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SOME CAYLEY GRAPHS WITH PROPAGATION TIME OF AT MOST TWO

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ABSTRACT. In this paper the zero forcing number as well as propagation time of $Cay(G, \Omega)$, where G is a finite group and $\Omega \subset G \backslash \{1\}$ is an inverse closed generator set of G is studied. In particular, it is shown that the propagation time of $Cay(G, \Omega)$ is at most two for some special generators.

1. INTRODUCTION

Let $\Gamma = (V, E)$ be a simple graph of order n and size m. For a vertex $v \in V$, the *open neighborhood* of v is the set $N_\Gamma(v) = \{u \in V \mid u \sim v\}$. Also, the *close neighborhood* of vertex v, $N_\Gamma[v]$, is $N_\Gamma[v] = N_\Gamma(v) \cup \{v\}$.The degree of a vertex v is $deg(v) = |N_\Gamma(v)|$. The minimum degree of a graph Γ denoted by $\delta(\Gamma)$. Let G be a non-trivial group with identity element 1 and let $\Omega \subseteq G$ such that $1 \notin \Omega$, $\Omega = \Omega^{-1} = \{\omega^{-1} \mid \omega \in \Omega\}$. The *Cayley graph* of G, $Cay(G, \Omega)$, is a graph with vertex set G and two vertices u and v are adjacent if and only if $uv^{-1} \in \Omega$.
Suppose that $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are two graphs with same order and $\mu : V_1 \to V_2$ is a bijection. Define the matching graph (H_1, H_2, μ) to be the graph constructed as the disjoint union of H_1, H_2 and perfect matching between V_1 and V_2 defined by μ. Let each vertex of a graph Γ be either "black" or "white". Let B denote the (initial) set of black vertices Γ. If the white vertex v is the only white neighbour of a black vertex u, then u changes the color of v to black (*color-change rule*) and we say "u forces v". The set B is said to a zero forcing set of

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Γ if all vertices of Γ will be turned black after finitely many applications of the color-change rule. The zero forcing number of Γ, $Z(\Gamma)$, is the minimum cardinality among all zero forcing sets. The notation of a zero forcing sets of G, as well as the associated zero forcing number of a graph was introduced by the "AIM Minimum Rank-Special Graphs Work Group" in (2008) [1]. They used the technique of zero forcing parameter of graph Γ and found an upper bound for the maximum nullity of Γ related to zero forcing sets. For more results in zero forcing number and Cayley graph, see [2, 4, 5, 6, 12].
Let $\Gamma = (V, E)$ be a graph and B a zero forcing set of Γ. Also let $B^{(0)} = B$ and for $t \geqslant 0$, $B^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \bigcup_{s=0}^{t} B^{(s)}$ such that w is the only neighbour of b not in $\bigcup_{s=0}^{t} B^{(s)}$. The propagation time of B in Γ, denoted by $Pt(\Gamma, B)$, is the smallest integer t_0 such that $V = \bigcup_{t=0}^{t_0} B^{(t)}$. The minimum propagation time of Γ is

$$Pt(\Gamma) = min\{Pt(\Gamma, B) \mid B \textit{ is a minimum zero forcing set of } \Gamma\}.$$

The propagation time of a zero forcing set was implicit in [3] and explicit in [10]. In 2012 Hogben et al. in [7] established some results regarding graphs having propagation time 1.
In this paper, the propagation time of $Cay(G, \Omega)$ is considered. Also it is shown that the propagation time of $Cay(G, \Omega)$ is at most two for some special generators.

2. **Preliminaries**

For investigating the propagation time of Cayley graphs, the following basic properties are useful.

Theorem 2.1. [2] *For any graph* Γ, $\delta(\Gamma) \leqslant Z(\Gamma)$.

Theorem 2.2. [6] *Let* Γ *be a connected graph of order* $n \geqslant 2$. *Then* $Z(\Gamma) = n - 1$ *if and only if* $\Gamma = K_n$.

Theorem 2.3. [7] *Let* Γ *be a graph. Then any two of the following conditions imply the third:*
1. $|\Gamma| = 2Z(\Gamma)$.
2. $Pt(\Gamma) = 1$.
3. Γ *is a matching graph.*

Lemma 2.4. *Let* $G = \langle \Omega \rangle$ *be a finite Abelian group,* $1 \notin \Omega = \Omega^{-1}$ *and* $G \setminus \Omega = \{x\} \cup H$ *such that* $x \notin H$. *If* H *is a subgroup of* G, *then* $o(x) = 2$, $|H| \bigm| |G|/2$ *and* $2 \bigm| [G : H]$.

Proof. Since $\Omega = \Omega^{-1}$ and H is a subgroup of G, $o(x) = 2$. So $N = \{1, x\}$ is a subgroup of G. Let $H = \{h_1 = 1, h_2, \dots, h_t\}$. Then for $i \neq j$ and $1 \leq i, j \leq t$, since $h_i h_j^{-1} \in H$, $Nh_i \neq Nh_j$ and so the cosets $N = Nh_1, Nh_2, \dots, Nh_t$ are distinct. If $G = \cup_{i=1}^{t} Nh_i$, then $[G : N] = t$. Otherwise, there is an $y_1 \in G \setminus \cup_{i=1}^{t} Nh_i$. It is easy to see that for $1 \leq i \leq t$ and $0 \leq j \leq 1$, the cosets $Nh_i y_j$ are distinct, where $y_0 = 1$. If $G = \cup_{j=0}^{1}(\cup_{i=1}^{t} Nh_i y_j)$, then $[G : N] = 2t$. Since G is a finite group, there is $\ell \in \mathbb{N}$ such that $Nh_i y_j$ for $1 \leq i \leq t$ and $0 \leq j \leq \ell$ are distinct and $G = \cup_{j=0}^{\ell}(\cup_{i=1}^{t} Nh_i y_j)$. Hence $[G : N] = t(\ell+1)$. Therefore $t \mid [G : N]$.
Similarly, if $G = H \cup Hx$, then $[G : H] = 2$. Otherwise, we can assume that there is a $y_1 \in G \setminus (H \cup Hx)$. Then for $0 \leq i \leq 1$ and $0 \leq j \leq 1$, the cosets $Hx_i y_j$ are distinct, where $x_0 = y_0 = 1$ and $x_1 = x$. Since G is a finite group, there is $\ell \in \mathbb{N}$ such that $Hx_i y_j$ for $0 \leq i \leq 1$ and $0 \leq j \leq \ell$ are distinct and $G = \cup_{i=0}^{1}(\cup_{j=0}^{\ell} Hx_i y_j)$. Hence $[G : H] = 2(\ell + 1)$. □

Lemma 2.5. *Let G be a group and H be a proper subgroup of G. Then $G = \langle G \setminus H \rangle$.*

Proof. It is clear that $G = H \cup \langle G \setminus H \rangle$. So $H \subseteq \langle G \setminus H \rangle$ or $\langle G \setminus H \rangle \subseteq H$. If $\langle G \setminus H \rangle \subseteq H$, then $G = H$, which is a contradiction. Thus $H \subseteq \langle G \setminus H \rangle$ and so $G = \langle G \setminus H \rangle$. □

Theorem 2.6. [11] *Let $K_{n_1,\dots,n_k}$ be a complete multipartite graph such that $n_i > 1$ for some $1 \leqslant i \leqslant k$. Then $Z(K_{n_1,\dots,n_k}) = n_1 + \cdots + n_k - 2$.*

Lemma 2.7. *Let $K_{n_1,\dots,n_k}$ $(n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k)$ be a complete multipartite graph. If $1 = n_1 = n_2 = \cdots = n_{k-1}$ and $2 \leqslant n_k$, then $Pt(K_{n_1,\dots,n_k}) = 2$. Otherwise, $Pt(K_{n_1,\dots,n_k}) = 1$.*

Proof. By Theorem 2.6, $Z(K_{n_1,\dots,n_k}) = k + n_k - 3 = n - 2$ where $n = n_1 + \cdots + n_k$. Let $V(K_{n_1,\dots,n_k}) = \bigcup_{i=1}^{k} V_i$ and $|V_i| = n_i$ for $1 \leqslant i \leqslant k$. Let $1 = n_1 = n_2 = \cdots = n_{k-1}$, $2 \leqslant n_k$ and $B = (\bigcup_{i=1}^{k} V_i) \setminus \{x, y\}$ be a zero forcing set for $K_{n_1,\dots,n_k}$. Then $x \in V_k$ and $y \notin V_k$ or $x \notin V_k$ and $y \in V_k$. Without loss of generality, we can assume that $x \in V_k$ and $y \in V_i$ for some $1 \leqslant i \leqslant k - 1$. Since y is not black vertex, x cannot be forced by any black vertex in the first stage. But every black vertex in V_k forces y and second stage x is forced by y. Thus $B^{(0)} = B$, $B^{(1)} = \{y\}$, $B^{(2)} = \{x\}$ and so $V(K_{n_1,\dots,n_k}) = B^{(0)} \cup B^{(1)} \cup B^{(2)}$. Hence for every zero forcing set B of $K_{n_1,\dots,n_k}$, we have $Pt(K_{n_1,\dots,n_k}, B) = 2$. Therefore $Pt(K_{n_1,\dots,n_k}) = 2$.
Let there exist $1 \leqslant i, j \leqslant k$ such that $2 \leqslant n_i \leqslant n_j$, $a \in V_i$, $b \in V_j$ and $B = (\bigcup_{i=1}^{k} V_i) \setminus \{a, b\}$ be the initial black vertices of $K_{n_1,\dots,n_k}$. Then every black vertex in V_i forces b and every black vertex in V_j

forces a, in the first stage. Hence, $B^{(0)} = B$, $B^{(1)} = \{a, b\}$ and so $V(K_{n_1,\dots,n_k}) = B^{(0)} \cup B^{(1)}$. Thus $Pt(K_{n_1,\dots,n_k}, B) = 1$ and therefore $Pt(K_{n_1,\dots,n_k}) = 1$. □

3. Propagation time for a finite group

In this section, the propagation time of Cayley graph for some groups with special generator set is considered.

Theorem 3.1. *Let G be a finite group of order n and $H \neq \{1\}$ a proper subgroup of G. Then $Pt(Cay(G, G \setminus H)) = 1$.*

Proof. Set $\Omega = G \setminus H$. By Lemma 2.5, $G = \langle \Omega \rangle$. Also we have $\Omega = \Omega^{-1}$ and $1 \notin \Omega$. Let $[G : H] = k$ and $Ha_1, Ha_2, \dots, Ha_k$ be the distinct cosets of H in G, where $a_1 = 1$. For h_1 and h_2 in H, we have $(h_1a_j)(h_2a_j)^{-1} = h_1h_2^{-1} \in H$ $(1 \leq j \leq k)$. Thus induced subgraphs on Ha_i in $Cay(G, \Omega)$ for $1 \leq i \leq k$ are empty graph. Also suppose that $(ha_j)(h^{'}a_\ell)^{-1} \in H$ for $ha_j \in Ha_j$ and $h^{'}a_\ell \in Ha_\ell$. Then $a_ja_\ell^{-1} \in H$ and so $Ha_j = Ha_\ell$. Which is a contradiction. Thus $(ha_j)(h^{'}a_\ell)^{-1} \notin H$. Hence ha_j is adjacent to $h^{'}a_\ell$. Therefore $Cay(G, \Omega)$ is isomorphic to $K_{n_1,\dots,n_k}$ and $n_1 = \cdots = n_k = |H| \geq 2$. By Lemma 2.7, $Pt\big(Cay(G, \Omega)\big) = 1$. □

Theorem 3.2. *Let $G = \langle \Omega \rangle$ be a group of order n, $x \in \Omega$ and $o(x) = 2$. If $H = \big(\Omega \setminus \{x\}\big) \cup \{1\}$ is a normal subgroup of G, then $Pt(\big(Cay(G, \Omega)\big) = 1$.*

Proof. Since $o(x) = 2$, so n is even. Let $H = \{1 = h_1, h_2, \dots, h_t\}$. Then $h_ih_j^{-1} \in H$ and $(h_ix)(h_jx)^{-1} \in H$ for each $1 \leq i, j \leq t$. So induced subgraphs on H and $Hx = xH$ in $Cay(G, \Omega)$ are isomorphic to complete graph K_t. Also for $1 \leq i \leq t$, we have $N_{Cay(G,\Omega)}[h_i] = H \cup \{xh_i\}$ and $N_{Cay(G,\Omega)}[xh_i] = \{h_i\} \cup Hx$. Since $Cay(G, \Omega)$ is a t-regular connected graph, $G = H \cup Hx = H \cup xH$, so $n = 2t$. Thus $Cay(G, \Omega)$ is a matching graph. Let $B = H$ be the initial black vertices in $Cay(G, \Omega)$. For each $1 \leqslant i \leqslant t$, xh_i is the only white neighbour of black vertex h_i, so xh_i is forced by h_i. Thus B is a zero forcing set of $Cay(G, \Omega)$ and so $Z(Cay(G, \Omega)) \leqslant t$. Then by Theorem 2.1, $Z(Cay(G, \Omega)) = t = \frac{n}{2}$. Hence by Theorem 2.3, we get $Pt\big(Cay(G, \Omega)\big) = 1$. □

Theorem 3.3. *Let G be an Abelian group of order n and H a proper subgroup of G such that $[G : H] = \alpha$. Let $x \in G \setminus H$, $o(x) = 2$, $G \setminus (H \cup \{x\}) = \Omega$ and $G = \langle \Omega \rangle$. Then $Pt(Cay(G, \Omega)) = 1$*

Proof. Let $g \in G \setminus H$. Then $Hg \subseteq \Omega \cup \{x\}$ and induced subgraphs on H and Hg in $Cay(G, \Omega)$ are empty. By Lemma 2.4, $\alpha = 2k$, for some $k \in \mathbb{N}$ and $G = \cup_{j=1}^{k} Hy_jx \cup_{j=1}^{k} Hy_j$, where the cosets Hy_jx and Hy_j are distinct ($y_1 = 1$). By definition of Cayley graph, every vertex $hy_jx \in Hy_jx$ is adjacent to all of the vertices of $G \setminus (Hy_jx \cup \{hy_j\})$. Let B be a zero forcing set of $Cay(G, \Omega)$ such that $Z(Cay(G, \Omega)) = |B|$. Since $Cay(G, \Omega)$ is a vertex transitive graph, we may assume that $1 \in B$ is the first forcing process. So there is $C \subseteq \Omega \cap B$ such that $|C| = |\Omega| - 1$. So $|\Omega| \leqslant Z(Cay(G, \Omega))$. If there are three white vertices in H, then each black vertex has at least two white vertices in its neighborhood. Thus the forcing process is stopped, which is not possible.
So $n - 4 \leqslant Z(Cay(G, \Omega))$. Let $B = G \setminus \{h_i, h_j, x, h_\ell x\}$ be the initial black vertices in $Cay(G, \Omega)$, where h_i, h_j and h_ℓ are distinct and belong to H. Since $h_\ell x$ is the only white neighbour of black vertex 1, so $h_\ell x$ is forced by 1. Since h_i is the only white neighbour of black vertex h_jx, so h_jx forces h_i. Similarly h_ix forces h_j. Also x is the only white neighbour of black vertex h_ℓ, so x is forced by h_ℓ. Thus $Z(Cay(G, \Omega)) = n - 4$. Furthermore we have $G = B^{(0)} \cup B^{(1)}$ and so $Pt(Cay(G, \Omega), B) = 1$. This shows that $Pt(Cay(G, \Omega)) = 1$. □

Corollary 3.4. *Let $G = \langle a \rangle$ be a cyclic group of order $2n$, where n is odd. If $\Omega = \{a^{2i+1} \mid 0 \leq i \leq n-1\} \setminus \{a^n\}$, then $Pt\big(\big(Cay(G, \Omega)\big)\big) = 1$.*

Proof. It is easy to see that if $\langle a^2 \rangle = H$, then $G \setminus \Omega = H \cup \{a^n\}$. The result follows by Theorem 3.3. □

Theorem 3.5. *Let $G = \langle \Omega \rangle$ be a finite group of order $n \geqslant 5$, $1 \notin \Omega = \Omega^{-1}$ and $Z(Cay(G, \Omega)) = |\Omega|$.*
1. If $Pt(Cay(G, \Omega)) = 1$, then $|G \setminus \Omega| \leq |\Omega|$.
2. If $Pt(Cay(G, \Omega)) = 1$ and $|G \setminus \Omega| = |\Omega|$, then G is not a simple group.

Proof. Let B be a zero forcing set for $Cay(G, \Omega)$ with minimum cardinality such that $Pt(Cay(G, \Omega), B) = 1$. Since $Cay(G, \Omega)$ is a vertex transitive graph, we may assume that $1 \in B$ is the first forcing process. Hence $B = \{1\} \cup \Omega \setminus \{a\}$, for some $a \in \Omega$. Since $Pt(Cay(G, \Omega), B) = 1$, for every $x \in \Omega \setminus \{a\}$ and $y \in G \setminus B$, we have $|N_{Cay(G,\Omega)}(x) \cap G \setminus B| \leq 1$ and $|N_{Cay(G,\Omega)}[y] \cap B| \geq 1$.Thus $|G \setminus B| \leq |B|$ and so $|G \setminus \Omega| \leq |\Omega|$.
Now let $Pt(Cay(G, \Omega)) = 1$ and $|G \setminus \Omega| = |\Omega|$. By Theorem 2.3, $Cay(G, \Omega)$ is a matching graph.
Let B be a zero forcing set for $Cay(G, \Omega)$ with minimum cardinality such that $Pt(Cay(G, \Omega), B) = 1$. We may assume that $B = \{1\} \cup \Omega \setminus \{a\}$, where $a \in \Omega$. Since $Cay(G, \Omega)$ is a $|\Omega|-$regular graph and $|\Omega| = |B|$, induced subgraphs on B and $G \setminus B$ are complete graph $K_{\frac{n}{2}}$.

Also $N_{Cay(G,\Omega)}[a] \cap \Omega = \{a\}$. We claim that $o(a) = 2$.
Let $o(a) = k$ and $k \neq 2$. Since a^2 is adjacent to a, $a^2 \notin \Omega$. Thus $k \neq 3$. If $k = 4$, then since $n \geq 5$, there is an $x \in B \setminus \{1, a^{-1}\}$. Thus x is adjacent to a^{-1} in $Cay(G,\Omega)$. So $xa \in \Omega$. It is clear that $(xa)a^{-1} = x \in \Omega$. Hence $xa \in \Omega$ is adjacent to a, in $Cay(G,\Omega)$, which is contract to this fact that $|N_{Cay(G,\Omega)}[a] \cap \Omega| = 1$. Now let $k \geq 5$. It is clear that a^2 is adjacent to a and so $a^2 \notin \Omega$. Thus a^3 is not adjacent to a in $Cay(G,\Omega)$. Hence $a^3 \in \Omega$. On the other hand a^3 is adjacent to a^2 and a^4. Thus $a^4 \in \Omega$. Also a^4 is adjacent to a, which is contract to this fact that $|N_{Cay(G,\Omega)}[a] \cap \Omega| = 1$. Therefore $o(a) = 2$. This shows that for every $x \in B$ we have $x^{-1} \in B$. Since induced subgraph on B is complete graph $K_{\frac{n}{2}}$, so $xy^{-1} \in B$, for every x and y belong to B. Therefore B is a subgroup of G. Furthermore $G = B \cup Ba$ or $[G : B] = 2$. Hence B is a normal subgroup of G and so G is not a simple group. □

Theorem 3.6. *Let $G =< a >$ be a cyclic group of order even $n \geq 6$ and let $\Omega = G \setminus \{1, a^2, a^{\frac{n}{2}}, a^{n-2}\}$. Then*

$$Pt(Cay(G,\Omega)) = \begin{cases} 1 & n \in \{8, 12\} \\ 2 & otherwise \end{cases}.$$

Proof. Let $n = 6$. Then $Cay(G,\Omega)$ is isomorphic to C_6. Hence, $Pt(Cay(G,\Omega)) = 2$.
Let $n = 8$. Then $G \setminus \Omega = \{1, a^2, a^4, a^6\}$ is a subgroup of G. By Theorem 3.1, $Pt(Cay(G,\Omega)) = 1$. Let $n = 12$ and B be a zero forcing set of $Cay(G,\Omega)$ with minimum cardinality. Since $Cay(G,\Omega)$ is a vertex transitive graph, we may assume that $1 \in B$ is the first forcing process. Then there exists $C \subseteq \Omega$ such that $C \subseteq B$ and $|C| = 7$. So $8 \leqslant |B|$. Also we have $N_{Cay(G,\Omega)}[a^2] = N_{Cay(G,\Omega)}[a^6] = N_{Cay(G,\Omega)}[a^{10}] = G \setminus \{1, a^4, a^8\}$. So there exist $D \subseteq \{a^2, a^6, a^{10}\}$ such that $D \subseteq B$ and $|D| = 2$. Hence, $10 \leqslant |B|$. Since $Cay(G,\Omega)$ is not a complete graph, we have $|B| = 10$, it is from Theorem 2.2. Suppose that $B = G \setminus \{a^4, a^{10}\}$, then $B^{(1)} = \{a^4, a^{10}\}$. Thus $G = B^{(0)} \cup B^{(1)}$ and so $Pt(Cay(G,\Omega), B) = 1$. Therefore $Pt(Cay(G,\Omega)) = 1$.
Now let $n \geqslant 10$ be even and $n \neq 12$. Then $a^{\frac{n}{2}-2} \in \Omega$, $a^{\frac{n}{2}-2}$ is not adjacent to a^{-2}, $a^{\frac{n}{2}-2}$ is not adjacent to $a^{\frac{n}{2}}$ and $a^{\frac{n}{2}-2}$ is adjacent to a^2. Also we have $a^{\frac{n}{2}+2} \in \Omega$, $a^{\frac{n}{2}+2}$ is not adjacent to a^2, $a^{\frac{n}{2}+2}$ is not adjacent to $a^{\frac{n}{2}}$ and $a^{\frac{n}{2}+2}$ is adjacent to a^{-2}.
If $X = G \setminus \{a^2, a^{\frac{n}{2}}, a^{-2}, a^{\frac{n}{2}+4}\}$ is initial black vertices in $Cay(G,\Omega)$, then 1 forces $a^{\frac{n}{2}+4}$ and $a^{\frac{n}{2}+2}$ forces a^{-2} in the first stage. Also we have $a^{\frac{n}{2}} \in N_{Cay(G,\Omega)}(a^4)$ and $a^2 \notin N_{Cay(G,\Omega)}(a^4)$. So $a^{\frac{n}{2}-2}$ forces a^2 and a^4 forces $a^{\frac{n}{2}}$ in the second stage. Hence, X is a zero forcing set of $Cay(G,\Omega)$ and so $Z(Cay(G,\Omega)) \leq |X| = n - 4$. By Theorem

2.1, $Z(Cay(G,\Omega)) = n-4$. Furthermore we have $X^{(0)} = X$, $X^{(1)} = \{a^{-2}, a^{\frac{n}{2}+4}\}$ and $X^{(2)} = \{a^2, a^{\frac{n}{2}}\}$. Hence, $Pt(Cay(G,\Omega), X) = 2$ and so $Pt(Cay(G,\Omega)) \leq 2$.
On the contrary, let B be a zero forcing set of $Cay(G,\Omega)$ with minimum cardinality such that $Pt(Cay(G,\Omega), B) = 1$. We may assume that $1 \in B$ is the first forcing process. Then there exists $a^\ell \in \Omega$ such that $a^\ell \notin B$ and 1 forces a^ℓ. Since $Z(Cay(G,\Omega)) = n-4$, so $\{a^2, a^{\frac{n}{2}}, a^{-2}\} \cap B = \phi$. Hence there exist a^j, a^k and a^r in Ω such that $a^\ell \notin N_{Cay(G,\Omega)}(a^k) \cup N_{Cay(G,\Omega)}(a^j) \cup N_{Cay(G,\Omega)}(a^r)$ and $a^j \in N_{Cay(G,\Omega)}(a^2)$, $a^k \in N_{Cay(G,\Omega)}(a^{\frac{n}{2}})$ and $a^r \in N_{Cay(G,\Omega)}(a^{-2})$.
Furthermore $a^j \notin N_{Cay(G,\Omega)}(a^{-2}) \cup N_{Cay(G,\Omega)}(a^{\frac{n}{2}})$, $a^k \notin N_{Cay(G,\Omega)}(a^{-2}) \cup N_{Cay(G,\Omega)}(a^2)$ and $a^r \notin N_{Cay(G,\Omega)}(a^{\frac{n}{2}}) \cup N_{Cay(G,\Omega)}(a^2)$. We have $a^2 \notin N_{Cay(G,\Omega)}(a^4) \cup N_{Cay(G,\Omega)}(a^{\frac{n}{2}+2}) \cup N_{Cay(G,\Omega)}(1)$. So $k \in \{4, \frac{n}{2}+2\}$. Since $a^k \in N_{Cay(G,\Omega)}(a^{\frac{n}{2}})$, $k = 4$. We know that $a^k \notin N_{Cay(G,\Omega)}(a^{-2})$, so a^4 is not adjacency to a^{-2}. Which is a contradiction. Therefore $Pt(Cay(G,\Omega)) = 2$. □

Let $U_{6n} =< a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} >$. Then $U_{6n} = \bigcup_{i=1}^{2}(V_i \cup V_i b \cup V_i b^2)$, where $V_i = \{a^{2k-i} \mid 1 \leqslant k \leqslant n\}$ for $i = 1, 2$. With this notations we prove the following results.

Theorem 3.7. *Let $G \cong U_{6n}$ and $\Omega = V_1 \cup V_1 b \cup V_1 b^2$. Then*

$$Pt(Cay(G,\Omega)) = 1.$$

Proof. By the definition of Cayley graph, the induced subgraph on Ω is empty. Since $Cay(G,\Omega)$ is $3n-$regular, so every vertex of Ω is adjacent to every vertex in $G \setminus \Omega$. Hence, $Cay(G,\Omega)$ is isomorphic to complete bipartite graph $K_{3n,3n}$. By Lemma 2.7, $Pt(Cay(G,\Omega)) = 1$. □

Theorem 3.8. *Let n be odd, $G \cong U_{6n}$ and $\Omega = V_2 \setminus \{1\} \cup V_2 b \cup V_2 b^2 \cup \{a^n\}$. Then $Pt(Cay(G,\Omega)) = 1$.*

Proof. Let $X = V_2 \cup V_2 b \cup V_2 b^2$ and $Y = V_1 \cup V_1 b \cup V_1 b^2$. Then the induced subgraphs on X and Y are isomorphic to complete graph K_{3n}. Also $Cay(G,\Omega)$ is isomorphic to graph in Figure 2.

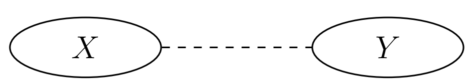

FIGURE 2: Dashed line: Every vertex of X is adjacent to exactly one vertex of Y.

Let X be the set of initial black vertices of $Cay(G,\Omega)$. Then for every $0 \leqslant k \leq n-1$, a^{2k} forces a^{2k+n}, $a^{2k}b$ forces $a^{2k+n}b$ and $a^{2k}b^2$ forces $a^{2k+n}b^2$. Hence, X is a zero forcing set of $Cay(G,\Omega)$ and so $Z(Cay(G,\Omega)) \leq |X| = 3n$. By Theorem 2.1, $Z(Cay(G,\Omega)) = 3n$. Since

$Cay(G, \Omega)$ is a matching graph and $|Cay(G, \Omega)| = 2Z(Cay(G, \Omega))$, by Theorem 2.3, $Pt(Cay(G, \Omega)) = 1$. $\square$

Question 3.9. *Which Cayley graphs have propagation time one?*

Acknowledgments

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