.

The first case: Cost efficiency evaluation under conditions of the identical input prices

In this case, prices are considered different for each input in all DMUs. This can be related to the situation where each DMU supplies its input from a separate source. The price information is listed in Table 6.

Table 5: Data for Pourmahmoud and Bagheri Article.

#DMU$_j$	$\tilde{x}_{1j}$	$\tilde{x}_{2j}$	$\tilde{x}_{3j}$	$\tilde{y}_{1j}$	$\tilde{y}_{2j}$	$\tilde{y}_{3j}$
1	$\mathscr{L}(4,7)$	$\mathscr{L}(5,8)$	$\mathscr{L}(4,8)$	$\mathscr{Z}(1,2,3)$	$\mathscr{Z}(1,2,4)$	$\mathscr{Z}(1,3,4)$
2	$\mathscr{L}(2,5)$	$\mathscr{L}(3,6)$	$\mathscr{L}(1,3)$	$\mathscr{Z}(4,7,10)$	$\mathscr{Z}(8,10,12)$	$\mathscr{Z}(9,11,13)$
3	$\mathscr{L}(3,4)$	$\mathscr{L}(4,6)$	$\mathscr{L}(3,7)$	$\mathscr{Z}(2,3,4)$	$\mathscr{Z}(2,6,8)$	$\mathscr{Z}(3,5,7)$
4	$\mathscr{L}(2,5)$	$\mathscr{L}(1,6)$	$\mathscr{L}(1,3)$	$\mathscr{Z}(4,6,8)$	$\mathscr{Z}(5,7,8)$	$\mathscr{Z}(5,7,9)$
5	$\mathscr{L}(1,6)$	$\mathscr{L}(1,3)$	$\mathscr{L}(1,3)$	$\mathscr{Z}(10,12,14)$	$\mathscr{Z}(5,6,7)$	$\mathscr{Z}(8,9,10)$
6	$\mathscr{L}(2,5)$	$\mathscr{L}(10,16)$	$\mathscr{L}(14,16)$	$\mathscr{Z}(1,4,6)$	$\mathscr{Z}(3,4,5)$	$\mathscr{Z}(2,4,8)$
7	$\mathscr{L}(2,6)$	$\mathscr{L}(2,9)$	$\mathscr{L}(4,6)$	$\mathscr{Z}(2,4,8)$	$\mathscr{Z}(5,6,7)$	$\mathscr{Z}(1,4,7)$
8	$\mathscr{L}(1,10)$	$\mathscr{L}(9,11)$	$\mathscr{L}(9,19)$	$\mathscr{Z}(3,6,12)$	$\mathscr{Z}(7,8,9)$	$\mathscr{Z}(5,7,9)$
9	$\mathscr{L}(3,6)$	$\mathscr{L}(6,10)$	$\mathscr{L}(12,17)$	$\mathscr{Z}(4,8,16)$	$\mathscr{Z}(12,13,14)$	$\mathscr{Z}(11,13,15)$
10	$\mathscr{L}(3,4)$	$\mathscr{L}(8,18)$	$\mathscr{L}(13,16)$	$\mathscr{Z}(5,10,20)$	$\mathscr{Z}(15,16,18)$	$\mathscr{Z}(25,26,28)$
11	$\mathscr{L}(5,6)$	$\mathscr{L}(1,2)$	$\mathscr{L}(6,7)$	$\mathscr{Z}(6,12,24)$	$\mathscr{Z}(10,20,25)$	$\mathscr{Z}(5,19,21)$
12	$\mathscr{L}(5,8)$	$\mathscr{L}(1,5)$	$\mathscr{L}(8,11)$	$\mathscr{Z}(7,10,13)$	$\mathscr{Z}(22,26,28)$	$\mathscr{Z}(31,33,38)$
13	$\mathscr{L}(3,7)$	$\mathscr{L}(5,7)$	$\mathscr{L}(11,15)$	$\mathscr{Z}(8,16,32)$	$\mathscr{Z}(1,5,8)$	$\mathscr{Z}(4,15,18)$
14	$\mathscr{L}(3,4)$	$\mathscr{L}(13,16)$	$\mathscr{L}(16,17)$	$\mathscr{Z}(9,11,18)$	$\mathscr{Z}(2,3,8)$	$\mathscr{Z}(33,36,37)$
15	$\mathscr{L}(9,16)$	$\mathscr{L}(11,26)$	$\mathscr{L}(3,4)$	$\mathscr{Z}(10,16,22)$	$\mathscr{Z}(5,6,9)$	$\mathscr{Z}(37,38,39)$
16	$\mathscr{L}(11,14)$	$\mathscr{L}(7,17)$	$\mathscr{L}(14,23)$	$\mathscr{Z}(11,16,28)$	$\mathscr{Z}(2,5,8)$	$\mathscr{Z}(8,9,10)$
17	$\mathscr{L}(9,17)$	$\mathscr{L}(10,27)$	$\mathscr{L}(4,9)$	$\mathscr{Z}(12,14,26)$	$\mathscr{Z}(7,9,11)$	$\mathscr{Z}(30,36,38)$
18	$\mathscr{L}(6,10)$	$\mathscr{L}(2,9)$	$\mathscr{L}(3,13)$	$\mathscr{Z}(9,11,13)$	$\mathscr{Z}(11,13,18)$	$\mathscr{Z}(29,33,37)$
19	$\mathscr{L}(2,7)$	$\mathscr{L}(7,15)$	$\mathscr{L}(13,19)$	$\mathscr{Z}(1,2,3)$	$\mathscr{Z}(1,3,8)$	$\mathscr{Z}(26,36,31)$
20	$\mathscr{L}(5,7)$	$\mathscr{L}(3,13)$	$\mathscr{L}(10,26)$	$\mathscr{Z}(2,3,6)$	$\mathscr{Z}(2,8,10)$	$\mathscr{Z}(12,14,19)$

Table 6: Cost of Uncertain inputs for 20 DMU.

DMU$_j$	p_{1j}	p_{2j}	p_{3j}
$j = 1,2,\ldots,n$	6	5	15

In order to provide information related to the proposed model, it is necessary to multiply the input prices by the input values of the DMUs. After doing so and applying model (14), the results are listed in Table 7.

The fourth column of Table 7 gives the results of the proposed model with identical prices for each input from all units and the sixth column indicates the results of the proposed model ignoring the input price. This table shows that comparing the two proposed models and Pourmahmood and Bagheri's model [18], the number of efficient units is different. In other words, in the proposed model, 5 units, i.e., No. 2, 5, 11, 12, and 15 are efficient, while in Pourmahmood and Bagheri's model, three DMUs, i.e.,

Table 7: Cost efficiency with same prices.

# DMU_j	x_i^*	x_{ik}	Proposed model with same prices		Efficiency without prices	
			Optimal value θ^*	**Efficiency**	Optimal value θ^*	**Efficiency**
1	17.9219	155.5	0.1153	Inefficient	0.1077	Inefficient
2	73.5	73.5	1.0000	Efficient	0.7767	Inefficient
3	40.425	121	0.3341	Inefficient	0.2945	Inefficient
4	52.26724	68.5	0.7630	Inefficient	.06111	Inefficient
5	61	61	1.0000	Efficient	1.0000	Efficient
6	31.43298	314	0.1001	Inefficient	0.1064	Inefficient
7	44.55469	126.5	0.3522	Inefficient	0.3031	Inefficient
8	61.05078	293	0.2084	Inefficient	0.2091	Inefficient
9	90.55	284.5	0.3359	Inefficient	0.3607	Inefficient
10	155.187	303.5	0.5113	Inefficient	0.5059	Inefficient
11	138	138	1.0000	Efficient	1.0000	Efficient
12	196.5	196.5	1.0000	Efficient	1.0000	Efficient
13	91.5	255	0.3588	Inefficient	.04688	Inefficient
14	205.5263	341	0.6027	Inefficient	0.5924	Inefficient
15	220	220	1.0000	Efficient	0.6559	Inefficient
16	9022917	412.5	0.2187	Inefficient	0.2580	Inefficient
17	205.0707	268	0.7652	Inefficient	0.5606	Inefficient
18	191.5068	195.5	0.9796	Inefficient	0.8789	Inefficient
19	186.7105	322	0.5798	Inefficient	0.5764	Inefficient
20	85.64557	346	0.2475	Inefficient	0.2595	Inefficient

No. 5, 11, and 12 are efficient. This shows that DMUs 2 and 15 are able to be efficient in the presence of prices in the case that the input prices were identical for all units, whereas they were not technically efficient in absence of prices. Also, despite assuming the same prices for inputs, the efficiency obtained for inefficient DMUs is different in the two models. Therefore, efficiency scores are affected by prices. In other words, the efficiency of the units is different in two cases once input prices are ignored and considered. Additionally, the ranking of inefficient units in these two modes is different. For example, DMU_{17} is in the 8-th position when prices are not included, but it ranks 7 when identical prices are included. It can be concluded that this difference was due to assigning identical prices to each input in all DMUs.

Second case: Cost efficiency evaluation under the conditions of different input prices

In this case, it is assumed that each DMU supplies its inputs from different sources. Thus, there are different prices for each input in each DMU. Considering the prices of inputs in different DMUs, columns 4 to 6 of Table 8 are obtained. The results of applying the proposed model in both cases are listed in columns 7 and 9, respectively.

Table 8: Cost efficiency with different prices

# DMU_j	p_{1j}	p_{2j}	p_{3j}	x_i^*	x_{ik}	Proposed model with different prices		Proposed model without prices	
						Optimal value θ^*	**Efficiency**	efficiency cost	**Efficiency**
1	6	5	15	21.0000	155.5	0.1350	Inefficient	0.1077	Inefficient
2	6	5	15	73.5000	73.5	1.0000	Efficient	0.7767	Inefficient
3	11	7	16	40.4250	153.5	0.2634	Inefficient	0.2945	Inefficient
4	11	6	17	63.0000	93.5	0.6738	Inefficient	0.6111	Inefficient
5	13	10	18	126.0000	160.5	0.7850	Efficient	1.0000	Efficient
6	6	5	15	39.3750	314	0.1254	Inefficient	0.1064	Inefficient
7	14	10	19	47.2500	199	0.2374	Inefficient	0.3031	Inefficient
8	15	8	20	70.8750	442.5	0.1602	Inefficient	0.2091	Inefficient
9	12	6	20	95.5500	392	0.2438	Inefficient	0.3607	Inefficient
10	9	5	18	164.6993	357.5	0.4607	Inefficient	0.5059	Inefficient
11	13	8	20	141.7500	213.5	0.6639	Inefficient	1.0000	Efficient
12	7	5	15	203.0000	203	1.0000	Efficient	1.0000	Efficient
13	11	10	16	189.0000	232	0.5851	Inefficient	0.4688	Inefficient
14	14	8	15	216.9220	405.5	0.5349	Inefficient	0.5924	Inefficient
15	16	8	19	237.8614	414.5	0.5739	Inefficient	0.6559	Inefficient
16	11	9	20	186.3750	615.5	0.3028	Inefficient	0.2580	Inefficient
17	12	7	20	222.5411	415.5	0.5356	Inefficient	0.5606	Inefficient
18	13	9	18	200.8861	297.5	0.6752	Inefficient	0.8789	Inefficient
19	12	10	17	193.9778	436	0.4449	Inefficient	0.5764	Inefficient
20	14	8	18	87.7185	472	0.1880	Inefficient	0.2595	Inefficient

This table shows that the number of efficient DMUs is different in both cases. Three DMUs, i.e., No. 5, 11, and 12 are efficient in absence of prices, and DMU_2 and DMU_{12} are efficient in the case where different prices are considered for inputs. The DMU_2 is inefficient in the absence of prices, but it is efficient in their presence, meaning that, despite the technical inefficiency, this unit can obtain its inputs from inexpensive sources and become efficient. Also, DMU_5 and DMU_{11} have been efficient in the absence of prices, but they have lost their efficiency with the introduction of different prices. The efficiency score of all DMUs except DMU_{12} has changed. Further, the efficiency score is different in two cases for inefficient DMUs. The ranking of inefficient units is also different in these two cases. For example, DMU_{19} ranks 9 in absence of prices, but ranks 12 in presence of different prices. It can be concluded that this difference is due to assigning different prices to each input in all DMUs. These results show that in order to improve inefficient units, their inputs should be chosen from sources with lower prices.

6 Conclusions

Although studies have been conducted to evaluate the efficiency of decision-making units, there is a challenge regarding the cost efficiency of units involves Liu-type uncertainty. To solve this challenge, in this study, a model is presented to evaluate the cost efficiency of units based on the Tone's model. The Tone's model is used for crisp data, but in the presented model, the input and output data are assumed to be constant and the prices are considered to be crisp.

The presented uncertain model converted into a crisp linear model that can be solved with the help of linear solvers. The equivalent linear model is applied on two numerical examples. In these examples, three uncertain inputs with linear distribution and three uncertain outputs with zigzag distribution are considered, the difference between the two examples being the number of decision-making units. The general conclusion of using the presented model on two examples shows the importance of paying attention to the input prices. For each example, these prices are considered in two cases. In the first case, it is assumed that all units purchase their inputs at the same price, and in the second case, each unit buys its inputs at different prices. In these examples, there were units that were technically efficient but lost their efficiency by applying prices, and there were units that were technically inefficient but were able to by preparing inputs with lower prices to work. In other words, inefficient units could become efficient by getting their inputs from cheaper sources. Also, in the presence of prices, the number of efficient units in each mode was different, and the efficiency scores of inefficient units were also different without the presence of prices and with the presence of prices. The superiority of this model over the previous models is in the opinion of experts. In other words, in some cases, there is no frequency and it can be used according to experts that none of the three non-crisp approaches are able to solve such problems. For future research, one may study ranking efficient units in the uncertain enviroment.

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