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MODULAR REPRESENTATION OF SYMMETRIC 2-DESIGNS

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Abstract. Complementary pairs of symmetric 2-designs are equivalent to coherent configurations of type $(2,2;2)$. D. G. Higman studied these coherent configurations and adjacency algebras of coherent configurations over a field of characteristic zero. These are always semisimple. We investigate these algebras over fields of any characteristic prime and the structures.

1. Introduction

Many researchers have studied the p-ranks of incidence matrices of combinatorial designs [1, 3, 8]. The p-ranks of incidence matrices of some 2-designs have been investigated in the majority of decodable codes because we can obtain a linear code having a relatively large number of information symbols from a 2-design whose incidence matrix having a relatively small p-rank [3]. Furthermore, these results help us classify 2-designs with the same parameters.

Complementary pairs of symmetric 2-designs are equivalent to coherent configurations of type $(2,2;2)$. The types of coherent configurations were considered in [5]. An algebra accompanies each coherent configuration. It is called adjacency algebra. We consider the structures of adjacency algebras of coherent configurations obtained from symmetric 2-designs. An adjacency algebra of a coherent configuration over a field of characteristic zero is always semisimple. This case was

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studied by Higman [6]. The author has considered symmetric 2-design in [4] and determined the structure of modular adjacency algebras of coherent configurations obtained from symmetric 2-designs over a field of characteristic 2.

This paper determined the structure of modular adjacency algebras of coherent configurations obtained from symmetric 2-designs over a field of any characteristic prime. We define a coherent configuration in Section 2 and consider structures of modular adjacency algebras of coherent configurations obtained from symmetric 2-designs in Section 3.

2. Preliminaries

We give some definitions of coherent configurations. The reader is referred to [5] for basic notation on coherent configurations. Let X be a finite nonempty set, C a set of nonempty binary relations on X so that $X^2 = \bigcup_{c\in C} c$ is a disjoint union of X^2. The pair (X, C) is called a *coherent configuration* if the following three axioms hold.

(C1) There is a subset C_0 of C such that $\bigcup_{f\in C_0} f = \{(x, x) \mid x \in X\}$,
(C2) if $c \in C$, then $c^* = \{(y, x) \mid (x, y) \in c\} \in C$,
(C3) for $a, b, c \in C$ and $(x,\ y) \in c$, a non-negative integer $p^c_{a,b} = \sharp\{z \in X \mid (x, z) \in a,\ (z, y) \in b\}$ is independent of the choice of x and y.

We put $X_f = \{x \in X \mid (x, x) \in f\}$ $(f \in C_0)$ and call X_f a *fiber*. A coherent configuration (X, C) is said to be *homogeneous* if $|C_0| = 1$. It is also called an *association scheme* in a sense of [2] and [10].

Let (X, C) be a coherent configuration with fibers $\{X_f | f \in C_0\}$. We denote by $\mathrm{Mat_X}(\mathbb{Z})$ the ring of matrices over $\mathbb{Z}$ whose rows and columns are indexed by X. For $c \in C$, we denote by A_c the *adjacency matrix* of c, namely

$$(A_c)_{x,y} = \begin{cases} 1 & (x, y) \in c, \\ 0 & \text{otherwise.} \end{cases}$$

The above three axioms are equivalent to the following condition in term of adjacency matrices $\{A_c | c \in C\}$ such that $\sum_{c\in C} A_c = J_{|X|}$, where $J_{|X|}$ is the all one matrix of order $|X|$.

(C1)' There is the subset C_0 of C such that $\sum_{f\in C_0} A_c = I_{|X|}$, where $I_{|X|}$ is the identity matrix of order $|X|$.
(C2)' $A_{c^*} = {}^tA_c \in \{A_c | c \in C\}$ for any $c \in C$, where ${}^tA_c$ is the transpose of the matrix A_c.

(C3)' For $a, b, c \in C$, there are integers $p_{a,b}^c$ such that

$$A_a A_b = \sum_{c \in C} p_{a,b}^c A_c.$$

$\mathbb{Z}C = \oplus_{c \in C} \mathbb{Z} A_c$ is a subalgebra of $\mathrm{Mat}_{\mathrm{X}}(\mathbb{Z})$ under the usual matrix multiplication by the above axioms. For a commutative ring R with the identity element, we can define $RC = R \otimes_{\mathbb{Z}} \mathbb{Z}C$ and call this R-algebra the *adjacency algebra* of (X, C) over R. We use the notation A_c for the corresponding element in RC. Since RC is defined as a subalgebra of $\mathrm{Mat}_{\mathrm{X}}(R)$, the inclusion map is a representation and we call it the *standard representation* of (X, C) over R. The corresponding RC-module is called the *standard module* of (X, C) over R. The standard module has a natural basis X, so we denote it by RX. A *modular adjacency algebra* FC is the adjacency algebra of (X, C) over a field F of positive characteristic p and a *modular standard module* FX is the standard module of (X, C) over F.

For $c \in C$, there is a unique pair $(f, g) \in {C_0}^2$ such that $A_f A_c A_g = A_c$. Subsets $C(f, g) = \{c \in C \mid A_f A_c A_g = A_c\}$ of C give a partition of C like $C = \bigcup_{f,g \in C_0} C(f, g)$. The sub-configuration $(X_f, C(f, f))$ is homogeneous and $RC(f, f) = \oplus_{c \in C(f,f)} RA_c$ is a subalgebra of RC (with non-common identity).

3. Types of adjacency algebras of symmetric 2-designs

We construct coherent configurations from symmetric 2-designs. The author studied the structure of modular adjacency algebras of coherent configurations obtained from 2-designs over a field of characteristic 2 in [4]. This paper considers the structure of modular adjacency algebras of coherent configurations obtained from symmetric 2-designs over a field of characteristic prime p.

Let $\mathfrak{D}$ be a symmetric 2-(v, ℓ, λ) design, that is, an incidence structures $(\mathfrak{P}, \mathfrak{B}, \mathfrak{F})$ consisting of disjoint sets $\mathfrak{P}$ and $\mathfrak{B}$, whose elements are called *points* and *blocks* respectively, and a subset $\mathfrak{F}$ of the Cartesian product $\mathfrak{P} \times \mathfrak{B}$, whose elements are called *flags*. A point ω and a block B are *incident* if (ω, B) is a flag. A *symmetric* 2*-design* $\mathfrak{D}$ with parameters v, b, r, ℓ, λ is an arrangement of v points $\mathfrak{P}$ into b blocks $\mathfrak{B}$ such that:

(D1) each block is incident with ℓ points (we assume that with $\ell < v$),
(D2) each point is incident with r blocks,
(D3) any two distinct points are incident with λ blocks, and
(D4) any two distinct blocks are incident with λ points.

Among parameters v, b, r, ℓ, λ, there are the following relations:

$$v = b,\ r = \ell \text{ and } \lambda(v-1) = \ell(\ell-1).$$

The incidence matrix N of $\mathfrak{D}$ will have rows indexed by the points and columns by the blocks, namely,

$$(N)_{\omega,B} = \begin{cases} 1 & (\omega, B) \in \mathfrak{F}(\subset \mathfrak{P} \times \mathfrak{B}), \\ 0 & \text{otherwise.} \end{cases}$$

For an incidence matrix N of $\mathfrak{D}$, the following equation holds.

$$N{}^tN = {}^tNN = (\ell - \lambda)I_v + \lambda J_v.$$

Associated with a symmetric 2-design $(\mathfrak{P}, \mathfrak{B}, \mathfrak{F})$ is the configuration (X, C) defined by $X = \mathfrak{P} \cup \mathfrak{B}$ $(\mathfrak{P} \cap \mathfrak{B} = \emptyset)$ and $C = \{c_i : 1 \le i \le 8\}$, where

$$c_1 = \{(x,x) \mid x \in \mathfrak{P}\},\ c_2 = \{(x,x) \mid x \in \mathfrak{B}\},\ c_3 = \mathfrak{P}^2 - c_1,$$
$$c_4 = \mathfrak{B}^2 - c_2,\ c_5 = \mathfrak{F},\ c_6 = \mathfrak{P} \times \mathfrak{B} - \mathfrak{F},$$
$$c_7 = {c_5}^* = \{(y,x) \mid (x,y) \in c_5\},$$
$$c_8 = {c_6}^* = \{(y,x) \mid (x,y) \in c_6\}.$$

Putting $A_i = A_{c_i}$ $(1 \le i \le 8)$, they can be written as block matrices:

$$A_1 = \begin{bmatrix} I_v & O \\ O & O \end{bmatrix}, A_2 = \begin{bmatrix} O & O \\ O & I_v \end{bmatrix}, A_3 = \begin{bmatrix} J_v - I_v & O \\ O & O \end{bmatrix},$$
$$A_4 = \begin{bmatrix} O & O \\ O & J_v - I_v \end{bmatrix},\ A_5 = \begin{bmatrix} O & N \\ O & O \end{bmatrix},\ A_6 = \begin{bmatrix} O & J_v - N \\ O & O \end{bmatrix},$$
$$A_7 = {}^tA_5 = \begin{bmatrix} O & O \\ {}^tN & O \end{bmatrix}, A_8 = {}^tA_6 = \begin{bmatrix} O & O \\ {}^tN & O \end{bmatrix}.$$

We provide tables of multiplications of algebras obtained by these configurations.

	A_1	A_3	A_5	A_6
A_1	A_1	A_3	A_5	A_6
A_3	A_3	$(v-1)A_1 + (v-2)A_3$	$(\ell-1)A_5 + \ell A_6$	$(v-\ell)A_5 + (v-\ell-1)A_6$
A_7	A_7	$(\ell-1)A_7 + \ell A_8$	$\ell A_2 + \lambda A_4$	$(\ell-\lambda)A_4$
A_8	A_8	$(v-\ell)A_7 + (v-\ell-1)A_8$	$(\ell-\lambda)A_4$	$(v-\ell)A_2 + (v-2\ell+\lambda)A_4$

TABLE 1. The first multiplication table of (X, C).

These tables show that the configuration (X, C) is a coherent configuration of type $(2, 2; 2)$. Consequently, we can prove that (X, C) is a coherent configuration of type $(2, 2; 2)$, where $C = \{c_i\}_{1 \le i \le 8}$. On the

	A_2	A_4	A_7	A_8
A_2	A_2	A_4	A_7	A_8
A_4	A_4	$(v-1)A_2+(v-2)A_4$	$(\ell-1)A_7+\ell A_8$	$(v-\ell)A_7+(v-\ell-1)A_8$
A_5	A_5	$(\ell-1)A_5+\ell A_6$	$\ell A_1+\lambda A_3$	$(\ell-\lambda)A_3$
A_6	A_6	$(v-\ell)A_5+(v-\ell-1)A_6$	$(\ell-\lambda)A_3$	$(v-\ell)A_1+(v-2\ell+\lambda)A_3$

TABLE 2. The second multiplication table of (X, C).

other hand, every coherent configuration of type $(2, 2; 2)$ is equivalent to complementary pairs of symmetric designs. Higman considered the types of coherent configurations and gave a method to compute irreducible ordinary characters of a coherent configuration by characters of its fibers [6]. We generalize them to modular representations [4]. In the rest of this paper, we assume that F is a field of characteristic a prime p and (K, R, F) is a p-modular system [7].

Let (X, C) be a coherent configuration defined by a symmetric 2-design $\mathfrak{D}$. Since $(X_1, C(c_1, c_1))$ and $(X_2, C(c_2, c_2))$ are complete graphs, the character table of (X, C) over a field characteristic zero is as follows:

	A_1	A_3	A_2	A_4	multiplicity[4]
χ_1	1	$v-1$	1	$v-1$	1
χ_2	1	-1	1	-1	$v-1$

.

Note that character values of A_i $(i = 5, 6, 7, 8)$ are zero and we omit them.

Suppose $p \nmid v$. Consider the central primitive idempotent corresponding to χ_1:

$$e_{\chi_1} = \frac{1}{v}\left(A_1 + A_3 + A_2 + A_4\right).$$

Then e_{χ_1} is also a central idempotent of FC. We can also quickly check that

$$e_{\chi_1} FC e_{\chi_1} \cong M_2(F).$$

Hence, there are two possibilities.
(A). The modular character table is

A_1	A_3	A_2	A_4	multiplicity
1	$v-1$	1	$v-1$	1
1	-1	1	-1	$v-1$

.

The decomposition and the Cartan matrices [4, 7] are

$$\mathcal{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case,

$$FC \cong M_2(F) \oplus M_2(F)$$

and this is semisimple.
(B). The modular character table is

A_1	A_3	A_2	A_4	multiplicity
1	$v-1$	1	$v-1$	1
1	-1	0	0	$v-1$
0	0	1	-1	$v-1$

The decomposition and the Cartan matrices are

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \ \mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

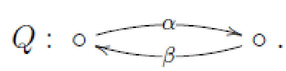
We can choose primitive idempotents $e_U = A_1 + A_2 + A_3 + A_4$, $e_V = A_3$ and $e_W = A_4$. Let us put $\alpha = A_5$ and $\beta = A_7$.

Then we have the following theorem.

Theorem 3.1. *The adjacency algebra of Type (B) is isomorphic to*

$$M_2(F) \oplus FQ/(\alpha\beta, \beta\alpha)$$

where FQ is a path algebra, Q is the following quiver:



Suppose $p \mid v$. Modular character table of fibers are as follows:

A_1	A_3	multiplicity
1	-1	v

A_2	A_4	multiplicity
1	-1	v

Hence, there are two possibilities.
(C). The modular character table is

A_1	A_3	A_2	A_4	multiplicity
1	-1	1	-1	v

The decomposition and the Cartan matrices are

$$\mathcal{D} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \mathcal{C} = \begin{pmatrix} 2 \end{pmatrix}.$$

In this case,

$$FC \cong M_2(F) \otimes_F F[x]/(x^2).$$

(D). The modular character table is

	A_1	A_3	A_2	A_4	multiplicity
	1	-1	0	0	v
	0	0	1	-1	v

.

The decomposition and the Cartan matrices are

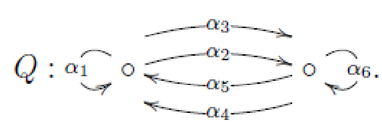
$$\mathcal{D} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \mathcal{C} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Only (D) is the non-trivial case. We can choose primitive idempotents $e_U = A_1$ and $e_V = A_2$. Put $\alpha_1 = A_1 + A_3$, $\alpha_2 = A_5$, $\alpha_3 = A_5 + A_6$, $\alpha_4 = A_7$, $\alpha_5 = A_7 + A_8$ and $\alpha_6 = A_2 + A_4$. We have the following theorem.

Theorem 3.2. *The adjacency algebra of Type (D) is isomorphic to*

$$FQ/(\{\alpha_i\alpha_j \mid\ 1 \leq i, j \leq 6\}),$$

where FQ is a path algebra, Q is the following quiver:



It is difficult to determine the structure of the standard module and we could not do that.

3.1. **Characterization of types by parameters of designs.**

Let (X, C) be a coherent configuration defined by a symmetric 2-(v, ℓ, λ) design. The Frame number [9] is

$$\mathcal{F}(C) = \frac{v^8(v-\ell)^2\ell^2}{(v-1)^2}.$$

We show that the following theorem.

Theorem 3.3. *Let (X, C) be a coherent configuration obtained from a symmetric 2-(v, ℓ, λ) design.*

(1) *Type (A) if and only if $\mathcal{F}(C) \not\equiv 0 \pmod p$,*
(2) *Type (B) if and only if $v \not\equiv 0 \pmod p$ and $\mathcal{F}(C) \equiv 0 \pmod p$,*
(3) *Type (C) if and only if $v \equiv 0 \pmod p$ and $\ell \not\equiv \lambda \pmod p$,*
(4) *Type (D) if and only if $v \equiv \ell \equiv \lambda \equiv 0 \pmod p$.*

Proof. Statements (1) and (2) are clear. Suppose $v \equiv 0 \pmod p$. Suppose that $\ell \equiv \lambda \pmod p$. Then by $\lambda(v-1) = \ell(\ell-1)$, we have $v \equiv \ell \equiv \lambda \equiv 0 \pmod p$. In this case, $FC(c_1, c_2) \subset \mathrm{Rad}(FC)$ and FC is of type (D), where $\mathrm{Rad}(FC)$ is the Jacobson radical of FC. Suppose that $\ell \not\equiv \lambda \pmod p$. Then A_7A_5 is not nilpotent, and so is not in $\mathrm{Rad}(FC)$. So FC is of type (C). □

3.2. **Structure of FX of type (A).** We will determine the structure of FC of type (A). There are two simple modules U and V with $\dim_F U = \dim_F V = 2$. The structure of the standard module is determined. We can write

$$FX \cong [U] \oplus (v-1)[V].$$

3.3. **Structure of FX of type (B).** We will determine the structure of FC of type (B). There are three simple modules U, V, and W with $\dim_F U = 2$, $\dim_F V = \dim_F W = 1$, and the Loewy structure of the projective covers is as follows:

$$P(U) = [U],\ P(V) = \begin{bmatrix} V \\ W \end{bmatrix},\ P(W) = \begin{bmatrix} W \\ V \end{bmatrix}.$$

The structure of the standard module is entirely determined. We can write

$$FX \cong [U] \oplus g_1[V] \oplus g_2 \begin{bmatrix} V \\ W \end{bmatrix} \oplus h_1[W] \oplus h_2 \begin{bmatrix} W \\ V \end{bmatrix}$$

for some non-negative g_1, g_2, h_1 and h_2.

By multiplicities $m_V = m_W = v-1$, we have

$$g_1 + g_2 + h_2 = v - 1, \tag{3.1}$$

$$g_2 + h_1 + h_2 = v - 1, \tag{3.2}$$

$g_2 = rank(\alpha) = rank(A_5)$, $h_2 = rank(\beta) = rank(A_7)$. Since ${}^tA_5 = A_7$, $g_2 = h_2$. We put $w = rank(A_5)$,

$$FX \cong [U] \oplus (v-2w-1)[V] \oplus w \begin{bmatrix} V \\ W \end{bmatrix} \oplus (v-2w-1)[W] \oplus w \begin{bmatrix} W \\ V \end{bmatrix}.$$

In this case, multiplicities must be a non-negative integer. Consequently, we know the upper ranks of N.

Corollary 3.4. *Let N be an incidence matrix of a symmetric* 2-(v, ℓ, λ) *design with $v \not\equiv 0 \pmod p$ and $\mathcal{F}(C) \equiv 0 \pmod p$. Then*

$$rank_p(N) \leq \frac{v-1}{2}.$$

3.4. **Structure of FX of type (C).** We determine the structure of FC of type (C). The module category of FC is Morita equivalent to the module category of $F[x]/(x^2)$. Hence, we know there are two isomorphic classes of indecomposable modules and $\dim_F \mathrm{Rad}(FC) = 4$. We know the fact that $A_1 + A_3$, $A_2 + A_4$, $A_5 + A_6$ and $A_7 + A_8$ are the basis of $\mathrm{Rad}(FC)$ and $\dim_F (FX)\mathrm{Rad}(FC) = 2$ by computation. According to these facts, we know the standard module FX structure. Then

$$FX \cong [U] \oplus (v-2)[V],$$

where $\dim_F U = 4$ and $\dim_F V = 2$.

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