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COMAXIMAL INTERSECTION GRAPH OF IDEALS OF RINGS

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ABSTRACT. The comaximal intersection graph $CI(R)$ of ideals of a ring R is an undirected graph whose vertex set is the collection of all non-trivial (left) ideals of R and any two vertices I and J are adjacent if and only if $I + J = R$ and $I \cap J \neq 0$. We study the connectedness of $CI(R)$. We also discuss independence number, clique number, domination number, chromatic number of $CI(R)$.

1. INTRODUCTION

In the past decade, many researchers have studied the interplay between ring structure and graph structure. They defined graphs whose vertices are elements in a ring or are ideals in the ring and edges are defined with respect to certain conditions on the elements of the vertex set. This idea was initially conceived by Beck[10] in 1988, where he introduced the zero-divisor graph $\Gamma(R)$ for a commutative ring R, whose vertex set is the set of elements in R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. After that, a lot of work was done in this area. In 1999, Anderson and Livingston in [3] modified the zero-divisor graph $\Gamma(R)$ by taking the vertex set as the set of non-zero zero-divisors of R. This modified graph $\Gamma(R)$ has better graph structure than the previous one. For more details about this graph one can refer to [2]. In 2011, Behboodi and Rakeei [15] defined a new graph called the annihilating-ideal graph $\mathbb{AG}(R)$ on a commutative ring R, where they used non-zero proper ideals as vertices instead of non-zero

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zero divisors of the ring. For more details about this graph one can refer to [7, 8, 15, 16].

In the year 1995, Sharma and Bhatwadekar [19] introduced a graph $\Omega(R)$ on a commutative ring R, whose vertex set is the set of elements of R and two distinct vertices x, y are adjacent if and only if $Rx + Ry = R$. In 2008, Maimani et al. [9] modified this graph by taking vertex set consists of non-unit elements of R and named this graph as the co-maximal graph of R. In 2012, Ye and Wu [18] introduced the graph $C(R)$, the co-maximal ideal graph on a commutative ring R with identity, whose vertices are the proper ideals of R that are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 = R$. Using the complement concept of this graph, Barman and Rajkhowa[1] introduced the non-comaximal graph of ideals of a ring R, whose vertex set is the collection of all non-trivial (left) ideals of R and any two distinct vertices I and J are adjacent if and only if $I + J \neq R$. They denoted this graph by $NC(R)$.

In 2009, Chakrabarty et al. [11] introduced the intersection graph of ideals of rings, denoted by $G(R)$, whose vertex set is the set of nontrivial left ideals of R and any two vertices I, J are adjacent if and only if $I \cap J \neq 0$. Utilising this insight, Rajkhowa and Saikia [13] introduced the prime intersection graph of ideals of a ring $G(R)$ by imposing one additional condition on the adjacency of two vertices I, J that one of I or J must be a prime ideal of R. For more details about intersection graph of ideals one can refer to [11, 13, 21, 22].

In this paper, we combine two concepts, the co-maximal ideal graph and the intersection graph of ideals of a ring and define a new graph called comaximal intersection graph $CI(R)$ of ideals of a ring R, whose vertex set is the collection of all non-trivial (left) ideals of R and two vertices I and J are adjacent if and only if $I + J = R$ and $I \cap J \neq 0$.

By G, we mean an undirected simple graph with the vertex set $V(G)$, unless otherwise mentioned. A walk in G is an alternating sequence of vertices and edges, $v_0e_1v_1 \cdots e_nv_n$, where each edge $e_i = v_{i-1}v_i$. If the beginning and the ending vertices of a walk are same then the walk is called a closed walk. In a walk, if all the vertices are distinct, it is called a path. A circuit is a closed walk in which all the vertices are distinct. The total number of edges in a circuit is called the length of the circuit. The length of a smallest circuit in G is called the girth of G and is denoted by $girth(G)$. If G does not contain a circuit then $girth(G) = \infty$. G is called a connected graph if for any two distinct vertices there is a path connecting them. A graph which is not a connected graph is called a disconnected graph. A graph that does not contain any edge is called a totally disconnected graph. In a connected graph G, the distance $d(u, v)$ between two vertices u and v is the length of the shortest uv-path in G. The greatest distance between any two vertices u and v in G is called the diameter of G and denoted by $diam(G)$. If G is not connected then $diam(G) = \infty$. The complement graph of G denoted by $\overline{G}$ is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. G is called a complete graph if every two distinct vertices in G are adjacent. A clique is a complete subgraph of G. The clique number of G, denoted by $\omega(G)$, is the cardinality of the maximum clique of G. If, in a set of vertices of G, no two vertices are mutually adjacent then it is called an independent set. The independence number of a graph G is the cardinality of a maximum independent set and is denoted by $\alpha(G)$. The chromatic number of G, denoted by $\chi(G)$ is the minimum number of colors assigning to the vertices of G so that no two adjacent vertices have the same color. The graph G is weakly perfect if $\omega(G) = \chi(G)$. A set D of vertices in G is called a dominating set of G if every vertex which is not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. A set D is called a global dominating set of G if it is a dominating set for both the graphs G and its complement $\overline{G}$. The minimum cardinality of a global dominating set is called the global domination number of G and is denoted by $\gamma_g(G)$. The domatic number of a graph G is the maximum order of partitions of vertices of G into disjoint dominating sets and is denoted by $d(G)$. The global domatic number of a graph G, denoted by $d_g(G)$ is equal to the maximum order of partitions of vertices into disjoint global dominating sets. Any undefined terminology can be obtained in [5, 6, 20]

Henceforth, R denotes a commutative with multiplicative identity unless otherwise specified. R is called local if it has a unique maximal ideal. R is said to be an artinian ring if every descending chain of ideals in R is stationary. A UFD is an integral domain in which every non-zero non-unit element can be written as a product of prime elements, uniquely up to order and units. R is said be an essential extension of an ideal I if for every non-zero ideal J of R , $I \cap J \neq 0$. Any undefined terminologies are available in [12, 14, 17]. In this paper, $J(R)$ is the Jacobson radical, $Min(R)$ set of minimal ideals, $Max(R)$

set of maximal ideals of R and $I(M)$, set of ideals of R contained in the maximal ideal M.

2. Connectedness of CI(R)

In this section, connectedness of $CI(R)$ is discussed. This section also contains results on diameter and girth. In [1], Theorem 2.3. states: "$NC(R)$ is totally disconnected if and only if every non-trivial ideal of R is maximal as well as minimal". In the following theorem, we establish the similar result for $CI(R)$.

Theorem 2.1. *$CI(R)$ is totally disconnected if and only if R is local or every non-trivial ideal of R is maximal (as well as minimal).*

Proof. Assume that $CI(R)$ is totally disconnected. Take two vertices I, J of $CI(R)$. Then either $I + J \neq R$ or $I \cap J = 0$. If $I + J \neq R$, then $I + J \subsetneqq M$, M is a maximal ideal of R. In this case, $I \subseteq M$, $J \subseteq M$ and so R is local. Also if $I \cap J = 0$, then there is nothing to prove whenever R is local. Assume that both I and J are not maximal. If I is not maximal, then we have a maximal ideal N such that $I \subsetneqq N$. So $J + N = R$ will imply that $I = N$, as $I + J = R$. But this is a contradiction since N is a maximal ideal. Hence every ideal is maximal. □

Theorem 2.2. *There is an isolated vertex I in $CI(R)$ if and only if I is contained in every maximal ideal of R or $I \cap M = 0$.*

Proof. If there exists an ideal I which is contained in every maximal ideal of R, then it is easy to notice that I is an isolated vertex in $CI(R)$. Similarly if there exists an ideal I which is not contained in a maximal ideal M of R with $I \cap M = 0$, then also I is an isolated vertex in $CI(R)$. For the converse part, if there exists an isolated vertex I in $CI(R)$ which is not contained in a maximal ideal M, then $I + M = R$. Thus $I \cap M = 0$. Hence the theorem. □

Corollary 2.3. *The ideals contained in $J(R)$ are isolated vertices in $CI(R)$.*

Theorem 2.4. *If R is an artinian ring, every ideal in $Min(R)$ is an isolated vertex of $CI(R)$.*

Proof. Let I be a minimal ideal in R. Then for any non-trivial ideal J of R, either $I \cap J = 0$ or $I \cap J \neq 0$. If $I \cap J \neq 0$ then $I \cap J = I \subseteq J$ and so $I + J = J \neq R$. □

Theorem 2.5. *Let R be a finite UFD. Then $CI(R)$ is disconnected if and only if $CI(R)$ has an isolated vertex.*

Proof. Assume that $CI(R)$ is disconnected and $p_1, p_2, \cdots, p_r, r \geq 1$ are the r number of prime elements of R. If $k_1, k_2, \cdots, k_r$ are the maximum exponents of $p_1, p_2, \cdots, p_r$ respectively, then $(p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r})$, $1 \leq j_l \leq k_l$, $l = 1, 2, \cdots, r$ is an isolated vertex. □

Theorem 2.6. *If R is an essential extension of each of the non-zero ideals of R, then $CI(R)$ is connected if and only if R is not a local ring.*

Proof. Assume that R is not a local ring. If I and J are two non-zero ideals of R, then I and J will be contained in M_1 and M_2 respectively, where M_1 and M_2 are two maximal ideals of R. If $M_1 = M_2$, then there is another maximal ideal M and so $I - M - J$ is a path between I and J, as $I \cap M \neq 0, J \cap M \neq 0$. Moreover, if $M_1 \neq M_2$, then $I - M_2 - M_1 - J$ is a path between I and J, as R is an essential extension of each of the non-zero left ideals of R. In the opposite direction, by contrary assume that R is local. But then $CI(R)$ is a disconnected graph, in fact a totally disconnected graph by Theorem 2.1. This completes the proof. □

In [22], Theorem 2.4 states: " For a ring R, the co-maximal ideal graph $\mathcal{C}(R)$ is a simple, connected graph with diameter less than or equal to three". We have established a similar result in the following theorem.

Theorem 2.7. *Let R be an essential extension of each of the non-zero ideals of R, then $diam(CI(R)) \leq 3$ or ∞.*

Proof. Suppose that $CI(R)$ is connected. Let I and J be any two ideals of R. If I and J are adjacent, then $diam(CI(R)) < 3$. If I and J are not adjacent, then either $I + J \neq R$ or $I \cap J = 0$. Since R is an essential extension of each of the non-zero ideals of R, so we must have $I + J \neq R$. This implies I and J are not maximal ideals of R. Let $I \subset M_1$ and $J \subset M_2$, where M_1 and M_2 are maximal ideals of R. If $M_1 = M_2$, then there is another maximal ideal M and so $I - M - J$ is a path between I and J, as $I \cap M \neq 0, J \cap M \neq 0$. Moreover, if $M_1 \neq M_2$, then $I - M_2 - M_1 - J$ is a path between I and J, as R is an essential extension of each of the non-zero ideals of R. Hence $diam(CI(R)) \leq 3$. Hence the theorem. □

Theorem 2.8. *If $J(R) \neq 0$, then $diam(CI(R)) = \infty$.*

Theorem 2.9. *If $J(R) \neq 0$, then $diam(\overline{CI(R)}) \leq 2$.*

Theorem 2.10. *If R is an artinian ring, then $diam(CI(R)) = \infty$.*

Theorem 2.11. *If R is an artinian ring, then $diam(\overline{CI(R)}) \leq 2$.*

Theorem 2.12. *$CI(R)$ is not a complete graph.*

Proof. If R is a local ring, then $CI(R)$ is totally disconnected. Assume that R is not a local ring. If $J(R) = 0$, then there exist maximal ideals which intersect trivially. Moreover, if $J(R) \neq 0$, then every non-trivial ideal is not maximal. Thus there is a non-trivial ideal which is properly contained in a maximal ideal. In either case, $CI(R)$ is not a complete graph. Hence the theorem. □

Theorem 2.13. *Let $J(R)$ be a minimal ideal. Then $CI(R)$ contains no circuit if and only if $|Max(R)| \leq 2$.*

Proof. For $|Max(R)| = 1$, it is obvious. Suppose $|Max(R)| = 2$. Our aim is to show $CI(R)$ contains no circuit. On the contrary, suppose $I_1 - I_2 - \cdots - I_n - I_1$ is a circuit in $CI(R)$. Then each I_i is contained in a maximal ideal M_i, $i = 1, 2$. Observe that no two ideals I_i and I_{i+1} are contained in a single maximal ideal. If this happens, then the corresponding ideals are not adjacent. But it is possible I_{i-1}, I_{i+1} are in same M_i, $i = 1, 2$. Let $I_{i-1}, I_{i+1} \subseteq M_1$ and $I_i \subseteq M_2$. Since $I_i - I_{i+1}$ is an edge, so $I_{i+1} \not\subseteq J(R)$. Therefore $I_{i+1} = M_1$ as $I_{i+1} \cap J(R) = 0$ implies $I_i - I_{i+1}$ not an edge. Similarly we will have $I_{i-1} = M_1$. Hence $n = 2$. Thus $CI(R)$ contains no circuit. Conversely, if $|Max(R)| \geq 3$, then we get a circuit. The proof is complete. □

In [22], Theorem 4.5. shows that C(R) is a (complete) bipartite graph if and only if R has exactly two maximal ideals. In the following theorems, we also establish the same results.

Theorem 2.14. *Let $J(R) \neq 0$. Then $CI(R)$ is a bipartite graph if and only if $|Max(R)| \leq 2$.*

Proof. If $|Max(R)| \geq 3$, then $M_1 - M_2 - M_3 - M_1$ is a cycle of length 3 in $CI(R)$, where $M_i \in Max(R)$. So, $CI(R)$ is not a bipartite graph. If $|Max(R)| = 2$, then from proof of Theorem 2.13; if $CI(R)$ contains a cycle, the length of the cycle should be even as no two ideals I_i and I_{i+1} are contained in a single maximal ideal. □

Theorem 2.15. *Let R be an essential extension of each of the non-zero left ideals of R, then $CI(R)$ is a complete bipartite graph if and only if $|Max(R)| = 2$.*

Theorem 2.16. *If $J(R) \neq 0$, then $girth(CI(R)) \leq 4$, whenever $CI(R)$ contains a circuit.*

Proof. If $Max(R) = 2$ and $CI(R)$ contains a circuit, then $girth(CI(R)) = 4$, which can be obtained from the proof of Theorem 2.13 and Theorem 2.14. If $|Max(R)| \geq 3$, then $M_1 - M_2 - M_3 - M_1$ is a circuit, where $M_i \in Max(R)$, $i = 1, 2, 3$. □

3. Independence number, clique number and domination number of CI(R)

In this section, we discuss independence number, clique number, chromatic number, domination number, global domination number and domatic number of $CI(R)$.

In the following theorem, we find the total number of maximal independent sets in $CI(Z_n)$ and the independence number of $CI(Z_n)$. Then we try to generalise the result.

Theorem 3.1. *The independence number of $CI(Z_n)$ is $|I(M_j)|$, where $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and j is corresponding to maximum value of k_j, $j = 1, 2, \cdots, r$.*

Proof. Here $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. So a maximal independent set of $CI(Z_n)$ is the collection of all ideals which are generated by multiple of $p_i, i = 1, 2, \cdots, r$. There are r maximal independent sets in $CI(Z_n)$. The cardinality of maximal independent set $I(M_1)$ which contains the ideals multiple of p_1 is $|I(M_1)| = k_1 + k_1(k_2 + k_3 + \cdots + k_r) + k_1 k_2 (k_3 + \cdots + k_r) + \cdots + k_1 k_2 \cdots k_r - 1$. Similarly, the cardinality of maximal independent set $I(M_2)$ which contains the ideals multiple of p_2 is $|I(M_2)| = k_2 + k_2(k_1 + k_3 + \cdots + k_r) + k_1 k_2 (k_3 + \cdots + k_r) + \cdots + k_1 k_2 \cdots k_r - 1$. Proceeding in the same way, the cardinality of maximal independent set $I(M_i)$ which contains the ideals multiple of p_i is $|I(M_i)| = k_i + k_i(k_1 + \cdots + k_{i-1} + k_{i+1} \cdots + k_r) + \cdots + k_1 k_2 \cdots k_r - 1$. The largest independent set is obtained for maximum value of k_i, $i = 1, 2, \cdots, r$. From this, it is easy to notice that the independence number of $CI(Z_n)$ is $|I(M_j)|$, where j is corresponding to maximum value of k_j, $j = 1, 2, \cdots, r$. □

Theorem 3.2. *For an artinian ring R that has a unique minimal ideal, $\alpha(CI(R)) = max\{|I(M)| : M \text{ is a maximal ideal of } R\}$.*

Proof. For any two ideals $I, I' \subseteq M$, M is a maximal ideal of R; $I - I'$ is not an edge in $CI(R)$ as $I + I' \neq R$. So $I(M)$, the set of ideals contained in a maximal ideal M of R is an independent set. Also for any ideal $J \nsubseteq I(M)$, $J - M$ is an edge in $CI(R)$, so $J \cup I(M)$ is not an independent set. Therefore, $I(M)$ is a maximal independent set in $CI(R)$. Hence $\alpha(CI(R)) = max\{|I(M)| : M \text{ is a maximal ideal of } R\}$. □

Theorem 3.3. *For an artinian ring R with a unique minimal ideal, $|I(J(R))| \leq \gamma(CI(R)) \leq |I(J(R)) \cup Max(R)|$.*

Proof. If R is a local ring, then $CI(R)$ is totally a disconneted graph and $J(R) = M$. Hence $\gamma(CI(R)) = |I(J(R))|$. Suppose R is a non local ring. Since R has unique minimal ideal, say m, so it is contained in every maximal ideal. So $m \subseteq J(R)$. Since a dominating set must contains all the isolated vertices, so by Corollary 2.3, a dominating set of $CI(R)$ contains at least $|I(J(R))|$ vertices. So $|I(J(R))| \leq \gamma(CI(R))$. Again for any ideal $I \nsubseteq I(J(R))$, there exist a maximal ideal M such that $I \nsubseteq M$. This implies $I - M$ is an edge. So the set $\{I(J(R)) \cup Max(R)\}$ of ideals form a dominating set for $CI(R)$. Hence $\gamma(CI(R)) \leq |I(J(R)) \cup Max(R)|$. □

In [5], Proposition 1 states: "A dominating set S of G is a global dominating set if and only if for each $v \in V - S$, there exists a $u \in S$ such that u is not adjacent to v". Using this proposition, we establish the following result.

Theorem 3.4. *For an artinian ring R with a unique minimal ideal, $|I(J(R))| \leq \gamma_g(CI(R)) \leq |I(J(R)) \cup Max(R)|$.*

Proof. Let D be a minimum dominating set of $CI(R)$. Then D contains vertices $I \subseteq J(R)$, as these are isolated vertices in $CI(R)$ by Corollary 2.3. Hence by Proposition 1 in [5], D is a global dominating set of $CI(R)$. Thus the result. □

Theorem 3.5. *If $R = R_1 \times R_2$; where R_i is not a field for $i = 1, 2$, then $\gamma(CI(R)) = 2 + |I(J(R))|$.*

Proof. Since $R = R_1 \times R_2$, so any ideal I of R is of the form $I = I_1 \times I_2$ where I_i is an ideal of R_i; $i = 1, 2$. The maximal ideals of R are $M_1 \times R_2$ and $R_1 \times M_2$, where M_i is a maximal ideal in R_i for $i = 1, 2$. The minimal ideals of R are $m_1 \times 0$ and $0 \times m_2$, where m_i is a minimal ideal in R_i for $i = 1, 2$. Now $J(R) = M_1 \times M_2$ and $Min(R) \subseteq J(R)$. Observe that any ideal $I \nsubseteq J(R)$ has the form $I_1 \times R_2$ or $R_1 \times I_2$, where $I_i \subseteq R_i$ for $i = 1, 2$. So $I_1 \times R_2$ is dominated by $R_1 \times M_2$ and $R_1 \times I_2$ is dominated by $M_1 \times R_2$. Hence the ideals that are not contained in $J(R)$ are dominated by two ideals. Also the induced subgraph $< I >$; $I \nsubseteq J(R)$, is not a complete subgraph. Thus $\gamma(CI(R)) = 2 + |I(J(R))|$. □

Theorem 3.6. *If $R = R_1 \times R_2$; where R_i is not a field for $i = 1, 2$, then $\gamma_g(CI(R)) = 2 + |I(J(R))|$.*

In [4], Proposition 4.1 states: "For any graph G, $d(G) \leq \delta(G) + 1$". Again in [5], Proposition 11 (ii) states: "For any graph G of order p, $d_g(G) \leq d(G)$". Using these two results we obtain the following theorem.

Theorem 3.7. *If $R = R_1 \times R_2$; where R_i is not a field for $i = 1, 2$, then $d(CI(R)) = d_g(CI(R)) = 1$.*

Theorem 3.8. *If $R = R_1 \times F$, where R_1 is a ring and F is a field, then $\gamma(CI(R)) = 1 + |I(J(R))|$.*

Proof. The maximal ideals of R are $M_1 \times F$ and $R_1 \times 0$, where M_1 is a maximal ideal in R_1. Again the minimal ideals of R take the form $m_1 \times 0$, where m_1 is a minimal ideal of R_1. Also any non zero ideal $I \subseteq M_1 \times F$ that is not contained in $J(R)$ is adjacent to $R_1 \times 0$. This implies the maximal ideal $R_1 \times 0$ dominates all the ideals that are not contain in $J(R)$. Hence $\gamma(CI(R)) = 1 + |I(J(R))|$. □

Theorem 3.9. *If $R = R_1 \times F$; R_1 is a ring and F is a field, then $\gamma_g(CI(R)) = 1 + |I(J(R))|$.*

Theorem 3.10. *If $R = R_1 \times F$; R_1 is a ring and F is a field, then $d(CI(R)) = d_g(CI(R)) = 1$.*

Theorem 3.11. *If $R = F_1 \times F_2$; where F_i is a field for $i = 1, 2$, then $\gamma(CI(R)) = 2$.*

Proof. Here R has only two non trivial ideals $F_1 \times 0$ and $0 \times F_2$, which are maximal as well as minimal. Hence by Theorem 2.1 and Theorem 3.3, $\gamma(CI(R)) = 2$. □

Theorem 3.12. *If $R = F_1 \times F_2$; where F_i is a field for $i = 1, 2$, then $\gamma_g(CI(R)) = 2$.*

Theorem 3.13. *If $R = F_1 \times F_2$; where F_i is a field for $i = 1, 2$, then $d(CI(R)) = d_g(CI(R)) = 1$.*

Theorem 3.14. *If $R = F_1 \times F_2 \times F_3 \times F_4 \times \cdots \times F_n$; $n \geq 3$, where F_i is a field for $i = 1, 2, ...n$, then $\gamma(CI(R)) = 2n - 1$.*

Proof. Any ideal of R is of the form $I = I_1 \times I_2 \times I_3 \times \cdots \times I_n$, where I_i is an ideal of R_i for $i = 1, 2, ...n$. The maximal ideals of R are $M_i = \prod_{i=1}^{n} F_i$ with $F_i = 0$. For an ideal $m_i = \prod_{j=1}^{n} F_j$ with $F_j = 0$ if $i \neq j$, we have $m_i + M_j \neq R$ and $m_i + M_i = R$ but $m_i \cap M_i = 0$. This implies that m_i is an isolated vertex of $CI(R)$. Now let us consider the ideal $m_{i,j} = \prod_{k=1}^{n} F_k$ with $F_k \neq 0$ if $k = i, j$. Then $m_{i,j}$ is dominated by M_i and M_j only. This asserts that the set $\{m_1, m_2, \cdots m_n, M_1, M_2, \cdots, M_{n-1}\}$ forms a minimum dominating set for $CI(R)$. Hence $\gamma(CI(R)) = 2n - 1$. □

Theorem 3.15. *If $R = F_1 \times F_2 \times F_3 \times F_4 \times \cdots \times F_n$; $n \geq 3$ and F_i is a field for $i = 1, 2, ...n$, then $\gamma_g(CI(R)) = 2n - 1$.*

Theorem 3.16. *If $R = F_1 \times F_2 \times F_3 \times F_4 \times \cdots \times F_n$; $n \geq 3$ and F_i is a field for $i = 1, 2, ...n$, then $d(CI(R)) = d_g(CI(R)) = 1$.*

Theorem 3.17. *If $J(R) \neq 0$, then*

$$\omega(CI(R)) = \chi(CI(R)) = |Max(M)|.$$

Theorem 3.18. *If R is an artinian ring with unique minimal ideal, then $\omega(CI(R)) = \chi(CI(R)) = |Max(M)|$.*

Proof. Consider an ideal I which is contained in a maximal ideal M, say. If we take another ideal I' such that $I' \subseteq M$, then they are not adjacent as $I + I' \neq R$. So the vertex set of a complete subgraph of $CI(R)$ can contain atmost one vertex from each $|I(M)|$ of R. That is a complete subgraph of $CI(R)$ can contain atmost $|Max(R)|$ vertices. This implies $\omega(CI(R)) \leq |Max(M)|$. Again $Max(R)$ forms a complete subgraph of $CI(R)$. Hence $\omega(CI(R)) = |Max(M)|$. Again the induced subgraph $< Max(R) >$ is a complete subgraph of $CI(R)$. So we need at least $|Max(R)|$ colours to colour the graph such that no two adjacent vertices have the same colour. This implies $|Max(M)| \leq \chi(CI(R))$. Also for any two ideals $I, J \subseteq M \in Max(R)$, we have $I - J$ not an edge. Hence $\chi(CI(R)) = |Max(M)|$. This completes the proof. □

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