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ON THE TOTAL RESTRAINED DOUBLE ITALIAN DOMINATION

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Abstract. A double Italian dominating (DID) function of a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that for every vertex $v \in V$, $\sum_{u \in N_G[v]} f(u) \geq 3$, if $f(v) \in \{0, 1\}$. A restrained double Italian dominating (RDID) function is a DID function f such that the subgraph induced by the vertices with label 0 has no isolated vertex. A total restrained double Italian dominating (TRDID) function is an RDID function f such that the set $\{v \in V : f(v) > 0\}$ induces a subgraph with no isolated vertex.

We initiate the study of TRDID function of any graph G. The TRDID and RDID functions of the middle of any graph G are investigated, and then, the sharp bounds for these parameters are established. Finally, for a graph H, we provide the minimum value of TRDID and RDID functions for corona graphs, $H \circ K_1$, $H \circ K_2$ and middle of them.

1. Introduction

For definitions and notations not given here we refer to [9, 25]. For a set $S \subseteq V$, the *open neighbourhood* is $N(S) = \bigcup_{v \in S} N(v)$ while the *closed neighbourhood* is $N[S] = N(S) \cup S$. The degree of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The maximum degree and minimum degree of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. For a subset D of vertices in a graph G, we denote by $G[D]$, as a subgraph induced

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by D. The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$, and two vertices of $M(G)$ are adjacent if and only if either they are adjacent edges of G or one is a vertex and the other is an edge of G, incident with it. A complete graph of order n, a complete bipartite graph with two partite sets of cardinalities m and n, and a star graph of order $n+1$ are denoted by K_n, $K_{m,n}$, $K_{1,n}$ respectively. A double star $S_{p,q}$ is a tree with only two support vertices such that one of them has p leaves and the other has q leaves. A graph with only one vertex is said to be a trivial graph. Finally C_n, P_n denote the cycle, path with n vertices respectively. The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 (where G_i is of order n_i) is defined as the graph G obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining by an edge of the ith vertex of G_1 to every vertex in the ith copy of G_2. For a simple graph G, an edge cover set of G is a set of edges such that every vertex of the graph is incident to at least one edge of the set. The minimum cardinality of any edge cover set is called edge cover number, denoted by $c = c(G)$ [25].

For a graph G with no isolated vertex, a set S of vertices of G such that every vertex of G is adjacent to at least one vertex in S is called a total dominating (TD) set of G. In the other words, S is a TD set of G if S is a dominating set of G and $G[S]$ has no isolated vertex. A minimum cardinality of a TD set of G is called total domination number denoted by $\gamma_t(G)$. A restrained dominating set is a subset R of V such that the subgraph induced by $V - R$ has no isolated vertex. A minimum size of any restrained dominating set of G is called restrained domination number denoted by $\gamma_r(G)$. Restrained domination was formally defined by Domke et al. in [7]. For more information on this parameter we refer the reader to the survey paper [10].

Roman domination was introduced by Cockayne et al. in [6], although this notion was inspired by the work of ReVelle et al. in [18], and Stewart in [19]. The original study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, 274-337 A.D. He decreed that for all cities in the Roman Empire, at most two legions should be stationed. For more details and some applications see [5, 22].

Chellali et al. [4] have introduced a Roman $\{2\}$-dominating function f. Roman $\{2\}$-domination is a generalization of Roman domination that has also studied by Henning et al. [11] with the name of Italian dominating function. In terms of the Roman Empire, the Italian dominating strategy requires that every location with no guard has a neighboring location with two guards, or at least two neighboring locations with one guard each.

Formally, an Italian dominating (ID) function $f\colon V \to \{0,1,2\}$ such that for every vertex $v \in V$, with $f(v) = 0$, $f(N(v)) \geq 2$, that is, either there is a vertex $u \in N(v)$ with $f(u) = 2$, or there exist at least two vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. Note that for an ID function f, it is possible that $f(N[v]) \geq 1$ for some vertex with $f(v) = 1$. The weight of an ID function is the sum $w(f) = \sum_{v \in V(G)} f(v)$ and the minimum weight of an ID function f is the Italian domination number, denoted by $\gamma_I(G)$, see also [20, 24].

Beeler et al. [3] have defined double Roman domination. What they propose is a stronger version of Roman domination that doubles the protection by ensuring that any attack can be defended by at least two legions. In Roman domination at most two Roman legions are deployed at any one location. But as we will see in what follows, the ability to deploy three legions at a given location provides a level of defense that is both stronger and more flexible, at less than the anticipated additional cost.

A double Roman dominating (DRD) function on a graph G is a function $f\colon V \to \{0,1,2,3\}$ such that the following conditions are met:

- **(a):** If $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.
- **(b):** If $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \cup V_3$.

The weight of a DRD function f on G is the sum $w(f) = \sum_{v \in V(G)} f(v)$ and the minimum weight of $w(f)$ for every double Roman dominating function f on G is called double Roman domination number of G denoted by $\gamma_{dR}(G)$ and a DRD function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$ function of G, see more in [8, 14, 16, 21].

Mojdeh and Volkmann [17] considered a variant of double Roman domination and Italian domination which they called double Italian domination. What they proposed, is a stronger version of Roman and Italian domination that support the protection by ensuring that any attack can be defended by at least three or more legions from one or more other locations.

A double Italian dominating DID (Roman $\{3\}$ dominating) function is a function $f\colon V(G) \to \{0,1,2,3\}$ having the property that for every vertex $u \in V$, if $f(u) \in \{0,1\}$, then $f(N[u] \geq 3)$. Formally, a DID function $f\colon V(G) \to \{0,1,2,3\}$ has the property that for every vertex $v \in V$, with $f(v) = 0$, there exist at last either three vertices in $V_1 \cap N(v)$ or one vertex in $V_1 \cap N(v)$ and one vertex in $V_2 \cap N(v)$ or two vertices in $V_2 \cap N(v)$ or one vertex $V_3 \cap N(v)$. The weight of the DID function is the sum $w(f) = f(v) = \sum_{v \in V} f(v)$, and the minimum weight of DID function f is the double Italian domination number, denoted by $\gamma_{dI}(G)$, see also [1, 2, 12, 13].

Here we define the restrained double Italian domination (double Italian dominating set for which each vertex with label 0 has a neighbor with label 0). In terms of the double Italian Empire, this defense strategy requires that every location with no legion has at least also a neighboring location with no legion for lessing the cost of expenses of Empire.

A *restrained double Italian dominating function* (RDID) function is a DID function such that $V_0 = \{v \in V : f(v) = 0\}$ induces a subgraph with no isolated vertex. A minimum weight of any RDID function f is called a restrained double Italian domination number denoted by $\gamma_{rdI}(G)$ [23]. It is necessary to note that, there have been done some research works on restrained double Roman domination so far, [15].

In terms of restrained double Italian Empire, as well if we assume that any location with guards can be adjacent to a location with a guard, equivalently, any vertex with positive label has a neighbor vertex of positive label. This defense strategy is for lessing the cost of expenses of empire and more security. Further, this strategy provides a more flexible and stronger level of defense which is named total restrained double Italian dominating function.

Definition 1.1. A total restrained double Italian dominating function (TRDID) Function is a restrained double Italian dominating function such that the subgraph induced by the set $\{v \in V\colon f(v) \neq 0\}$ has no isolated vertex. A minimum weight of any TRDID function f is called a total restrained double Italian domination number denoted by $\gamma_{trdI}(G)$.

In the other words, a (TRDID) Function f is a DID function such that at the same time the sets $\{v \in V(G) : f(v) = 0\}$ and $\{v \in V(G) : f(v) > 0\}$ induce subgraphs without isolated vertices.

It is obvious that $\gamma_{dI}(G) \leq \gamma_{rdI}(G) \leq \gamma_{trdI}(G)$.
Our motivation for using middle graphs is to expand the two concepts RDID, TRDID functions, and make the defense strategy as favorable as possible.

This paper is organized as follows. The exact values of RDID function of Middle of standard graphs are established in Section 2. We study the TRDID function for standard graphs and middle of them and determine the precise value of $\gamma_{trdI}(G)$ for these graphs in Section 3. We peruse the exact bound on the RDID and TRDID function of $M(G)$ for any graph G in Section 4, and finally, these parameters are provided with the precise value for corona graphs $G = H \circ K_1$, $G = H \circ K_2$ and middle of them in Section 5.

2. RDID function of middle of standard graph

In this section we determine the restrained double Italian domination number of middle graph for cycles, paths, complete, star, complete bipartite and double star graphs

Volkmann [23] showed that.

Observation 2.1. ([23] Observation 2) *If $n \geq 3$ is an integer, then $\gamma_{rdI}(C_n) = n$.*

Proposition 2.2. *For any cycle C_n, $\gamma_{rdI}(M(C_n)) = \lceil \frac{3n}{2} \rceil$.*

Proof. Let C_n be a cycle with vertex set $V(C_n) = \{v_1, v_2, \cdots, v_n\}$. Let $M(C_n)$ be the middle of C_n and $V(C_n) \cup \{u_{1,2}, u_{2,3}, \cdots, u_{n-1,n}, u_{n1}\}$ be the set of vertices of $M(C_n)$ where $u_{i,i+1}$ is the new vertex corresponding to edge $e = v_i v_{i+1}$. For n even, devoting value 3 to each vertex in the set $\{u_{2i-1,2i} : 1 \leq i \leq \frac{n}{2}\}$ and zero otherwise (Figure 1 $M(C_6)$) and for n odd, devoting value 3 to each vertex in the set $\{u_{2i-1,2i} : 1 \leq i \leq \frac{n-1}{2}\}$, value 2 to v_n and zero otherwise (Figure 1 $M(C_7)$), show that $\gamma_{rdI}(M(C_n)) \leq \lceil \frac{3n}{2} \rceil$. On the other hand, for any RDID function f and any set

$$A = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, u_{i,i+1}, u_{i+1,i+2}, u_{i+2,i+3}, u_{i+3,i+4}\}$$

or

$$A = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, u_{i-1,i}, u_{i,i+1}, u_{i+1,i+2}, u_{i+2,i+3}\},$$

$f(A) \geq 6$. Sinc $M(C_n)$ has $2n$ vertices, $f(M(C_n)) \geq 6\frac{2n}{8} = \frac{3n}{2}$. Therefore $\gamma_{rdI}(M(C_n)) \geq \lceil \frac{3n}{2} \rceil$. It proves the result. □

From Observation 2.1 and Proposition 2.2, we have.

Corollary 2.3. *For any cycle C_n, $\gamma_{rdI}(M(C_n)) = \gamma_{rdI}(C_n) + c(C_n)$, where $c(C_n)$ is the edge cover number of cycle C_n.*

In [23] author showed that.

Observation 2.4. ([23] Observation 3) *If $n \geq 4$ is an integer, then $\gamma_{rdI}(P_n) = n + 2$.*

Now we have.

Proposition 2.5. *For any $n \geq 2$ and for path P_n, $\gamma_{rdI}(M(P_n)) = \lceil \frac{3n}{2} \rceil + 1$.*

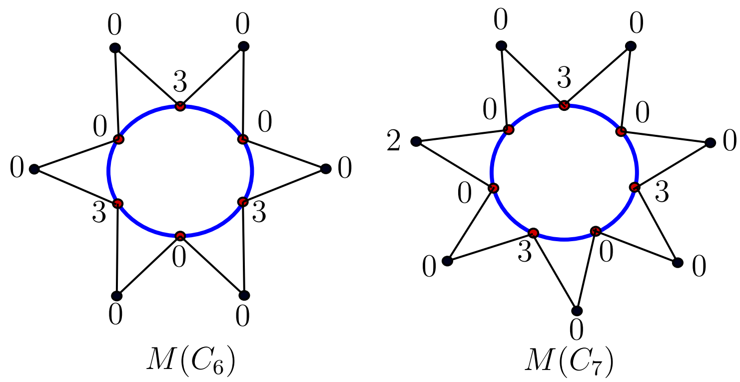

FIGURE 1. Restrained double Italian domination of $M(C_n)$

Proof. Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. Let $M(P_n)$ be the middle of P_n and $V(P_n) \cup \{u_{1,2}, u_{2,3}, \cdots, u_{n-1,n}\}$ be the vertex set of $V(M(P_n))$ where $u_{i,i+1}$ is the vertex corresponding to edge $e = v_iv_{i+1}$ where $1 \leq i \leq n-1$. If n is even, we devote the value 2 to the vertices v_1, v_n and value 3 to the vertices $u_{2i,2i+1}$ for $1 \leq i \leq \frac{n-2}{2}$ (Figure 2, $M(P_4)$).

For n odd, we devote the value 1 to the vertices v_n, value 2 to v_1 and value 3 to the vertices $u_{2i,2i+1}$ for $1 \leq i \leq \frac{n-1}{2}$ (Figures 2, $M(P_5)$). These show that $\gamma_{rdI}(M(P_n)) \leq \lceil \frac{3n}{2} \rceil + 1$, for $n \geq 2$.

Conversely, on the contrary, $\gamma_{rdI}(M(P_n)) \leq \lceil \frac{3n}{2} \rceil$. First we show that any $u_{i,i+1}$ cannot be devoted by value 2 or 1 under any γ_{rdI}-function. Let f be a such function and $f(u_{i,i+1}) = 2$. Then $f(v_i)$ and $f(v_{i+1})$ must be positive. Because, if each of them is assigned 0, then $f(u_{i+1,i+2}) = 2$ or $f(u_{i-1,i}) = 2$. This is impossible, since it is restrained. Now we bring up some cases.
Case 1. Let $f(v_1) = f(v_n) = 2$. Then $f(u_{n-1,n}) \leq 1$ and $f(u_{1,2}) \leq 1$. In this situation the weights of vertices adjacent to $u_{n-1,n}$ or $u_{1,2}$ do not related to the weights of $u_{n-1,n}$ or $u_{1,2}$. From $M(P_n)$ via adding vertex $u_{n,1}$ and make adjacent it to $v_n, v_1, u_{1,2}$ and $u_{n-1,n}$, a. $M(C_n)$ is formed. If we define g on $M(C_n)$ with $g(u_{n,1}) = 3$, $g(\{v_1, v_n, u_{1,2}, u_{n-1,n}\}) = 0$ and $g(x) = f(x)$ otherwise, then $w(g) \leq w(f) - 1 < \lceil \frac{3n}{2} \rceil$ a contradiction.
Case 2. Let $f(v_1) = 1$ or $f(v_n) = 1$. In this part, $f(u_{n,1})$ or $f(u_{1,2})$ is labeled by value 3. If $f(u_{1,2}) = 3$, then we define g on $M(C_n)$, via $g(v_1) = g(u_{1,2}) = 0$ and $g(x) = f(x)$ otherwise, then $w(g) \leq w(f) - 1 < \lceil \frac{3n}{2} \rceil$ a contradiction. These conflicts show that $\gamma_{rdI}(M(P_n)) \geq \lceil \frac{3n}{2} \rceil + 1$. Therefore the result is observed. $\square$

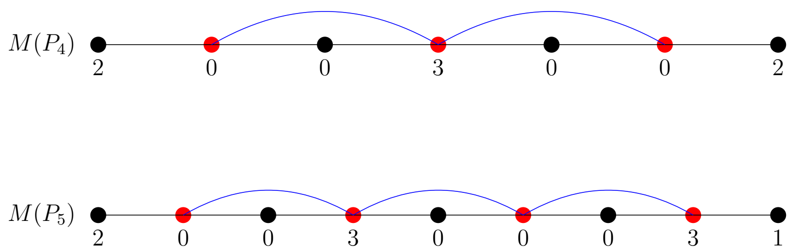

FIGURE 2. Restrained double Italian domination of $M(P_n)$

From Observation 3.6 and Proposition 2.5, we have.

Corollary 2.6. $\gamma_{rdI}\left(M(P_n)\right) = \gamma_{rdI}(P_n) + \lceil \frac{n-2}{2} \rceil$.

In [23] we have the following.

Observation 2.7. ([23] Observation 1) *(i)* $\gamma_{rdI}(K_n) = 3$ *for* $n \geq 2$.
(ii) $\gamma_{rdI}(K_{1,n}) = n + 2$ *for* $n \geq 1$.
(iii) $\gamma_{rdI}(K_{2,2}) = 4$, $\gamma_{rdI}(K_{2,3}) = 5$ *and* $\gamma_{rdI}(K_{m,n}) = 6$ *for* $m, n \geq 2$ *and* $m + n \geq 6$.

Here we investigate γ_{rdI} number of middle of $K_n, K_{1,n}$ and $K_{m,n}$.

Proposition 2.8. *For any complete graph* K_n, $\gamma_{rdI}\left(M(K_n)\right) = \lceil \frac{3n}{2} \rceil$.

Proof. Let K_n be a complete graph and $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ be its vertex set. Let $M(K_n)$ be the middle of K_n, with vertex set $V(K_n) \cup \{u_{i,j} : 1 \leq i < j \leq n\}$ where $u_{i,j}$ is the vertex corresponding to edge $e = v_iv_j$. For n even, devoting value 3 to the $u_{1,2}, u_{3,4}, \ldots, u_{n-1,n}$ and zero otherwise and for n odd, devoting value 3 to the $u_{1,2}, u_{3,4}, \ldots, u_{n-2,n-1}$, value 2 to v_n and zero otherwise, gives us an RDID function f with $w(f) = \lceil \frac{3n}{2} \rceil$. This shows that, $\gamma_{rdI}\left(M(K_n)\right) \leq \lceil \frac{3n}{2} \rceil$.

Conversely, for any two vertices v_i, v_j, define $V_{i,j} = \{v_i, v_j\} \cup \{u_{i,k}, u_{j,l} : k \geq i+1, \ and\, l \geq j+1\}$. Let n be even, and f be a $\gamma_{rdI}(M(K_n))$ function. Then $f(V_{2i-1,2i}) \geq 3$ for $1 \leq i \leq \frac{n}{2}$. Therefore $\gamma_{rdI}(M(K_n)) \geq \frac{3n}{2}$. If n is odd, and f is a $\gamma_{rdI}(M(K_n))$ function, then $f(V_{2i-1,2i}) \geq 3$, for $1 \leq i \leq \frac{n-1}{2}$ and in this case $f(V_n)$ must be at least 2. Therefore $\gamma_{rdI}(M(K_n)) \geq \frac{3(n-1)}{2} + 2 = \lceil \frac{3n}{2} \rceil$. Thus the result holds. □

In [23] we have for star graph $K_{1,n}$, $\gamma_{rdI}(K_{1,n}) = n + 1$.
We now study restrained double Italian domination of middle of star graph, $M(K_{1,n})$.

Proposition 2.9. *For star* $K_{1,n}$ *we have* $\gamma_{rdI}M(K_{1,n}) = 2n + 2$.

Proof. Let $V(K_{1,n}) = \{v_0, v_1, \cdots, v_n\}$. Let $M(K_{1,n})$ be the middle of K_1, n with vertex set $V(M(K_{1,n})) = V(K_{1,n}) \cup \{u_{0,1}, u_{0,2}, \cdots, u_{0,n}\}$ where $u_{0,i}$ is the vertex corresponding to edge $e_i = v_0 v_i$. $M(K_1, n)$ is formed from K_{n+1} with vertices v_0 and $u_{0,i}$ for $1 \leq i \leq n$, such that the vertex v_i is a leaf neighbor of $u_{0,i}$ for $1 \leq i \leq n$. Let f be a γ_{rdI} function. Then $f(v_i) + f(u_{0,i}) \geq 2$, $\cup_{j=1}^{k} f(u_{0,j}) \cup f(v_0) \geq 2$ and $f(v_i) \geq 1$ $(1 \leq i \leq k)$. These show that $\gamma_{rdI} M(K_{1,n}) \geq 2n + 2$ (Figure 3). On the other hand, the assignment 2 to each v_i for $0 \leq i \leq n$ gives us an RDID function of $M(K_{1,n})$ of weight $n + 1$. Therefore $\gamma_{rdI} M(K_{1,n}) \leq 2n + 2$. Thus the proof is observed. □

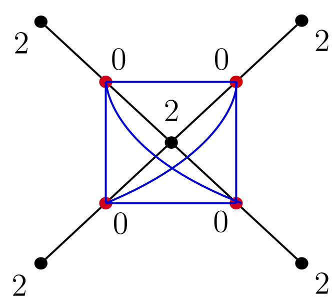

FIGURE 3. Restrained double Italian domination of $M(K_{1,4})$

As a prompt result we have $\gamma_{rdI}(M(K_{1,n})) = 2\gamma_{rdI}(K_{1,n})$.

Proposition 2.10. *Let $K_{m,n}$ be a complete bipartite graph with $m \geq n$. Then $\gamma_{rdI}\left(M(K_{m,n})\right) = 2m + n$.*

Proof. Let M, N be two partite sets of graph $K_{m,n}$, with the set of vertices $M = \{v_1, v_2, \cdots, v_m\}$ and $N = \{w_1, w_2, \cdots, w_n\}$. Let $M(K_{m,n})$ be the middle of $K_{m,n}$ with vertex set $V(M(K_{m,n})) = V(K_{m,n}) \cup \{u_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$, where $u_{i,j}$ is the vertex corresponding to edge $e = v_i w_j$. Now devoting value 3 to the vertex $u_{j,j}$ for $1 \leq j \leq n$, value 2 to $r = m - n$ vertices like v_{n+k} where $1 \leq k \leq m - n$ in M, deduce that $\gamma_{rdI}\left(M(K_{n,m})\right) \leq 3n + 2(m - n) = 2m + n$.

Let now $V_{k,k} = \{v_k, w_k\} \cup \{u_{k,k}, u_{k,j}, u_{i,k} : k+1 \leq j \leq n, k+1 \leq i \leq m\}$ where $1 \leq k \leq n$ and $V_{n+l} = \{v_{n+l}\}$ for $1 \leq l \leq m - n$. If f is a $\gamma_{rdI}(M(K_{m,n}))$ function, then $f(V_{k,k}) \geq 3$ and so $f(V_{m+l})$ must be at least 2. Hence $\gamma_{rdI}(M(K_{m,n})) \geq 3n + 2(m - n) = 2m + n$. Therefore $\gamma_{rdI}(M(K_{m,n})) = 2m + n$. □

Let $S_{p,q}$ be a double star and $p, q \geq 2$. It is easy to see that.

Observation 2.11. *For double star $S_{p,q}$, $\gamma_{rdI}(S_{p,q}) = p + q + 4$.*

Proposition 2.12. *For double star $S_{p,q}$, with $p, q \geq 2$, $\gamma_{rdI}(M(S_{p,q})) = 2(p+q)+3$.*

Proof. Let $V(S_{p,q}) = \{u_0, u_1, \cdots, u_p, v_0, v_1, \cdots, v_q\}$. Let $M(S_{p,q})$ be the middle of $S_{p,q}$ and $V(M(S_{p,q})) = V(S_{p,q}) \cup \{x_{0,1}, x_{0,2}, \cdots, x_{0,p}\} \cup \{y_{0,1}, y_{0,2}, \cdots, y_{0,q}\} \cup \{z_0\}$ where $x_{0,i}$, $y_{0,j}$ is the vertex corresponding to edge $e = u_0u_i$, v_0v_j and z_0 is a vertex corresponding to $e = u_0v_0$. As we see in Figure 4, $M(S(3,4))$, the set $\{u_0, x_{0,1}, x_{0,2}, \cdots, x_{0,p}\}$ induces a clique of order $p+1$, also the set $\{v_0, y_{0,1}, y_{0,2}, \cdots, y_{0,p}\}$ induces a clique of order $q+1$ in $M(S_{p,q})$. The vertex z_0 is adjacent to all vertices of these two cliques. The vertex u_i is a leaf neighbor of $x_{0,i}$ for $1 \leq i \leq p$ and vertex v_j is a leaf neighbor of $y_{0,j}$ for $1 \leq j \leq q$ in $M(S_{p,q})$. It is clear, under any RDID function, any leaf has a positive weight and summation of the value of any leaf and value of its support must be at least 2, furthermore, if we devote label 1 to a leaf, then its support must be devoted by value 2. On the other hand, for any RDID function f of $M(S_{p,q})$, $f(N[z_0]) \geq 3$. Thus $\gamma_{rdI}(M(S_{p,q})) \geq 2(p+q)+3$. Since assignments 2 to any leaf and 3 to z_0 leads to an RDID function of weight $2(p+q)+3$, thus $\gamma_{rdI}(M(S_{p,q})) \leq 2(p+q)+3$. Therefore we observe the result. □

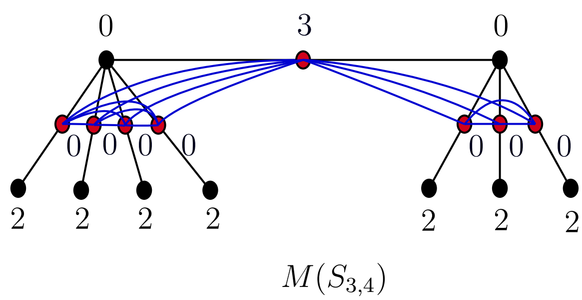

FIGURE 4. Restrained double Italian domination of $M(S_{3,4})$

3. TRDID FUNCTION OF MIDDLE OF STANDARD GRAPH

In this section we determine the total restrained double Italian domination number of middle graph for cycles, paths, complete, star, complete bipartite and double star graphs.

Proposition 3.1. *For any cycle C_n, $\gamma_{trdI}(M(C_n)) = 2n$.*

Proof. Let C_n be a cycle with vertex set $V(C_n) = \{v_1, v_2, \cdots, v_n\}$. Let $M(C_n)$ be the middle of C_n with vertex set as stated in Proposition 2.2. Assigning value 1 to the all vertices in $M(C_n)$ shows that $\gamma_{trdI}(M(C_n)) \leq 2n$ (Figure 5).

On the other hand, we prove that for any TRDID function f of $M(C_n)$, $w(f) \geq 2n$. First, we show that, any vertex of $M(C_n)$ under any TRDID function f cannot be necessarily devoted by label 2. On the contrary, let $f(v_i) = 2$. Then $f(u_{i,i+1}) \geq 1$ or $f(u_{i-1,i}) \geq 1$. If two of them are positive, then we change $f(v_i) = 1$.
So let $f(v_i) = 2$ and $f(u_{i-1,i}) \geq 1$ or $f(u_{i,i+1}) = 0$. If $f(u_{i-1,i}) \geq 2$, then we change $f(v_i) = 1$.
Let $f((v_i)) = 2$, $f(u_{i-1,i}) = 1$ and $f(u_{i,i+1}) = 0$. Then $f(v_{i+1}) = 0$, $f(u_{i+1,i+2}) = 3$, $f(v_{i+2}) = 1$ and $f(u_{i+2,i+3}) \geq 0$. If $f(v_{i-1}) \geq 2$, then we change $f(u_{i-1,i}) = 3$, $f(v_{i-1}) = 1$, $f(v_i) = 0$. Hence, it is not necessarily, a vertex of $M(C_n)$ is assigned by label 2.
Now we display that, all vertices are devoted by only value 1, or all v_is are devoted by 1 or 0, and all $(u_{i,i+1})$s by values 0 or 3, for $(1 \leq i \leq n)$, $(mod\ n)$.
If for a vertex v_i, $f(v_i) = 3$, then $f(u_{i,i+1}) + f(u_{i-1,i}) \geq 1$. But in this mood, we change $f(v_i)$ to 1 or 2. Thus $f(v_i) \leq 1$.
If $f(u_{i,i+1}) = 3$, then $f(v_i) = 1$, $f(u_{i+1,i+2}) = f(v_{i+1}) = 0$. This assignments can be continued with $f(u_{i+2,i+3}) = 3$, $f(v_{i+3}) = 1$, $f(u_{i+3,i+4}) = f(v_{i+4}) = 0$, $(mod\ n)$. This devoting leads to $w(f) = 2n$.
If $f(u_{i,i+1}) = 3$, $f(v_i) = 1$, $f(u_{i+1,i+2}) = f(v_{i+1}) = 1$, but $f(u_{i+2,i+3}) = 3$, $f(v_{i+3}) = 1$, then we change $f(u_{i+1,i+2}) = f(v_{i+1}) = 0$.
If $f(u_{i,i+1}) = 3$, $f(v_i) = 1$, $f(u_{i+1,i+2}) = f(v_{i+1}) = f(u_{i+2,i+3}) = f(v_{i+3}) = 1$, then we change $f(u_{i+1,i+2}) = f(v_{i+1}) = 0$ and $f(u_{i+2,i+3}) = 3$.
If $f(u_{i,i+1}) = 1$, it is clear that, $f(v_{i+1}) = f(u_{i+1,i+2}) = f(v_i) = f(u_{i-1,i}) = 1$ and so all vertices are devoted by value 1. Now there exits two situations.

1. Let n be even (Figure 5, $M(C_6)$). We must assign the label 1, to all vertices, or $f(u_{2i-1,2i}) = 3$, $f(v_{2i}) = 1$, and $f(u_{2i,2i+1}) = f(v_{2i+1}) = 0$, for $1 \leq i \leq \frac{n}{2}$. Therefore $w(f) \geq 2n$. This proves f is a γ_{trdI} function with $w(f) = 2n$ for n even.

2. Let n be odd (Figure 5, $M(C_7)$). We display all vertices should be devoted by only value 1 under any γ_{trdI} function f. For this, assume that there is a TRDID function f such that, $f(u_{1,2}) = 3$, $f(v_2) = 1$, $f(u_{2,3}) = f(v_3) = 0$. This process leads to $f(u_{2i+1,2i+2}) = 3$, $f(v_{2i}) = 1$, $f(u_{2i+2,2i+3}) = f(v_{2i+3}) = 0$, for $1 \leq i \leq \frac{n-1}{2}$. Therefore $f(u_{n-2,n-1}) = 3$, $f(v_{n-1}) = 1$, $f(u_{n-1,n}) = f(v_n) = 0$. In this status, we should have $f(u_{n,1}) = 3$, $f(v_1) = 1$. This indicates $w(f) = 2n + 1 > 2n$. Therefore, all vertices must be assigned label 1. Thus the desired result holds. $\square$

By Observation 2.1 and since any TRDID function is an RDID function, we obviously have.

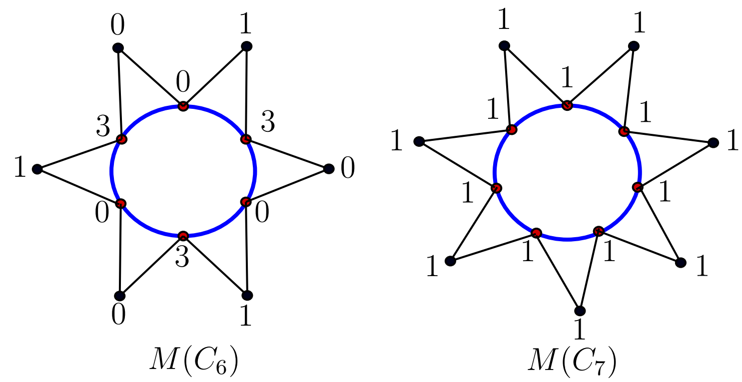

FIGURE 5. Total restrained double Italian domination of $M(C_n)$

Observation 3.2. *For any cycle graph C_n, $\gamma_{trdI}(C_n) = n = \gamma_{rdI}(C_n)$.*

From Proposition 2.2 we have.

Observation 3.3. *For any cycle C_n, $\gamma_{trdI}(M(C_n)) = 2\gamma_{trdI}(C_n)$.*

Corollary 3.4. *$\gamma_{trdI}(M(C_n)) = \gamma_{rdI}(C_n) + \alpha(C_n)$ where $\alpha(C_n)$ is the independence number of C_n.*

Now we discuss on TRDID function of $M(P_n)$.

Proposition 3.5. *For any path P_n, $\gamma_{trdI}(M(P_n)) = \begin{cases} 2n & \text{if } 2 \mid n \\ 2n+1 & \text{if } 2 \nmid n \end{cases}$.*

Proof. Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. Let $M(P_n)$ be the middle of P_n with vertex set as stated in Proposition 2.5 Any TRDRD function f devote labels positive weight to the vertices v_1, $u_{1,2}$, $u_{n-1,n}$ and v_n, such that $f(v_1) + f(u_{1,2}) + f(v_2) \geq 4$ and $f(v_n) + f(u_{n-1,n}) \geq 3$. Thus we bring up two situations.

1. Let n be an even integer. We devote value 1 to any vertex in the set vertex $\{v_1, v_{2i} : 2 \leq i \leq \frac{n}{2}\}$, value 3 to any vertex of the set vertex $\{u_{2i-1,2i} : 1 \leq i \leq \frac{n}{2}\}$, and zero otherwise (Figure 6, $M(P_6)$). Thus $\gamma_{trdI}(M(P_n)) \leq 2n$. On the other hand, let f be a TRDID function with $w(f) \leq 2n-1$. By adding a vertex $u_{n,1}$ and make adjacent the vertex $u_{n,1}$ to vertices $v_1, v_n, u_{1,2}$ and $u_{n-1,n}$. The resulted graph is $M(C_n)$. Now define g on the resulted $M(C_n)$ with $g(u_{n,1}) = g(v_1) = 0$, $g(u_{1,2}) = 3$, $g(v_2) = 1$ and $g(x) = f(x)$ otherwise. This g is a TRDID function with $w(g) = w(f) \leq 2n-1$ a contradiction with Proposition 3.1.

2. Let n be an odd integer. We devote any vertex of the set vertex $\{v_1, v_n, v_{2i} : 2 \leq i \leq \frac{n-1}{2}\}$ value 1, and any vertex of the set vertex

$\{u_{2i-1,2i} : 1 \leq i\frac{n-1}{2}\}$ value 3, and value 2 to the vertex $u_{n-1,n}$, value 0 to otherwise (Figure 6, $M(P_5)$). Thus $\gamma_{trdI}(M(P_n)) \leq 2n+1$.
On the other hand, let f be a TRDID function with $w(f) \leq 2n$. It is well known that $f((u_{n-1,n}) + f(v_n) \geq 3$ and $f(v_1) + f((u_{1,2}) + f(v_2) \geq 4$. By adding a new vertex $u_{n,1}$ to $M(P_n)$ and make adjacent the vertex $u_{n,1}$ to vertices $v_1, v_n, u_{1,2}$ and $u_{n-1,n}$. The resulted graph is $M(C_n)$. Now define g on the resulted $M(C_n)$ with $g(u_{n,n+1}) = 0 = g(v_1)$, $g(u_{1,2}) = 3$, $g(v_2) = 1$, and $g(x) = f(x)$ otherwise. This g is a TRDID function with $w(g) = w(f) \leq 2n$ where some vertices assigned by 2 or 3, a contradiction with Proposition 3.1. Therefore, the result is proved. □

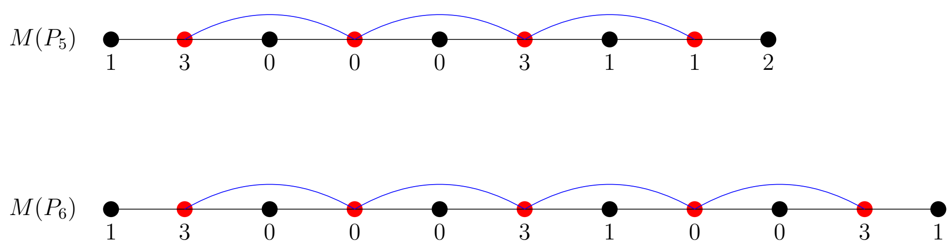

FIGURE 6. Total restrained double Italian domination of $M(P_n)$

From Observation 3.6 and devoting label 2 to the vertices v_1, v_2 and 1 otherwise of path P_n, we have.

Observation 3.6. *For any path P_n $(n \geq 4)$, $\gamma_{trdI}(P_n) = n+2$.*

Due to Proposition 3.5 and Observation 3.6, we obtain.

Observation 3.7. *For any path P_n,*
1. $\gamma_{trdI}(M(P_n)) = 2\gamma_{trdI}(P_n) - 4$ for n even.
2. $\gamma_{trdI}(M(P_n)) = 2\gamma_{trdI}(P_n) - 3$ for n odd.

From Propositions 2.5 and 3.5 we get the following outcome.

Corollary 3.8. $\gamma_{trdI}(M(P_n)) = \gamma_{rdI}(M(P_n)) + \lceil\frac{n-1}{2}\rceil$.

Now we moot on TRDID of $M(K_n)$.

Proposition 3.9. *For any complete graph K_n,*

$$\gamma_{trdI}(M(K_n)) = \begin{cases} 3\lceil\frac{n}{2}\rceil + \lceil\frac{n}{4}\rceil & \text{if } 2 \mid n \\ 3\lceil\frac{n}{2}\rceil + \lfloor\frac{n}{4}\rfloor & \text{if } 2 \nmid n \end{cases}.$$

Proof. Let K_n be a complete graph with $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Let $M(K_n)$ be the middle of K_n, with vertex set as stated in Proposition 2.8. For n even, devoting value 3 to the $u_{1,2}, u_{3,4}, \ldots, u_{n-1,n}$, value 1 to $u_{1,3}, u_{5,7}, \ldots, u_{n-3,n-1}$ and zero otherwise improve a TRDID function f with $w(f) \leq 3\lceil \frac{n}{2} \rceil + \lceil \frac{n}{4} \rceil$. For n odd, devoting value 3 to the $u_{1,2}, u_{3,4}, \ldots, u_{n-2,n-1}, u_{n-1,n}$, value 1 to $u_{1,3}, u_{5,7}, \ldots, u_{n-4,n-2}$, and zero otherwise, improves a TRDID function f with $w(f) \leq 3\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{4} \rfloor$.

Conversely, for any two vertices v_i, v_j, define $V_{i,j} = \{v_i, v_j\} \cup \{u_{i,k}, u_{j,l} : k \geq i+1, \ and \ l \geq j+1\}$. Let n be even, and f be a $\gamma_{trdI}(M(K_n))$ function. Then $f(V_{2i-1,2i}) \geq 3$ for $1 \leq i \leq \frac{n}{2}$. Furthermore, for any two sets $V_{2i-1,2i}, V_{2i+1,2i+2}$, the totality of f requires that we need one vertex like $u_{2k,2k+1}$ with positive weight for odd k where $1 \leq k \leq \frac{n}{2}$ $(mod\, n)$. Therefore $\gamma_{trdI}(M(K_n)) \geq \frac{3n}{2} + \lceil \frac{n}{4} \rceil$.
If n is odd, and f is a $\gamma_{trdI}(M(K_n))$ function, then $f(V_{2i-1,2i}) \geq 3$, for $1 \leq i \leq \frac{n-1}{2}$ and in this status $f(V_{n,n+1}) \geq 3$. Furthermore, for any two sets $V_{2i-1,2i}, V_{2i+1,2i+2}$, the totality of f requires that we need one vertex like $u_{2k,2k+1}$ with positive weight under f for odd k where $1 \leq k \leq \frac{n-1}{2}$ $(mod\, n)$. Therefore $\gamma_{trdI}(M(K_n)) \geq \frac{3(n+1)}{2} + \lfloor \frac{n}{4} \rfloor = 3\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{4} \rfloor$. Thus the result is proved. □

As an prompt outcome from Propositions 2.8 and 3.9, we get.

Corollary 3.10. *$\gamma_{trdI}(M(K_n)) = \gamma_{rdI}(M(K_n)) + \lceil \frac{n}{4} \rceil$ for any complete graph K_n.*

From matriculate of Proposition 2.9, we get the following outcome, which has a simple proof. Because of, for any TRDID function f of $M(K_{1,n})$, all leaves and support vertices must be devoted by positive value and if v is a leaf and w of its support, then $f(v) + f(w) \geq 3$. See Figure 7.

Proposition 3.11. *For star $K_{1,n}$, $\gamma_{trdI}(M(K_{1,n})) = 3n + 1$.*

As an instant result from Propositions 2.9 and 3.11 one can have.

Corollary 3.12. $\gamma_{trdI}(M((K_{1,n}))) = \gamma_{rdI}(M((K_{1,n}))) + (n-1)$.

In [23] RDID of complete bipartite graphs has been studied. In section 2, RDID of middle of complete bipartite graphs was perused. Here we would like to check TRDID number on middle of complete bipartite graphs.

Proposition 3.13. *Let $K_{m,n}$ be a complete bipartite graph with $m \geq n \geq 2$. Then*

$$\gamma_{trdI}\left(M\left(K_{m,n}\right)\right) = \begin{cases} 3m & if\, m \geq 2n \\ 3m + \lceil \frac{2n-m}{2} \rceil & otherwise \end{cases}.$$

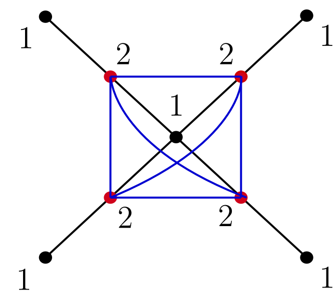

FIGURE 7. Total restrained double Italian domination of $M(K_{1,4})$

Proof. Let M, N be two partite sets of graph $K_{m,n}$ and $n \geq m$ where $M = \{v_1, v_2, \cdots, v_m\}$ and $N = \{w_1, w_2, \cdots, w_n\}$. Let $M(K_{m,n})$ be the middle of $K_{m,n}$ with vertex set as stated in 2.10. Further, in $M(K_{m,n})$ we have m cliques of size n, or n cliques of size m. Now we bring forth some cases.

Case 1. Let $m \geq kn$ where $k \geq 2$. Now devoting value 3 to the vertex $u_{j,j}$ for $1 \leq j \leq n$, and $u_{tn+r,r}$ where $1 \leq t \leq k$ and $1 \leq r < n$, $(tn+r \leq m)$, and zero otherwise infer that $\gamma_{trdI}\,(M(K_{m,n})) \leq 3n+3(m-n) = 3m$ (Figure 8, $M(K_{5,2})$).
On the other hand, we have m clique and for total restrained double Italian dominating requires at least two vertices of positive weights which sum of them at least 3. This grantees a TRDID function of weight $3m$ on $M(K_{m,n})$. Thus $\gamma_{trdI}(M(K_{m,n})) \geq 3m$ and it establishes $\gamma_{trdI}(M(K_{m,n})) = 3m$ for $m \geq 2n$.

Case 2. Let $n \leq m < 2n$ and $p = m - n$. Let now $V_{k,k} = \{v_k, w_k\} \cup \{u_{k,k}, u_{k,j}, u_{i,k} : k+1 \leq j \leq n, k+1 \leq i \leq m\}$ where $1 \leq k \leq n$ and $V_{n+r} = \{v_{n+r}\} \cup \{u_{n+r,j} : 1 \leq j \leq n\}$ for $1 \leq r \leq p$. If f is a $\gamma_{trdI}(M(K_{m,n}))$ function, then $f(V_{k,k}) \geq 3$ and also $f(V_{n+r}) \geq 3$. If we bring up $f(u_{k,k}) = 3 = f(u_{n+k,k})$, then the set $V_{k,k}$ for $1 \leq k \leq p$ has the totality property, but the set $V_{k,k}$ for $p+1 \leq k \leq n$ has no totality property. Thence by setting at least one common vertex of positive weight betwee two such $V_{k,k}$ for $p+1 \leq k \leq n$ and since $n-p = 2n-m$, $w(f) \geq 3m + \lceil \frac{2n-m}{2} \rceil$. This grantees $\gamma_{rdI}(M(K_{m,n})) \geq 3m + \lceil \frac{2n-m}{2} \rceil$ (Figure 8, $M(K_{4,3})$).
On the other hand, we allocate label 3 to vertices $u_{k,k}$, $1 \leq k \leq n$, $u_{n+r,r}$, $(1 \leq r \leq p)$. Furthermore, if $n-p$ is even, allocate label 1 to $u_{p+1,p+2}, u_{p+3,p+4}, \cdots, u_{n-1,n}$, and if $n-p$ is odd, allocate 1 to $u_{p+1,p+2}, u_{p+3,p+4}, \cdots, u_{n-2,n-1}, u_{n-1,n}$. This grantees $\gamma_{rdI}(M(K_{m,n})) \leq 3m + \lceil \frac{2n-m}{2} \rceil$. $\square$

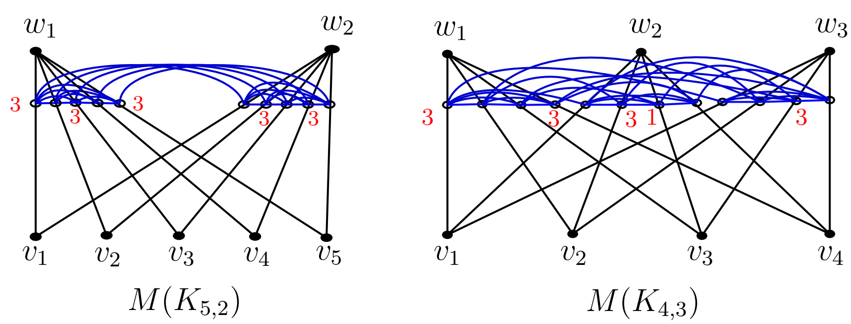

FIGURE 8. Total restrained double Italian domination of $M(K_{m,n})$

From the Propositions 2.10 and 3.13 we have.

Corollary 3.14. *Let $m \geq n \geq 2$. Then*

$$\gamma_{trdI}(K_{m,n}) = \begin{cases} \gamma_{rdI}(K_{m,n}) + m - n & if\, m \geq 2n \\ \gamma_{rdI}(K_{m,n}) + \lceil \frac{m}{2} \rceil & otherwise \end{cases}.$$

Let $S_{p,q}$ be a double star and $p, q \geq 2$. Via Observation 2.11, it is observed that.

Observation 3.15. *For double star $S_{p,q}$, $\gamma_{trdI}(S_{p,q}) = p + q + 4$.*

Proposition 3.16. *If $p, q \geq 2$, then $\gamma_{trdI}(M(S_{p,q})) = 3(p+q)$.*

Proof. Any TRDID function f devotes positive value to all leaves and their support vertices and for any leaf v and its support u, $f(v) + f(u) \geq 3$. Now using symbolization of Proposition 2.12 and Figure 9, we have $\gamma_{trdI}(M(S_{p,q})) \geq 3(p+q)$. On the other hand, it is forthright, devoting label 1 to all leaves and label 2 to support vertices and 0 otherwise, deduce a TRDID function of weight $3(p+q)$ (Figure 9). This establishes, $\gamma_{trdI}(M(S_{p,q})) \leq 3(p+q)$. Therefore the proof is explicit. □

As an immediate of Propositions 2.12 and 3.16, we have.

Corollary 3.17. $\gamma_{trdI}(S_{p,q}) = \gamma_{rdI}(S_{p,q}) + p + q - 3.$

4. BOUND ON THE γ_{rdI} AND γ_{trdI} OF MIDDLE OF A GRAPH

In this section we center our attention on presenting bounds for middle of any graph. Assume that Q denote the edge cover set of minimum size c.

Theorem 4.1. *Let G be a graph of order n. Then $\gamma_{rdI}(G) \leq 3c + \ell$ where c is the edge cover number and ℓ is the number of leaves. This bound is sharp.*

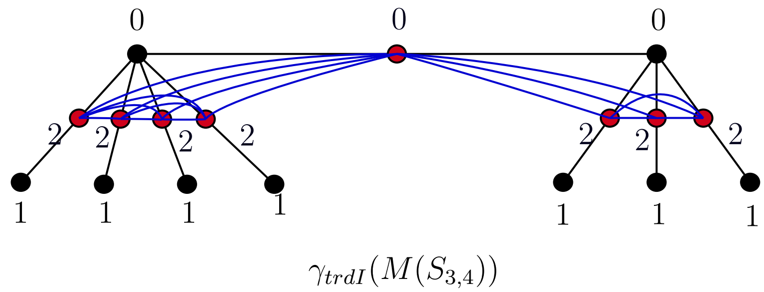

FIGURE 9. Total restrained double Italian domination of $M(S_{3,4})$

Proof. Let Q be an edge cover set of minimum cardinality c. We bring up two parts.

1. Each vertex has an incident edge, which is not in Q. In this position, we devote label 3 to each edge in Q and 0 to edges which are not in Q and vertices of degree at least 2, and label 1 to the leaves. These assignments grantee an RDID function of $M(G)$ of weight at most $3c + \ell$. Thence $\gamma_{rdI}(G) \leq 3c + \ell$.

2. There are vertices, such that all their incident edges are in Q. We devote label 0 to these edges and 2 to their end vertices. Further, we devote label 3 to another edges in Q, label 0 to their end vertices of degree at least two, and label 1 to their end vertices of degree one. Let v_k be the only vertex such that all k edges incident to v_k be in Q. Then by the above assignments, $\gamma_{rdI}(G) \leq 3(c-k) + 2k + 2 + \ell \leq 3c + \ell$. If the number of such vertices are more than one, then this inequality is not hard to see. Therefore these assignments grantee an RDIDF of $M(G)$ of weight at most $3c + \ell$. Thus we observe the desired result.

For seeing the sharpness, bring up the graph G, constructed from C_m and P_n with odd n and m, where one of the end vertices of P_n make adjacent to one vertex of C_m. Then $\gamma_{rdI}(M(G)) = 3c + 1 = 3c + \ell$. □

Theorem 4.2. *Let G be a graph of order n, then $\gamma_{trdI}(G) \leq 4c$; where c is the edge cover number of G. This bound is sharp.*

Proof. Let Q be the edge cover set of minimum cardinality c. By the definition of edge cover set, we deduce, each vertex has an incident edge in Q and Each pendant edge stands in Q. For establishing the result, we bring up two parts.

1. Each vertex has an incident edge, which is not in Q. In this position, we devote label 3 to each edge in Q and label 1 to an incident vertex (specially to leaves) and zero to another edges which are not in Q and the corresponding vertices. These assignments make a commitment a TRDID function of $M(G)$ of weight at most $4c$. Thence $\gamma_{trdI}(G) \leq 4c$.

2. There are vertices, such that all their incident edges are in Q. We devote label 2 to these edges and 1 to their end vertices. Further, we label 3 to another edges in Q, label 1 to one of incident vertices (specially to leaves) and zero to another edges which are not in Q and the corresponding vertices. With the same method of proving the part 2 of Theorem 4.1 since for $k \geq 1$, $3k+1 \leq 4k$, these assignments grantee a TRDID function of $M(G)$ of weight at most $4c$. For seeing the sharpness, bring up the graph G, constructed from C_m and P_n with even n, m, where one of the end vertices of P_n make adjacent to one vertex of C_m. Then from Propositions 3.1 and 3.5 $\gamma_{trdI}(M(G)) = 4c$. Thus we observe the desired result. □

5. RDID and TRDID of middle of corona of K_1 and K_2 of a graph

Now we investigate the RDID and TRDID function of corona of K_1 and K_2 of a graph.

5.1. Corona of K_1.

Theorem 5.1. *Let H be a connected graph of order $n \geq 2$ with ℓ leaves and $G = H \circ K_1$. Then $\gamma_{rdI}(M(G)) = 3c(G) + \lfloor \frac{n+\ell}{2} \rfloor$.*

Proof. For $n = 2$ it is trivial. Let $n \geq 3$ Let $V(H) = \{v_1, v_2, \cdots, v_n\}$ and $V(G) = V(H) \cup \{w_1, w_2 \cdots, w_n\}$. Let $u_{i,j}$ be the vertex in $M(G)$ corresponding to $e = v_iv_j$ of H in G where $1 \leq i < j \leq n$ and $u_{k,k}$ be the vertex in $M(G)$ corresponding to edge $e = w_kv_k$. So $V(M(G)) = V(G) \cup \{u_{i,j} : 1 \leq i < j \leq n\} \cup \{u_{k,k} : 1 \leq k \leq n\}$. Since w_i is a leaf in G, this vertex in $M(G)$ is also of degree 1 (a leaf). So under any RDID function, these vertices are devoted by positive label. It is manifest, for any i, vertex v_i with edges incident to v_i form a clique q_i of degree $\deg(v_i)$ and w_i is adjacent to $u_{i,i}$ in $M(G)$ (Figure 10). For any RDID function f, $f(N[v_i]) \geq 2$ and if $f(v_i) \in \{0, 1\}$, then $f(N[v_i]) \geq 3$ and $f(w_i) \geq 1$. Therefore we can assume that, if $f(w_i) = 1$, then $f(u_{i,i}) = 2$ and if $f(w_i) = 2$, then $f(u_{i,i}) \leq 1$. Although, in the position of $f(w_i) + f(u_{i,i}) = 3$, the situation of $f(w_i) = 1$, and $f(u_{i,i}) = 2$ is more affordable. So we bring up the following mode.

Let $f(w_i) = 1$ and $f(u_{i,i}) = 2$. Then $f(N[v_i] - \{u_{i,i}\}) \geq 1$, and the restrained of dominating, convince us to assume $f(v_i) \geq 1$, whenever $\deg(v_i) = 2$ and $f(N[v_i]) + f(w_i) \geq 4$. Since H has ℓ leaves, the graph G has ℓ vertex v_i of degree 2 and $N_{M(G)}[v_i]$ induces a subgraph K_3 in $M(G)$ where $\deg_{M(G)}(v_i) = 2$. Since the graph H is connected and $n \geq 3$, for other v_js every clique has a common vertex with at least one other clique in $M(G)$. If we devote value 1 to the common vertex between two cliques, then $\gamma_{rdI}(M(G)) \geq 3c(G) + \ell + \lceil \frac{n-\ell}{2} \rceil = 3c(G) + \lceil \frac{n+\ell}{2} \rceil$. On the other hand, suppose that $q_1, q_2, \cdots, q_k$ are cliques of order at least 4 in $M(G)$ where there is a common vertex $x_{i,i+1}$ between q_i and q_{i+1}. For even k, devoting 1 to w_i, and $x_{2j-1,2j}$ $(1 \leq j \leq \frac{k}{2})$, 2 to $u_{i,i}$, and zero otherwise, and for odd k, devoting 1 to w_i, and $x_{2j-1,2j}$ $(1 \leq j \leq \frac{k-1}{2})$, and $x_{k-1,k}$, 2 to $u_{i,i}$, and zero otherwise, represent an RDID function of size $3c(G) + \lceil \frac{n+\ell}{2} \rceil$. Thence $\gamma_{rdI}(M(G)) \leq 3c(G) + \lceil \frac{n+\ell}{2} \rceil$. Therefore we have the proof. □

For example see the Figure 10, the graph $M(P_8 \circ K_1)$ with its RDID function and RDID number.

Theorem 5.2. *Let H be a connected graph of order $n \geq 3$ with ℓ leaves and $G = H \circ K_1$. Then $\gamma_{trdI}(M(G)) = \gamma_{rdI}(M(G)) = 3c(G) + \lfloor \frac{n+\ell}{2} \rfloor$.*

Proof. From the symbolization in Theorem 5.1, any vertex with positive value in clique is adjacent to $u_{i,i}$ (Figure 10). Therefore the γ_{rdI} function of $M(G)$ is a γ_{trdI} function of $M(G)$. The proof is observed. □

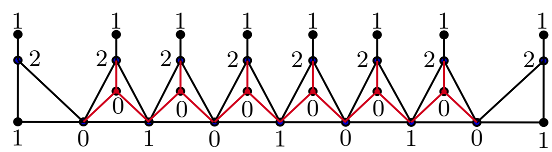

FIGURE 10. (Total) restrained double Italian domination of $M(P_8 \circ K_1)$.

Now we investigate the RDID and TRDID function of corona K_2 of a graph.

5.2. **Corona of K_2.**

Theorem 5.3. *Let H be a graph of order n. Let G be a corona K_2 of H, that is $G = H \circ K_2$. Then $\gamma_{rdI}(M(G)) = 5n$.*

Proof. Let $V(H) = \{v_1, v_2, \cdots, v_n\}$ and the set of vertices of G be $V(G) = V(H) \cup \{x_1, y_1, x_2, y_2, \cdots, x_n, y_n\}$ where x_i, y_i are the vertices of corona K_2 corresponding to vertex v_i. Let $u_{i,j}$ be the vertex in $M(G)$ corresponding to $e = v_iv_j$ where $1 \leq i < j \leq n$ and $z_{k,k}$, $x_{k,k}$ and $y_{k,k}$ be the vertices in $M(G)$ corresponding to edges $e = x_ky_k$, $e = x_kv_k$ and $e = y_kv_k$ respectively. So $V(M(G)) = V(G) \cup \{u_{i,j} : 1 \leq i < j \leq n\} \cup \{z_{k,k}, x_{k,k}, y_{k,k} : 1 \leq k \leq n\}$.

Let f be an RDID function on $M(G)$. Then $f(v_k) + f(x_k) + f(y_k) + f(x_{k,k}) + f(y_{k,k}) + f(z_{k,k}) \geq 5$, because theses vertices induce a middle graph of K_3 ($M(K_3)$). Since there are n such independent subgraphs in $M(G)$, then $w(f) \geq 5n$ and so $\gamma_{rdI}(M(G)) \geq 5n$.

On the other hand, it is manifest, devoting label 3 to the vertex $z_{i,i}$, label 2 to v_i and label 0 otherwise grantee $\gamma_{rdI}(M(G)) \leq 5n$), (Figure 11). Therefore $\gamma_{rdI}(M(G)) = 5n$. □

For example see Figure 11, the graph $M(C_4 \circ K_2)$ with RDID function and RDID number.

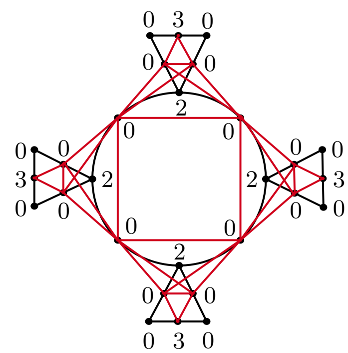

FIGURE 11. Restrained double Italian domination of $M(C_4 \circ K_2)$

As a prompt result from Theorem 5.3 and considering that for the given graph G in Theorem 5.3, the edge cover number c is at least $n + \lceil \frac{n}{2} \rceil$, we have.

Corollary 5.4. *Let H be a graph of order n and $G = H \circ K_2$. Then $\gamma_{rdI}(M(G)) \leq \frac{10c(G)}{3}$. This bound is sharp for $H \in \{C_n, P_n\}$ with even n.*

Theorem 5.5. *Let H be a graph of order n and $G = H \circ K_2$. Then $\gamma_{trdI}(M(G)) = 6n$.*

Proof. From the symbolization in Theorem 5.3, let f be a TRDID function on $M(G)$. Then $f(v_k)+f(x_k)+f(y_k)+f(x_{k,k})+f(y_{k,k})+f(z_{k,k}) \geq 6$, because these vertices induce a middle graph of K_3 ($M(K_3)$). Since there are n such independent subgraphs in $M(G)$, then $w(f) \geq 6n$ and thus $\gamma_{trdI}(M(G)) \geq 6n$.

On the other hand, it is apparent, devoting label 3 to the vertex $x_{i,i}$ and $y_{i,i}$, label 0 otherwise, grantee $\gamma_{trdI}(M(G)) \leq 6n$) (Figure 12). Therefore $\gamma_{trdI}(M(G)) = 6n$. □

For example see Figure 12, the graph $M(P_9 \circ K_2)$ with its TRDID function and TRDID number.

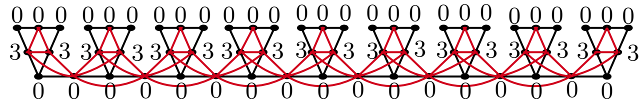

Figure 12. Total restrained double Italian domination of $M(P_9 \circ K_2)$

As a prompt result from Theorem 5.5 and considering that for the given graph G in Theorem 5.5, the edge cover number c is at least $n + \lceil \frac{n}{2} \rceil$ we have.

Corollary 5.6. *Let H be a graph of order n and $G = H \circ K_2$. Then $\gamma_{trdI}(M(G)) \leq 4c$. This bound is sharp for $H \in \{C_n, P_n\}$ with even n.*

6. Conclusion and problems

In this manuscript, we perusal double Italian domination of middle of any graph. We perused RID and TRID on some custom graph, and established bounds on the γ_{rdI} and γ_{trdI} of Middle of any graph. Also RDID and TRDID of middle of corona of K_1 and K_2 of a graph have been investigated.

There are several construction of graphs for instance Kneser graphs and Mycielski graphs for which the double Italian domination on them have not studied yet. Characterization of graphs G for which to achieve the bounds of Middle of a graph stated in Theorems 4.1 and 4.2 are problems. The RID and TRID function of corona of two graphs can be also attended as problems.

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