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A NOTE ON GENERALIZED DERIVATIONS AND LEFT IDEALS OF PRIME RINGS

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Abstract. Let R be a prime ring and $Z(R)$ denotes the center of R. In this study, we expose the commutativity of R as a consequence of specific differential identities involving derivations acting on left ideals of R. Finally, we give examples that demonstrate the necessity of hypotheses taken in the theorems.

1. Motivation

The investigation of polynomial constraints on a ring that finally imply commutativity has its roots in the first half of the twentieth century. A well-organized survey of the commutativity theorems in rings during 1950-2005 is given by James Pinter-Lucke [11]. These studies were stimulated by Jacobson's famous result [10, Theorem 11] and were extensively developed by Herstein, Bell, Yaqub, Quadri, Ashraf. Perhaps motivated by the work of Jacobson and Herstein, Posner [15] proved a surprising result called *Posner's Second Therem*, which is expressed as: If a 2-torsion free prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. In 1984, Mayne [12] obtained automorphism analogy of the Posner's result. Since then, many commutativity theorems in rings have been obtained as a consequence of various identities involving mappings like derivations, generalized derivations, automorphisms, endomorphisms etc., for a good cross-section we refer the reader to [4], [5], [8], [9], [14], [16], [17], [18].

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In 1997, Hongan [9] proved that if R is a 2-torsion free semiprime ring, I a nonzero ideal of R and $d : R \to R$ is a derivation of R such that $d([x,y]) \pm [x,y] \in Z(R)$ for all $x, y \in I$, then R is commutative. Ashraf and Rehman [3] explored the commutativity of a prime ring R that admits a nonzero derivation d satisfying the identities $d(xy) \pm xy \in Z(R)$, $d(xy) \pm yx \in Z(R)$, $d(x)d(y) \pm xy \in Z(R)$ for all $x, y \in I$, a nonzero ideal of R. In 2007, Ashraf et al. [4] extended these result by taking a generalized derivation in place of derivation. Motivated by these studies, many significant papers appeared in the recent literature obtaining commutativity of rings in more general situations, see [1], [7], [13], [19]. Recently, Al-Omary and Nauman [2] investigated the following differential identities: (i) $d(x) \circ y = d(xy)$, (ii) $F(x \circ y) = F(x) \circ y - F(y) \circ x$, (iii) $F([x,y]) = F(x) \circ y - F(y) \circ x$, (iv) $F([x,y]) = [F(x), y] + [F(y), x]$. Motivated from these studies, in this paper our aim is to establish commutativity of prime rings and describe possible forms of derivations satisfying the following identities: $d_1(xy) \pm d_2(x) \circ y \in Z(R)$, $F([x,y]) \pm ([G(x), y] \pm [x, H(y)]) \in Z(R)$ and $F(x \circ y) \pm (G(x) \circ y \pm x \circ H(y)) \in Z(R)$ over a nonzero left ideal of R, where d_1, d_2 are derivations of R and $(F,d), (G,g), (H,h)$ are generalized derivations of R.

2. Notions and preliminaries

A ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ (resp. $aRa = \{0\}$) implies $a = 0$ or $b = 0$ (resp. $a = 0$), for any $a, b \in R$. A mapping $d : R \to R$ is a derivation of R if d is additive and satisfies $d(xy) = d(x)y + xd(y)$ for every $x, y \in R$. A more general mapping $F : R \to R$ is called generalized derivation if F is additive and satisfies $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is a unique derivation of R associated to F; for the sake of brevity it can be denoted as an order pair (F, d). Obviously, the concept of generalized derivations includes the concept of derivations. A mapping $\xi : R \to R$ is called a multiplier if $\xi(xy) = \xi(x)y = x\xi(y)$ for all $x, y \in R$. The Lie product of any two elements $x, y \in R$ is denoted by $[x,y]$ and defined by $xy - yx$; while their Jordan product is denoted by $x \circ y$ and defined by $xy + yx$. The basic identities related to Lie product and Jordan product as given as follows:

$$[xy, z] = x[y,z] + [x,z]y, [x, yz] = y[x,z] + [x,y]z,$$

$$x \circ yz = (x \circ y)z - y[x,z] = y(x \circ z) + [x,y]z,$$

$$xy \circ z = x(y \circ z) - [x,z]y = (x \circ z)y + x[y,z].$$

These identities along with the following Lemmas shall be used in the sequel.

Lemma 2.1. [6, Lemma 3.1] *Let R be a 2-torsion free semiprime ring and λ is a nonzero left ideal of R. If $a, b \in R$ such that $axb + bxa = 0$ for all $x \in \lambda$, then $axb = 0 = bxa$ for all $x \in \lambda$.*

Lemma 2.2. (Brauer's trick) *A group G cannot be written as union of two of its proper subgroups.*

Lemma 2.3. *Let R be a 2-torsion free prime ring and λ be a nonzero left ideal of R. If d is a derivation of R such that $\lambda[d(\lambda), \lambda] = (0)$, then $d = 0$ or R is commutative.*

Proof. Assume that

$$u[d(x), y] = 0, \ \forall \ x, y, u \in \lambda. \tag{2.1}$$

Changing y with ry in (2.1) and using it, we find

$$ur[d(x), y] + u[d(x), r]y = 0, \ \forall \ x, y, u \in \lambda, \ r \in R. \tag{2.2}$$

Replacing r by ry in (2.2), we get

$$ur[d(x), y]y = 0, \ \forall \ x, y, u \in \lambda, \ r \in R. \tag{2.3}$$

Primeness of R forces $[d(x), y]y = 0$ for all $x, y \in \lambda$. Linearizing y in the last relation, we find

$$[d(x), t]y + [d(x), y]t = 0, \ \forall \ x, y, t \in \lambda. \tag{2.4}$$

Substituting t for tu in (2.4), to obtain

$$[d(x), t]uy = 0, \ \forall \ x, y, u, t \in \lambda.$$

Writing pu by u in the above relation, we get $[d(x), t]Ruy = (0)$ for all $x, y, t, u \in \lambda$. It forces $[d(x), t] = 0$ for all $x, t \in \lambda$. Replacing t by rt, where $r \in R$, we find $[d(x), r]t = 0$ for all $x, t \in \lambda$ and $r \in R$. Taking st for t, where $s \in R$, we get $[d(x), R]Rt = (0)$ for all $x, t \in \lambda$. It forces $[d(x), r] = 0$ for all $x \in \lambda$ and $r \in R$. Replacing x by sx, where $s \in R$ in the last expression, we get

$$[d(s)x, q] + [s, q]d(x) = 0, \ \forall \ x \in \lambda, \ s, q \in R. \tag{2.5}$$

Substituting q by sq in (2.5) and sing it, we obtain

$$[d(q)sx, q] = 0, \ \forall \ x \in \lambda, \ s, q \in R. \tag{2.6}$$

Replacing x by xt in (2.6), to get

$$d(q)sx[t, q] = 0, \ \forall \ x, t \in \lambda, \ s, q \in R. \tag{2.7}$$

That is, $d(q)Rx[t, q] = 0$ for all $x, t \in \lambda$ and $q \in R$. It implies that for each $q \in R$, we have either $d(q) = 0$ or $\lambda[\lambda, q] = (0)$. Set $A = \{q \in R :$

$d(q) = 0\}$ and $B = \{q \in R : \lambda[\lambda, q] = (0)\}$. Note that A and B both are additive subgroups of $(R, +)$ and $R = A \cup B$. Invoking Brauer's trick (Lemma 2.2), we conclude that either $R = A$ or $R = B$, i.e., either $d(q) = 0$ for all $q \in R$ or $x[y, q] = 0$ for all $x, y \in \lambda$ and $q \in R$, which assures commutativity of R. □

3. Results

Proposition 3.1. *Let R be a prime ring and λ be a nonzero left ideal of R. If R admit derivations d_1, d_2 and nonzero multipliers ϱ, ς such that $\varrho(x)d_1(y) \pm \varsigma(y)d_2(x) \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:*

(i) *R is commutative,*
(ii) *$\lambda d_1(\lambda) = (0) = \lambda d_2(\lambda)$.*

Proof. By our assumption, we have

$$\varrho(x)d_1(y) - \varsigma(y)d_2(x) \in Z(R), \ \forall\ x, y \in \lambda. \tag{3.1}$$

Case 1. Let $Z(R) = (0)$. In this case our situation reduces to

$$\varrho(x)d_1(y) - \varsigma(y)d_2(x) = 0, \ \forall\ x, y \in \lambda. \tag{3.2}$$

Replacing x by rx in (3.2), we get

$$r\varrho(x)d_1(y) - (\varsigma(y)rd_2(x) + \varsigma(y)d_2(r)x) = 0, \ \forall\ x, y \in \lambda, \ r \in R. \tag{3.3}$$

Pre-multiplying (3.2) by r and subtracting from (3.3), it follows that

$$[r, \varsigma(y)]d_2(x) - \varsigma(y)d_2(r)x = 0, \ \forall\ x, y \in \lambda, \ r \in R. \tag{3.4}$$

Taking sy for y in (3.4), we get

$$[r, s]\varsigma(y)d_2(x) = 0, \ \forall\ x, y \in \lambda, \ r, s \in R. \tag{3.5}$$

It gives $[r, s]R\varsigma(y)d_2(x) = (0)$ for all $x, y \in \lambda$ and $r, s \in R$. In view of primeness of R, we have either R is commutative or $\varsigma(\lambda)d_2(\lambda) = (0)$. In the latter case, our hypothesis (3.2) assures $\varrho(\lambda)d_1(\lambda) = (0)$. Note that since ς and ϱ are nonzero multiplier, it is not difficult to obtain $\lambda d_2(\lambda) = (0)$ and $\lambda d_1(\lambda) = (0)$. It completes our conclusion in this case.

Case 2. Let $Z(R) \neq (0)$. Then there exists $0 \neq c \in Z(R)$. Replacing y by $cy = yc$ in (3.1), we see that

$$\begin{aligned}(\varrho(x)d_1(y) - \varsigma(y)d_2(x))c + \varrho(xy)d_1(c) \\ = \varrho(xy)d_1(c) \in Z(R), \ \forall\ x, y \in \lambda.\end{aligned} \tag{3.6}$$

Since $Z(R)$ is a domain, it forces $\varrho(xy) \in Z(R)$, i.e., $[\varrho(xy), r] = 0$ for all $x, y \in \lambda$ and $r \in R$. Changing x by qx in the last expression, we get $[q, r]\varrho(x)y = 0$ for all $x, y \in \lambda$ and $r, q \in R$. Replacing y by $d_1(w)sy$

in the last expression, we find $[q,r]\varrho(x)d_1(w)Ry = 0$ for all $x, y \in \lambda$ and $r, q \in R$. Since R is prime ring and λ is a nonzero left ideal of R, we get $[R,R]\varrho(\lambda)d_1(\lambda) = (0)$. It forces that either R is commutative or $\varrho(\lambda)d_1(\lambda) = (0)$. Clearly in view of the latter case, it follows from our hypothesis that $[\varsigma(y)d_2(x), r] = 0$ for all $x, y \in \lambda$. Substituting py for y in the last relation, we get $[p,r]\varsigma(y)d_2(x) = 0$ for all $x, y \in \lambda$ and $p, r \in R$. Hence primeness of R yields $\varsigma(\lambda)d_2(\lambda) = (0)$. And hence $\lambda d_2(\lambda) = (0)$ and $\lambda d_1(\lambda) = (0)$.

By repeating the same argument with slight variations, we can prove the same conclusion for $\varrho(x)d_1(y) + \varsigma(y)d_2(x) \in Z(R)$ for all $x, y \in \lambda$. □

Corollary 3.2. *Let R be a prime ring and λ be a nonzero left ideal of R. If R admit derivations d_1 and d_2 such that $xd_1(y) \pm yd_2(x) \in Z(R)$ for all $x, y \in \lambda$, then one of the following holds:*

(i) *R is commutative,*
(ii) $\lambda d_1(\lambda) = (0) = \lambda d_2(\lambda)$.

Corollary 3.3. *Let R be a prime ring and I be a nonzero ideal of R. If R admit nonzero derivations d_1 and d_2, then the following assertions are equivalent:*

(i) *$xd_1(y) \pm yd_2(x) \in Z(R)$ for all $x, y \in I$.*
(ii) *R is commutative.*

Theorem 3.4. *Let R be a 2-torsion free prime ring and λ be a nonzero left ideal of R. If R admit derivations $d_1 \neq 0$ and $d_2 \neq 0$, then the following assertions are equivalent:*

(i) *$d_1(xy) - d_2(x) \circ y \in Z(R)$ for every $x, y \in \lambda$.*
(ii) *$d_1(xy) + d_2(x) \circ y \in Z(R)$ for every $x, y \in \lambda$.*
(iii) *R is commutative.*

Proof. $(i) \Rightarrow (iii)$: Let us suppose that

$$d_1(xy) - d_2(x) \circ y \in Z(R),\ \forall\ x, y \in \lambda. \tag{3.7}$$

We split the proof into the following two parts:
Case 1. Let $Z(R) = (0)$. Then our situation reduces to

$$d_1(xy) - d_2(x) \circ y = 0,\ \forall\ x, y \in \lambda. \tag{3.8}$$

It implies

$$\begin{aligned} 0 &= d_1(xyu) - d_2(x) \circ yu \\ &= (d_1(xy) - d_2(x) \circ y)u + (xyd_1(u) + y[d_2(x), u]) \\ &= xyd_1(u) + y[d_2(x), u],\ \forall\ x, y, u \in \lambda. \end{aligned} \tag{3.9}$$

Taking ty in place of y in (3.9) and using it, we get

$$[t,x]yd_1(u) = 0, \ \forall\ x,y,u,t \in \lambda. \tag{3.10}$$

Replacing y by ry in (3.10), where $r \in R$, we find $[t,x]Ryd_1(u) = (0)$ for all $x,t,y,u \in \lambda$. It implies either $[\lambda,\lambda] = (0)$ or $\lambda d_1(\lambda) = (0)$. in the first case, we have $0 = [rx,y] = [r,y]x$ for all $x,y \in \lambda$ and $r \in R$. Substituting sx for x in the last relation, where $s \in R$, we have $[r,y]Rx = (0)$ for all $x,y \in \lambda$. Since λ is a nonzero left ideal of R, we find that $[r,y] = 0$ for all $y \in \lambda$ and $r \in R$. Now replacing y by py, we get $0 = [r,p]y$ for all $y \in \lambda$ and $r,p \in R$. It forces $[r,p] = 0$ for all $r,p \in R$, i.e., R is commutative.

On the other hand, we have $\lambda d_1(\lambda) = (0)$. From (3.9), we have $\lambda[d_2(\lambda),\lambda] = 0$ for all $x,y,u \in \lambda$. Invoking Lemma 2.3, we have $d_2 = 0$ or R is commutative. Consider $d_2 = 0$, from (3.8) we obtain $d_1(xy) = 0$ for all $x,y \in \lambda$. Substituting rx for x in the last relation, we find $d_1(r)xy = 0$ for all $x,y \in \lambda$ and $r \in R$. It forces $d_1 = 0$.

Case 2. Let $Z(R) \neq (0)$. Replacing y by yc in (3.7), where $0 \neq c \in Z(R)$, we get $xyd_1(c) \in Z(R)$ for all $x,y \in \lambda$. It yields $[xy,r]d_1(c) = 0$ for all $x,y \in \lambda$ and $r \in R$. But center of a prime ring is free from zero divisors, therefore, we have $[xy,r] = 0$ for all $x,y \in \lambda$ and $r \in R$. Replacing x by px in the last relation, we get $[p,r]xy = 0$ for all $x,y \in \lambda$ and $r,p \in R$. It forces R commutative.

$(ii) \Rightarrow (iii)$: In the same way, we can prove implication. □

Corollary 3.5. *Let R be a 2-torsion free prime ring and λ be a nonzero left ideal of R. If R admit derivations d_1 and d_2 such that, then the following assertions are equivalent:*

(i) $d_1(xy) + d_2(x) \circ y = 0$ *for every* $x,y \in \lambda$.
(ii) $d_1(xy) - d_2(x) \circ y = 0$ *for every* $x,y \in \lambda$.
(iii) $d_1 = d_2 = 0$.

Proof. By Theorem 3.4, either $d_1 = d_2 = 0$ or R is commutative. Let us assume that R is a commutative ring, then obviously λ becomes a two-sided ideal of R. By the hypothesis, we have $d_1(xy) \pm 2d_2(x)y = 0$ for all $x,y \in R$. Replacing y by yr, we get $xyd_1(r) = 0$ for all $x,y \in \lambda$. It forces $d_1 = 0$. Substituting $d_1 = 0$ in the last expression, we find $2d_2(x)y = 0$ for all $x,y \in \lambda$. In light of assumption of torsion of R, we find $d_2(x)y = 0$ for all $x,y \in \lambda$ and hence $d_2 = 0$. □

Corollary 3.6. [2, Theorem 2.1] *Let R be a prime ring. If R admits a derivation d such that $d(xy) \pm d(x) \circ y = 0$ for all $x,y \in R$, then $d = 0$.*

Theorem 3.7. *Let R be a 2-torsion free prime ring and λ be a nonzero left ideal of R. If (F,d), (G,g) and (H,h) are generalized derivations* *of R such that $xg(y) \neq \pm xh(y)$ for all $x, y \in \lambda$, then the following assertions are equivalent:*

(i) $F([x,y]) - ([G(x),y] \pm [x,H(y)]) \in Z(R)$ *for every* $x, y \in \lambda$.
(ii) $F([x,y]) + ([G(x),y] \pm [x,H(y)]) \in Z(R)$ *for every* $x, y \in \lambda$.
(iii) *R is commutative.*

Proof. $(i) \Rightarrow (iii)$: Assume that

$$F([x,y]) - ([G(x),y] \pm [x,H(y)]) \in Z(R),\ \forall\ x,y \in \lambda. \tag{3.11}$$

Case 1. Let $Z(R) = (0)$. Then our situation is

$$F([x,y]) - ([G(x),y] \pm [x,H(y)]) = 0,\ \forall\ x,y \in \lambda. \tag{3.12}$$

Replacing x by xt in (3.12) in order to get

$$\begin{aligned}[x,y]d(t) + F(x)[t,y] + xd([t,y]) - \Big(G(x)[t,y]\\ +[x,y]g(t) + x[g(t),y] \pm x[t,H(y)]\Big) = 0,\ \forall\ x,y,t \in \lambda.\end{aligned} \tag{3.13}$$

In particular for $t = y$, we have

$$[x,y]d(y) - \Big([x,y]g(y) + x[g(y),y] \pm x[y,H(y)]\Big) = 0,\ \forall\ x,y \in \lambda.$$

Substituting rx for x in in the last expression, we see that

$$[r,y]x(d-g)(y) = 0,\ \forall\ x,y \in \lambda,\ r \in R. \tag{3.14}$$

It gives $[r,y]Rx(d-g)(y) = (0)$ for all $x, y \in \lambda$ and $r \in R$. Primeness of R yields that for each $y \in \lambda$, either $[R,y] = (0)$ or $\lambda(d-g)(y) = (0)$. An application of Brauer's trick yields that either $[R,\lambda] = (0)$, which forces R is commutative or $xd(y) = xg(y)$ for all $x, y \in \lambda$. Let us consider $xd(y) = xg(y)$ for all $x, y \in \lambda$. Using the fact $xg([t,y]) = x[g(t),y] + x[t,g(y)]$ for all $x, t, y \in \lambda$ in (3.13), we get

$$\begin{aligned}\Big([x,y]d(t) - [x,y]g(t)\Big) + (F(x) - G(x))[t,y] + \Big(xd([t,y])\\ -xg([t,y])\Big) + x[t,g(y)] \mp x[t,H(y)] = 0,\ \forall\ x,y,t \in \lambda.\end{aligned}$$

Our assumption reduces it to

$$(F(x) - G(x))[t,y] + x[t,g(y)] \mp x[t,H(y)] = 0,\ \forall\ x,y,t \in \lambda. \tag{3.15}$$

In particular, it implies

$$x[y,g(y)] \mp x[y,H(y)] = 0,\ \forall\ x,y \in \lambda.$$

That is, $x[y,(g\mp H)(y)]=0$ for all $x,y\in\lambda$. Linearizing this equation, we get

$$x[y,(g\mp H)(t)]+x[t,(g\mp H)(y)]=0,\ \forall\ x,y,t\in\lambda. \tag{3.16}$$

Changing y by yw in (3.16), we obtain

$$\begin{aligned} xy[w,(g\mp H)(t)]+x(g\mp H)(y)[t,w]+xy[t,(g\mp h)(w)]\\ +x[t,y](g\mp h)(w)=0,\ \forall\ x,y,t,w\in\lambda. \end{aligned} \tag{3.17}$$

In particular, we have

$$xy[t,(g\mp h)(t)]+x[t,y](g\mp h)(t)=0,\ \forall\ x,y,t\in\lambda. \tag{3.18}$$

Replacing y by xy in (3.18), we find $x[t,x]y(g\mp h)(t)=0$ for all $x,y,t\in\lambda$. It yields $x[t,x]Ry(g\mp h)(t)=(0)$ for all $x,y,t\in\lambda$. It implies that for each $t\in\lambda$, we have either $x[t,x]=0$ for all $x\in\lambda$ or $\lambda(g\mp h)(t)=(0)$. Applying Brauer's trick, we obtain either $x[t,x]=0$ for all $x,t\in\lambda$ or $xg(y)=\pm xh(y)$ for all $x,y\in\lambda$, which is not possible.

Thus, we have $x[t,x]=0$ for all $x,t\in\lambda$. From this, one can easily obtain $\lambda[\lambda,\lambda]=(0)$. Replacing x by xu in (3.15), we find

$$xu[t,(g\mp H)(y)]=0,\ \forall\ x,y,t,u\in\lambda.$$

It can be seen as

$$u[x,\theta(y)]=0,\ \forall\ x,y,u\in\lambda, \tag{3.19}$$

where $\theta=g\mp H$ is a generalized derivation of R with associated derivation $\vartheta=g\mp h$. Replacing y by yt in (3.19), to get

$$u\theta(y)[x,t]+uy[x,\vartheta(t)]=0,\ \forall\ x,y,u,t\in\lambda. \tag{3.20}$$

Replacing x by xk in (3.20) in order to obtain

$$u\vartheta(w)x[k,t]+uwx[k,\vartheta(t)]=0,\ \forall\ x,u,t,w,k\in\lambda. \tag{3.21}$$

Also replacing u by ux in (3.20) gives

$$ux\vartheta(w)[k,t]+uxw[k,\vartheta(t)]=0,\ \forall\ x,u,t,w,k\in\lambda. \tag{3.22}$$

Comparing (3.21) and (3.22), we get $u[\vartheta(w),x][k,t]=0$ for all $x,u,t,k,w\in\lambda$. Putting $k=rv$, where $r\in R$ and $v\in\lambda$ in the last relation, we find

$$0=u[\vartheta(w),x]r[v,t]+u[\vartheta(w),x][r,t]v,\ \forall\ x,u,t,w,v\in\lambda,\ r\in R. \tag{3.23}$$

Substituting tv for t in (3.23) and using it, we get

$$u[\vartheta(w),x]t[r,v]v=0,\ \forall\ x,u,t,w,v\in\lambda,\ r\in R. \tag{3.24}$$

It forces that either $u[\vartheta(w), x] = 0$ for all $x, u, w \in \lambda$ or $t[r, v]v = 0$ for all $t, v \in \lambda$ and $r \in R$. Let us suppose that $t[r, v]v = 0$ for all $t, v \in \lambda$ and $r \in R$ and linearizing it in order to get

$$t[r, u]v + t[r, v]u = 0, \ \forall\ u, v, t \in \lambda, \ r \in R. \tag{3.25}$$

Writing vw for v in (3.25), it follows that

$$t[r, v][w, u] + tv[r, w]u = 0, \ \forall\ u, v, t, w \in \lambda, \ r \in R.$$

It implies

$$-tvr[w, u] + tv[r, w]u = 0, \ \forall\ u, v, t, w \in \lambda, \ r \in R.$$

From this, we obtain

$$vr[w, u] = v[r, w]u, \ \forall\ u, v, w \in \lambda, \ r \in R. \tag{3.26}$$

Replacing u by su in (3.26), we see that

$$vrs[w, u] + vr[w, s]u = v[r, w]su, \ \forall\ u, v, w \in \lambda, \ r, s \in R. \tag{3.27}$$

On the other hand taking rs instead of r in (3.26), we find

$$vrs[w, u] = v[r, w]su + vr[s, w]u, \ \forall\ u, v, w \in \lambda, \ r, s \in R. \tag{3.28}$$

Comparing (3.27) and (3.28), we have

$$vr[w, s]u = vr[s, w]u, \ \forall\ u, v, w \in \lambda, \ r, s \in R.$$

It yields $2vr[w, s]u = 0$ for all $u, v, w \in \lambda$ and $r, s \in R$. Since R is 2-torsion free, we get $\lambda R[\lambda, R]\lambda = (0)$. It forces that $[\lambda, R] = (0)$, hence R is commutative, as desired.

On the other hand, we now consider $y[\vartheta(w), x] = 0$ for all $x, y, w \in \lambda$. By Lemma 2.3, we conclude that R is commutative.

Case 2. Let $Z(R) \neq (0)$. In that case, there exists $0 \neq c \in Z(R)$. Replacing y by yc in (3.11), we find $[x, y](d(c) \pm h(c)) \in Z(R)$ for all $x, y \in \lambda$. It implies $[[x, y], r](d(c) \pm h(c)) = 0$ for all $x, y \in \lambda$ and $r \in R$. Since $Z(R)$ is a domain, we obtain $[[x, y], r] = 0$. Substituting xy for x in the last relation to get $[x, y][y, r] = 0$ for all $x, y \in \lambda$ and $r \in R$. It implies $[x, y]R[y, r] = (0)$ for all $x, y \in \lambda$ and $r \in R$. Primeness of R implies that either λ is commutative or $\lambda \subseteq Z(R)$. Thus, it is not difficult to see that both of these cases imply commutativity of R.
$(ii) \Rightarrow (iii)$: In the same way, we can prove this assertion. □

The following example justifies our hypotheses:

(i) R is 2-torsion free,
(ii) $xg(y) \neq \pm xh(y)$ for all $x, y \in \lambda$ in the above theorem.

Example 3.8. Let $R = \left\{ \begin{pmatrix} x & y \\ t & z \end{pmatrix} : x, y, t, z \in \mathbb{Z}_2 \right\}$ and

$$\lambda = \left\{ \begin{pmatrix} 0 & u \\ 0 & v \end{pmatrix} : u, v \in \mathbb{Z}_2 \right\}.$$

Note that R is a prime ring with nonzero left ideal λ.

- Let $F = 0, G = id$ and $H = id$ be the generalized derivations with associated derivations $d = 0, g = 0$ and $h = 0$ respectively. Then one can check that the conditions $F([x,y]) - ([G(x),y] + [x,H(y)]) \in Z(R)$, $F([x,y]) + ([G(x),y] + [x,H(y)]) \in Z(R)$ are satisfied on λ, but R is not commutative.
- Let $F = 0, G = id$ and $H = id$ be the generalized derivations with associated derivations $d = 0, g = 0$ and $h = 0$ respectively. Then one can check that the conditions $F([x,y]) - ([G(x),y] - [x,H(y)]) \in Z(R)$, $F([x,y]) + ([G(x),y] - [x,H(y)]) \in Z(R)$ are satisfied on λ, but R is not commutative.

Thus, we conclude that the assumptions taken are not superfluous in Theorem 3.7.

Corollary 3.9. *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If (F,d) and $(G, g \neq 0)$ are generalized derivations of R, then the following assertions are equivalent:*

(i) $F([x,y]) - [G(x),y] \in Z(R)$ *for every* $x, y \in I$.
(ii) $F([x,y]) + [G(x),y] \in Z(R)$ *for every* $x, y \in I$.
(iii) R *is commutative.*

Corollary 3.10. *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If $(G, g \neq 0)$ and $(H, h \neq 0)$ are generalized derivations of R, then the following assertions are equivalent:*

(i) $[G(x),y] - [x,H(y)] \in Z(R)$ $(g \neq -h)$ *for every* $x, y \in I$.
(ii) $[G(x),y] + [x,H(y)] \in Z(R)$ $(g \neq h)$ *for every* $x, y \in I$.
(iii) R *is commutative.*

Corollary 3.11. *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If $(G, g \neq 0)$ is a generalized derivation of R, then the following assertions are equivalent:*

(i) $[G(\lambda), \lambda] \subseteq Z(R)$.
(ii) R *is commutative.*

Theorem 3.12. *Let R be a 2-torsion free prime ring and λ be a nonzero left ideal of R. If (F,d), (G,g) and (H,h) are generalized derivations of R such that $xg(y) \neq \pm xh(y)$ for all $x, y \in \lambda$, then the following assertions are equivalent:*

(i) $F(x \circ y) - G(x) \circ y \pm x \circ H(y) \in Z(R)$ *for every* $x, y \in \lambda$.
(ii) $F(x \circ y) + G(x) \circ y \pm x \circ H(y) \in Z(R)$ *for every* $x, y \in \lambda$.
(iii) R *is commutative.*

Proof. $(i) \Rightarrow (iii)$: Assume that

$$F(x \circ y) - (G(x) \circ y \pm x \circ H(y)) \in Z(R), \ \forall \ x, y \in \lambda. \tag{3.29}$$

Case 1. Let $Z(R) = (0)$. Then our situation is

$$F(x \circ y) - (G(x) \circ y \pm x \circ H(y)) = 0, \ \forall \ x, y \in \lambda. \tag{3.30}$$

Replacing x by xt in (3.30) in order to get

$$(x \circ y)d(t) + F(x)[t, y] + xd([t, y]) - (G(x)[t, y] + (x \circ y)g(t) + x[g(t), y] \\ \pm x[t, H(y)]) = 0, \ \forall \ x, y, t \in \lambda. \tag{3.31}$$

In particular for $t = y$, we have

$$(x \circ y)d(y) - ((x \circ y)g(y) + x[g(y), y] \pm x[y, H(y)]) = 0, \ \forall \ x, y \in \lambda.$$

Substituting rx for x in the last expression, we see that

$$[r, y]x(d - g)(y) = 0, \ \forall \ x, y \in \lambda, \ r \in R. \tag{3.32}$$

As Theorem 3.7, it implies R commutative or $\lambda(d - g)(\lambda) = (0)$. Using the latter case in (3.31), we find

$$F(x)[t, y] + xd([t, y]) - (G(x)[t, y] + x[g(t), y] \\ \pm x[t, H(y)]) = 0, \ \forall \ x, y, t \in \lambda. \tag{3.33}$$

That is,

$$(F(x) - G(x))[t, y] + x[t, g(y)] \mp x[t, H(y)] = 0, \ \forall \ x, y, t \in \lambda,$$

and hence the conclusion follows from Theorem 3.7.
Case 2. Let $Z(R) \neq (0)$. In that case, there exists $0 \neq c \in Z(R)$. Replacing y by yc in (3.29), we find $(x \circ y)(d(c) \pm h(c)) \in Z(R)$ for all $x, y \in \lambda$. It implies $[x \circ y, r](d(c) \pm h(c)) = 0$ for all $x, y \in \lambda$ and $r \in R$. Since $Z(R)$ is a domain, we obtain $[x \circ y, r] = 0$. Substituting xy for x in the last relation to get $(x \circ y)[y, r] = 0$ for all $x, y \in \lambda$ and $r \in R$. It implies $(x \circ y)R[y, r] = (0)$ for all $x, y \in \lambda$ and $r \in R$. Now it is not difficult to see that either λ is commutative or $\lambda \subseteq Z(R)$, and hence R is commutative in each case.
$(ii) \Rightarrow (iii)$: In the same way, we can prove this assertion. □

The following example justifies our hypotheses:

(i) R is 2-torsion free,
(ii) $xg(y) \neq \pm xh(y)$ for all $x, y \in \lambda$ in the above theorem.

Example 3.13. Let $R = \left\{ \begin{pmatrix} x & y \\ t & z \end{pmatrix} : x, y, t, z \in \mathbb{Z}_2 \right\}$ and

$$\lambda = \left\{ \begin{pmatrix} 0 & u \\ 0 & v \end{pmatrix} : u, v \in \mathbb{Z}_2 \right\}.$$

Note that R is a prime ring with nonzero left ideal λ.

- Define $F = 0$ and
$$G\begin{pmatrix} x & y \\ t & z \end{pmatrix} = H\begin{pmatrix} x & y \\ t & z \end{pmatrix} = \begin{pmatrix} t+y & z \\ z & 0 \end{pmatrix}$$
with associated derivation $d = 0$ and
$$g\begin{pmatrix} x & y \\ t & z \end{pmatrix} = h\begin{pmatrix} x & y \\ t & z \end{pmatrix} = \begin{pmatrix} y & 0 \\ z-x & -y \end{pmatrix}$$
respectively. Then we see that the conditions $F(x \circ y) - (G(x) \circ y + x \circ H(y)) \in Z(R)$, $F(x \circ y) + (G(x) \circ y + x \circ H(y)) \in Z(R)$ are satisfied on λ, but R is not commutative.
- Define $F = 0$,
$$G\begin{pmatrix} x & y \\ t & z \end{pmatrix} = \begin{pmatrix} t+y & z \\ z & 0 \end{pmatrix}$$
and $H = -G$ with associated derivation $d = 0$,
$$g\begin{pmatrix} x & y \\ t & z \end{pmatrix} = \begin{pmatrix} y & 0 \\ z-x & -y \end{pmatrix}$$
and $h = -g$ respectively. Then we see that the conditions $F(x \circ y) - (G(x) \circ y - x \circ H(y)) \in Z(R)$, $F(x \circ y) + (G(x) \circ y - x \circ H(y)) \in Z(R)$ are satisfied on λ, but R is not commutative.

Thus, we conclude that the assumptions taken are not superfluous in Theorem 3.12.

Corollary 3.14. *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If (F, d) and $(G, g \neq 0)$ are generalized derivations of R, then the following assertions are equivalent:*

(i) $F(x \circ y) - (G(x) \circ y) \in Z(R)$ *for every* $x, y \in I$.
(ii) $F(x \circ y) + G(x) \circ y) \in Z(R)$ *for every* $x, y \in I$.
(iii) R *is commutative.*

Corollary 3.15. *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If $(G, g \neq 0)$ and $(H, h \neq 0)$ are generalized derivations of R, then the following assertions are equivalent:*

(i) $G(x) \circ y - x \circ H(y) \in Z(R)$ *for every* $x, y \in I$.
(ii) $G(x) \circ y + x \circ H(y) \in Z(R)$ *for every* $x, y \in I$.
(iii) R *is commutative.*

Corollary 3.16. *Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If $(G, g \neq 0)$ is a generalized derivation of R, then the following assertions are equivalent:*

(i) $G(\lambda) \circ \lambda) \subseteq Z(R)$.
(ii) *R is commutative.*

We conclude this paper with the following example which exhibits that the hypothesis of primeness in Theorem 3.7 and Theorem 3.12 is essential.

Example 3.17. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ and

$$\lambda = \left\{ \begin{pmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : k \in \mathbb{Z} \right\}.$$

It can be easily seen that λ is a nonzero left ideal of R, and R is not a prime ring as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define $(F, d), (G, g), (H, h) : R \to R$ as

$$F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix},$$

$$G \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \; g \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$H \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; h \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

one may verify that $(F, d), (G, g)$ and (H, h) are generalized derivations which satisfy the identities:

- $F([x, y]) - ([G(x), y] \pm [x, H(y)]) \in Z(R)$,
- $F([x, y]) + ([G(x), y] \pm [x, H(y)]) \in Z(R)$,
- $F(x \circ y) - (G(x) \circ y \pm x \circ H(y)) \in Z(R)$,
- $F(x \circ y) + (G(x) \circ y \pm x \circ H(y)) \in Z(R)$

for all $x, y \in \lambda$ and $xg(y) \neq \pm xh(y)$ for all $x, y \in \lambda$. But R is not commutative.

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