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ON PRIME IDEAL BUNDLES OF LIE ALGEBRA BUNDLES

M. V. MONICA AND R. RAJENDRA*

Abstract. In this paper, prime ideal bundles and semi-prime and irreducible ideal bundles of a Lie algebra bundle are defined and their relation with prime ideal bundles is studied.

1. Introduction

Prime ideals play an important role in the theory of associative algebras. Kawamoto [3] and F. A. M Aldosray [2] studied this notion of prime ideals in Lie algebras He discussed the conditions for ideals to be prime and also its relation with semi prime, irreducible and maximal ideals.

Lie algebra bundles were defined and studied in [4], [5] and [6]. Some recent advancements in Lie and associative algebra bundles are discussed in [1], [7] and [8]. It is of interest how the notion of prime ideals can be applied to Lie algebra bundles. In this paper we define prime ideal bundle of a Lie algebra bundle and characterize it in terms of sections of Lie algebra bundle. Interrelations between prime, semi prime and irreducible Lie algebra bundles are deduced.

A Lie algebra bundle is a vector bundle $\xi = (E, p, B)$ in which each fibre ξ_x is a Lie algebra and for each x in B, there is an open neighbourhood U of x, a Lie algebra L and a homeomorphism $\phi : U \times L \to p^{-1}(U)$ such that for each y in U, $\phi_y : L \to p^{-1}(y)$ is a Lie algebra isomorphism. A Lie algebra bundle with a semisimple Lie algebra structure

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*Corresponding author .

on each of its fibre is called a semisimple Lie algebra bundle. A section S of a Lie algebra bundle ξ is a continuous map $S : B \to E$ such that $p \circ S = id_B$. An ideal bundle of a Lie algebra bundle is a Lie algebra bundle whose fibres at any point x in B are ideals of ξ_x.

We assume that the base space B of any Lie algebra bundle is compact Hausdorff and all underlying vector spaces are real and finite dimensional.

For a Lie algebra L, an ideal R of L is called a prime ideal if for ideals I, J of L such that $[I, J] \subseteq R$ implies $I \subseteq R$ or $J \subseteq R$ (See[3]).

2. Prime, Semiprime and Irreducible ideal bundles

Definition 2.1. An ideal bundle P of a Lie algebra bundle ξ is said to be a prime ideal bundle if for each x in B, P_x is a prime ideal of ξ_x.

Then for ideal bundles H, K of ξ, $[H,K] \subseteq P$ implies $H \subseteq P$ or $K \subseteq P$. For a Lie algebra L and an element x in L, $\langle x^L \rangle$ denotes the smallest ideal of L containing x.

We now characterize prime ideal bundles using sections of Lie algebra bundle.

Theorem 2.2. *Let P be an ideal bundle of a Lie algebra bundle ξ. Then the following conditions are equivalent.*

1. *P is a prime ideal bundle.*
2. *If $[S(x), H_x] \subseteq P_x$ for some section S of ξ and ideal bundle H of ξ then either $S(x) \in P_x$ or $H_x \subseteq P_x$.*
3. *If $[S_1(x), \langle S_2(x)\rangle] \subseteq P_x$ for sections S_1, S_2 of ξ then $S_1(x) \in P_x$ or $S_2(x) \in P_x$.*

Proof. (1) $\implies$ (3): For $x \in B$, define

$$(V_0)_x = (S_1(x)),$$
$$(V_i)_x = [\dots[(S_1(x)), \underbrace{\xi_x], \dots, \xi_x]}_{i \text{ times}}, \text{ for all } i.$$

If for $S_1, S_2 \in \Gamma(\xi)$ and $x \in B$, $[S_1(x), \langle S_2(x)\rangle] \subseteq P_x$, we assert that $[(V_i)_x, \langle S_2(x)\rangle] \subseteq P_x$ for all $i \geq 0$. We prove this by induction.

For $i = 0$, $[S_1(x), \langle S_2(x)\rangle] \subseteq P_x$ implies $[(S_1(x)), \langle S_2(x)\rangle] \subseteq P_x$ so that $[(V_0)_x, \langle S_2(x)\rangle] \subseteq P_x$. Let us assume the assertion is true for $i-1$, $i \geq 1$. Then

$$[(V_i)_x, \langle S_2(x)\rangle] = [[(V_{i-1})_x, \xi_x], \langle S_2(x)\rangle].$$

Using Jacobi identity,

$$[(V_i)_x, \langle S_2(x)\rangle] \subseteq [[(V_{i-1})_x, \langle S_2(x)\rangle], \xi_x] + [(V_{i-1})_x, [\xi_x, \langle S_2(x)\rangle]$$

$$\subseteq [P_x, \xi_x] + [(V_{i-1})_x, \langle S_2(x)\rangle]$$
$$\subseteq P_x.$$

By definition of $\langle S_1(x)\rangle$, $\langle S_1(x)\rangle = \sum_{i=0}^{\infty}(V_i)_x$. Therefore,

$$\begin{aligned}[\langle S_1(x)\rangle, \langle S_2(x)\rangle] &= \left[\sum_{i=0}^{\infty}(V_i)_x, \langle S_2(x)\rangle\right] \\ &= \sum_{i=0}^{\infty}[(V_i)_x, \langle S_2(x)\rangle] \\ &\subseteq P_x.\end{aligned}$$

Since P_x is a prime ideal in ξ_x, $\langle S_1(x)\rangle \subseteq P_x$ or $\langle S_2(x)\rangle \subseteq P_x$ from which it follows that $S_1(x) \in P_x$ or $S_2(x) \in P_x$. This proves (3).

(3) $\implies$ (2): Let S_1 be a section of ξ such that $S_1(x) \notin P_x$ for all $x \in B$. In fact, such a section exists. Let $\{U_i\}_{i\in I}$ be an open cover for B. Then it has a finite subcover $\{U_i\}_{i=1}^{n}$. Define a map $S_1 : U_i \to p^{-1}(U_i)$ such that $S_1(x) = u_x \in \xi_x \setminus P_x$. By pasting lemma, $S_1 : B \to E$ is a continuous function, $p \circ S_1 = id_B$ and hence S_1 is a section of ξ. Suppose $[S_1(x), K_x] \subseteq P_x$ for $x \in B$. Then for any section S_2 of ξ, $S_2(x) \in K_x$, we have , $[S_1(x), \langle S_2(x)\rangle] \subseteq P_x$ and from (3), $S_1(x) \in P_x$ or $S_2(x) \in P_x$ which gives $S_1(x) \in P_x$ or $H_x \subseteq P_x$. Therefore (2) is proved.

(2) $\implies$ (1): Let H, K be ideal bundles of ξ such that $[H_x, K_x] \subseteq P_x$ and $H_x \nsubseteq P_x$. If S is a section of ξ, $S(x) \in H_x \setminus P_x$ we have $[S(x), K_x] \subseteq P_x$ and from (ii), $K_x \subseteq P_x$ which proves (1). □

Definition 2.3. An ideal subbundle Q of a Lie algebra bundle ξ is said to be semiprime if for ideal bundle H of ξ, $[H, H] \subseteq Q$ then $H \subseteq Q$.

Definition 2.4. An ideal bundle N of a Lie algebra bundle ξ is said to be irreducible if $N = H \cap K$ for ideal bundles H and K of ξ gives $N = H$ or $N = K$.

Lemma 2.5. *Let ξ be a Lie algebra bundle. Then*

1. *Any prime ideal bundle is a semiprime ideal bundle.*
2. *Any prime ideal bundle is irreducible.*

Proof. (1) follows by definition.

(2) Let P be a prime ideal bundle and $P = H \cap K$, H, K are ideal bundles of ξ. In the fibre ξ_x, $x \in B$, $[H_x, K_x] \subseteq H_x \cap K_x$ so that $[H, K]$ is contained in $H \cap K$. P is prime implies $H \subseteq P$ or $K \subseteq P$. By

hypothesis, $P = H \cap K$ is contained in both H and K which gives $P = H$ or $P = K$. Therefore P is irreducible. □

Definition 2.6. A Lie algebra bundle ξ is said to satisfy the minimal condition for ideal bundles if for every chain of ideal bundles $\eta_1 \supseteq \eta_2 \supseteq \cdots$, there exists an index m such that $\eta_i = \eta_k$ for every i, $k > m$.

We write $\xi \in Min - \triangleleft$ when ξ satisfies the minimal condition for ideal bundles.

Similarly, we have

Definition 2.7. A Lie algebra bundle ξ is said to satisfy the maximal condition for ideal bundles if for every chain of ideal bundles $\eta_1 \subseteq \eta_2 \subseteq \cdots$, there exists an index m such that $\eta_i = \eta_k$ for every i, $k > m$.

We write $\xi \in Max - \triangleleft$ when ξ satisfies the maximal condition for ideal bundles.

Proposition 2.8. *Let ξ be a Lie algebra bundle and P be a prime ideal bundle of ξ.*

1. *P is prime if and only if P is irreducible and semiprime.*
2. *Let $\xi \in Min - \triangleleft$. Then an ideal bundle P of ξ is prime if and only if there exists the smallest ideal bundle M which contains P, $M \neq P$ such that M/P is not abelian.*

Proof. (1) Direct part is obtained in Lemma 2.5. Let H, K be ideal bundles of ξ such that $[H, K] \subseteq P$. For each $x \in B$, set

$$N_x = (H_x + P_x) \cap (K_x + P_x)$$

then

$$N_x^2 = [(H_x + P_x) \cap (K_x + P_x), (H_x + P_x) \cap (K_x + P_x)].$$

Let $z \in N_x^2$. Then z is a linear combination of $[x, y]$ such that x, $y \in (H_x+P_x) \cap (K_x+P_x)$. It follows that $[x, y] \in [H_x+P_x, K_x+P_x]$ which gives $z \in [H_x+P_x, K_x+P_x]$. Therefore, $N_x^2 \subseteq [H_x+P_x, K_x+P_x] \subseteq P_x$. Since P is semiprime, $N_x \subseteq P_x \ \forall\, x \in B$. Now, $P_x \subseteq H_x + P_x, K_x + P_x$ and hence $P_x \subseteq (H_x + P_x) \cap (K_x + P_x)$. This gives $P_x \subseteq N_x$ for all $x \in B$. Therefore,

$$P_x = N_x = (H_x + P_x) \cap (K_x + P_x).$$

Then $P = \cup_{x \in B}(H_x+P_x) \cap (K_x+P_x) = (H+P) \cap (K+P)$. Since P is irreducible, $P_x = H_x + P_x$ or $P_x = K_x + P_x$. So $H_x \subseteq P_x$ or $K_x \subseteq P_x$. This proves that P is prime.

(2) Let $\xi \in Min - \triangleleft$. Then there exists a descending chain of ideal bundles $A_1 \supseteq A_2 \supseteq \cdots$, which contain P and there exists a minimal

ideal bundle M which contains P properly. We assert that this M is unique, for if there exists two such ideal bundles M_1 and M_2, $M_1 \neq M_2$ then $[M_1, M_2] \subseteq M_1 \cap M_2 = P$. Since P is prime, $M_1 \subseteq P$ or $M_2 \subseteq P$ - not possible. Therefore M is unique. We assert that M/P is not an abelian subbundle of ξ. Otherwise $(M/P)^2 = (0)$. It follows that $[y + P_x, z + P_x] = 0 + P_x$ that is,$[y, z] \in P_x$ for $y, z \in M_x$, $x \in B$. Then $M_x \subseteq P_x$ which is not possible. The assertion thus follows.

Conversely, let M be the ideal bundle satisfying the given conditions. We assert that P is a prime ideal bundle. If P is not prime, for some $x \in B$ there exists ideals H_x, K_x of ξ_x such that $H_x \nsubseteq P_x$ and $K_x \nsubseteq P_x$ and $[H_x, K_x] \subseteq P_x$. Since $H_x \nsubseteq P_x$ and $K_x \nsubseteq P_x$, $P_x \subset H_x + P_x$ and $P_x \subset K_x + P_x$. M is the minimal ideal bundle containing P implies $M_x \subset H_x + P_x$ and $M_x \subset K_x + P_x$. Also from $[H_x, K_x] \subseteq P_x$ it follows that $M_x^2 \subseteq P_x$. Now for $y + P_x$, $z + P_x \in M_x/P_x$, $[y + P_x, z + P_x] = [y, z] + P_x = P_x$. We obtain M_x/P_x is abelian which contradicts our assumption that M/P is not abelian. This proves the assertion. □

Definition 2.9. An ideal bundle M of Lie algebra bundle ξ, $M \neq \xi$, is said to be maximal if for each x in B, M_x is a maximal ideal of ξ_x.

Then, if M' is any ideal bundle of ξ and $M \subseteq M' \subseteq \xi$ we have $M = M'$ or $M' = \xi$.

Theorem 2.10. *Let M be a maximal ideal bundle of a Lie algebra bundle ξ. Then the following conditions are equivalent.*

(1) *M is a prime ideal bundle.*
(2) *M is a semiprime ideal bundle.*
(3) *rank$(\xi/M) > 1$.*

Proof. (1) $\implies$ (2) is obvious.

(2) $\implies$ (3): Suppose $rank(\xi/M) = 1$ locally. Then there exists an open set U in B such that $dim(\xi/M)_x = 1$ for all $x \in U$. Then $(\xi/M)_x = \{0 + M_x, y + M_x\}$, $y \neq 0$ and $y \in \xi_x$. We prove that $y \notin M_x$. If $y^2 + M_x \neq 0 + M_x$ then $y^2 \notin M_x$ and $M_x \cup \{y^2\}$ is an ideal of ξ_x which contains M_x properly. This contradicts the maximality of M_x. Therefore, ξ/M is abelian and hence $\xi^2 \subseteq M$. Since M is semiprime, $\xi \subseteq M$ which contradicts the maximality of M. It follows that $dim(\xi/M)_x > 1$. This proves (3).

(3) $\implies$ (1): If M is not prime there exists ideal bundles H, K of ξ such that $H \nsubseteq M$, $K \nsubseteq M$ and $[H, K] \subseteq M$. M is maximal gives, $H + M = K + M = \xi$ and $\xi^2 = [H, K] + M \subseteq M$ because $[H, K] \subseteq M$. Therefore, ξ/M is abelian. Since M is a maximal ideal bundle, the

only ideal bundle of ξ/M is itself so that $rank(\xi/M) = 1$ which is not true. Therefore M is a prime ideal bundle. □

As in the context of Lie algebras, we define perfect Lie algebra bundles as follows.

Definition 2.11. A Lie algebra bundle ξ is said to be perfect if $\xi = [\xi, \xi]$.

A semisimple Lie algebra bundle is perfect.

Corollary 2.12. *Let ξ be a perfect Lie algebra bundle. Then a maximal ideal bundle of ξ is prime.*

Proof. Let M be a maximal ideal bundle of ξ. We assert that M is semiprime for otherwise, there exists an ideal bundle H of ξ such that $H^2 \subseteq M$ and H is not contained in M. Since M is a maximal ideal bundle, $H + M = \xi$ and $\xi^2 = [H, H] + M \subseteq M$ so that $\xi^2 \subseteq M$. By assumption ξ is perfect and hence $\xi \subseteq M$ which gives $\xi = M$, not possible. Therefore M is semiprime. By Theorem 2.10, M is prime. □

Corollary 2.13. *Let ξ be a semisimple Lie algebra bundle and H be an ideal subbundle of ξ, $H \neq \xi$. Then the following are equivalent.*

(1) *H is a prime ideal bundle.*
(2) *H is a maximal ideal bundle.*
(3) *H is an irreducible ideal bundle.*

Proof. By Corollary 2.12, we get a maximal ideal bundle is prime. From Lemma 2.5, prime ideal bundles are irreducible. Let H be an irreducible bundle of ξ and H is different from ξ. Since ξ is semisimple,

$$\xi = H \oplus \xi_1 \oplus \ldots \oplus \xi_k,$$

where $\xi_1, \xi_2, \ldots, \xi_k$ are simple ideal bundles of ξ.
If $k > 1$, then by applying the methods of [3] to each fibre of ξ, there exists ξ_i, ξ_j, both distinct such that $H \oplus \xi_i$ and $H \oplus \xi_j$ are different from H. Therefore,

$$H = (H \oplus \xi_i) \cap (H \oplus \xi_j),$$

which is a contradiction to the irreducibility of H. Therefore,

$$\xi = H \oplus \xi_\lambda,$$

where ξ_λ is simple. Therefore, H is maximal. □

For a Lie algebra L and an ideal H of L, $r(H)$ denotes the intersection of all prime ideals of L containing H. Then the intersection of all prime ideals of L is given by $r(0)$ and is called the prime radical of L (See [3]). We shall denote this as L^{rad}.

Let ξ be a Lie algebra bundle with local triviality given by $\phi : U \times L \to p^{-1}(U)$ where L is a Lie algebra. η be an ideal subbundle of ξ with local triviality $\phi : U \times I \to p^{-1}(U)$, I is an ideal of L. Denote by $r(I)$ the intersection of prime ideals P of L containing I and $r(\eta_x)$ the intersection of prime ideals of ξ_x containing the ideal fibre η_x. Since $ker\phi_x = (0)$, P contains $ker\phi_x$ and ϕ_x being an isomorphism, $\phi_x(P)$ is a prime ideal of ξ_x. Now, $I \subseteq P$ implies $\eta_x = \phi_x(I) \subseteq \phi_x(P)$ so that $\phi_x(P)$ is a prime ideal of ξ_x containing η_x. We have $\phi_x(\cap_{i\in I} P) = \cap_{i\in I}\phi_x(P) \subseteq r(\eta_x)$ which gives $\phi_x(r(I)) \subseteq r(\eta_x)$. Similarly, $\phi_x^{-1}(r(\eta_x)) \subseteq r(I)$. Then $\phi \restriction_{U\times r(I)} : U \times r(I) \to \cup_{x\in B} r(\eta_x)$ is an isomorphism and hence $r(\eta) = \cup_{x\in B} r(\eta_x)$ is an ideal subbundle of ξ.

In particular, when η is the zero ideal bundle, η_x for $x \in B$ is the intersection of all prime ideals of ξ_x. Then $r(0)$ is the union of all prime radicals of the fibres of ξ and we call $r(0)$ as the prime radical of ξ and denote it as $r(\xi)$.

Theorem 2.14. *$Rad(\xi)$ is contained in $r(\xi)$. If $\xi \in Max - \triangleleft$ then $Rad(\xi) = r(\xi)$.*

Proof. Let η be a solvable ideal bundle of ξ. Then there exists an integer k, $k \geq 0$ such that $\eta^{(k)} = (0)$. If P is a prime ideal bundle of ξ then $\eta^{(k)} = (0) \subseteq P$. This gives $Rad(\xi) \subseteq r(\xi)$.

Let $\xi \in Max - \triangleleft$. Then $Rad(\xi)$ is the unique maximal solvable ideal bundle of ξ. We assert that $r(\xi)$ is solvable. If not, $r(\xi)^{(n)} = \cup_{x\in B}(r(\xi))_x^{(n)} \neq (0)$ for any n. This implies there exists a x in B such that $r(\xi)_x^{(n)} \neq (0)$. We use the methods of [3] to arrive at a contradiction.

Let $\mathbb{C}$ denote a collection of ideals H in ξ_x such that $r(\xi)_x^{(n)} \not\subseteq H$ for all $n \geq 0$. $(0) \in \mathbb{C}$ and hence $\mathbb{C}$ is a non empty collection of ideals. For any ascending chain of ideals in ξ_x since $\xi \in Max - \triangleleft$, $\mathbb{C}$ has a maximal ideal P_x. We claim that P_x is a prime ideal in ξ_x. Otherwise, there are ideals H_x, K_x in ξ_x such that $H_x \not\subseteq P_x$, $K_x \not\subseteq P_x$ and $[H_x, K_x] \subseteq P_x$. P_x is a maximal element of $\mathbb{C}$ and hence $H_x + P_x$, $K_x + P_x \notin \mathbb{C}$. Therefore, $r(\xi)_x^{(n)} \subseteq H_x + P_x$, $r(\xi)_x^{(m)} \subseteq K_x + P_x$ for some integers n and m, both positive.

Let $k = max\{n, m\}$. Then $r(\xi)_x^{(k)} \subseteq r(\xi)_x^{(m)}, r(\xi)_x^{(n)} \subseteq H_x + P_x$, $K_x + P_x$. Thus we obtain, $r(\xi)_x^{(k+1)} \subseteq [H_x + P_x, K_x + P_x] \subseteq P_x$. It follows that $P_x \notin \mathbb{C}$ which is a contradiction. This proves our claim that P_x is a prime ideal. Now, $r(\xi)_x \not\subseteq P_x$ which contradicts the definition of $r(\xi)_x$. Therefore $r(\xi)_x^{(n)} = (0)$ for each x in B and for some $n \geq 0$ and hence $r(\xi)$ is solvable and $r(\xi) \subseteq Rad(\xi)$. This completes the proof. $\square$

The intersection of two ideal bundles of a semisimple Lie algebra bundle ξ is an ideal bundle of ξ (See [5]). By induction it can be proved that finite intersection of ideal bundles of ξ is an ideal bundle of ξ.

Theorem 2.15. *Let ξ be a semisimple Lie algebra bundle. Then the finite intersection of prime ideal bundles of ξ is a semiprime ideal bundle of ξ.*

Proof. Let ξ be a Lie algebra bundle with local trivialization $\phi : U \times L \to p^{-1}(U)$, U is an open set in B and $\xi_1, \xi_2, \dots, \xi_n$ be prime ideal bundles of ξ. For each $i = 1, 2, \dots, n$, let

$$\phi_i : U \times I_i \to p^{-1}(U)$$

be the local triviality of ξ_i, I_i is a prime ideal of L. Then for each x in U, $(\phi_i)_x : \{x\} \times I_i \to p^{-1}(x)$ is an isomorphism.

Let $\eta = \cap_{i=1}^{n} \xi_i$. Then η is an ideal bundle of ξ whose local triviality is given by

$$\psi : U \times (\cap_{i=1}^{n} I_i) \to p^{-1}(U).$$

Now, if ζ is any ideal bundle of ξ and $\zeta^2 \subseteq \eta$ then $\zeta_x^2 \subseteq \eta_x$ for x in B. By the isomorphism $\psi_x : \{x\} \times (\cap_{i=1}^{n} I_i) \to p^{-1}(x)$, $\zeta_x^2 \subseteq \cap_{i=1}^{n} I_i$, I_i is a prime ideal of L. It follows that $\zeta_x \subseteq \eta_x$ for all x in B. Thus η is a semiprime ideal bundle of ξ. □

Corollary 2.16. *The prime radical bundle of any ideal bundle in ξ is a semiprime ideal bundle of ξ.*

Corollary 2.17. *Let Q be an ideal bundle of ξ. If $A^2 \subseteq Q$ for any ideal bundle A of ξ then $A \subseteq r(Q)$.*

Lemma 2.18. *Let η be an ideal bundle of a Lie algebra bundle ξ. Then $r(\xi/\eta) = r(\eta)/\eta$.*

Proof. We have

$$\begin{aligned} r(\xi/\eta) &= \cup_{x\in B}\, r\big((\xi/\eta)_x\big) = \cup_{x\in B}\Big(\cap\{\text{prime ideals of } (\xi/\eta)_x\}\Big) \\ &= \cup_{x\in B}\,\Big(\cap\{P_x/\eta_x : P_x \text{is a prime ideal of} \xi_x\}\Big) \\ &= \cup_{x\in B}\,\Big((\cap\{P_x : P_x \text{is a prime ideal of } \xi_x\})/\eta_x\Big) \\ &= \cup_{x\in B}\,\big(r(\eta_x)/\eta_x\big) \\ &= r(\eta)/\eta. \end{aligned}$$

□

Theorem 2.19. *Let $\xi \in Max - \lhd$ and η be an ideal bundle of ξ. Then there exists $n \in \mathbb{N}$ such that $(r(\eta))^{(n)} \subseteq \eta$.*

Proof. By Lemma 2.18, $r(\xi/\eta) = r(\eta)/\eta$. By Theorem 2.14, $rad(\xi/\eta) = r(\xi/\eta)$ which is solvable. Therefore there exists a $n \in \mathbb{N}$ such that $(r(\xi/\eta))^{(n)} = (0)$ that is, $(r(\eta)/\eta)^{(n)} = (0)$ or $(r(\eta))^{(n)} \subseteq \eta$. □

Theorem 2.20. *Let $\xi \in Max - \triangleleft$ and η be an ideal bundle of ξ. Then η has a finite number of minimal prime ideal bundles containing it.*

Proof. If η is a prime ideal bundle then the assertion follows. We assume that η is not a prime ideal bundle of ξ. Then there exists ideal bundles ζ_1 and ζ_2 of ξ such that $\zeta_1 \not\subseteq \eta$, $\zeta_2 \not\subseteq \eta$ but $[\zeta_1, \zeta_2] \subseteq \eta$. Let us suppose that η has infinite number of minimal prime ideal bundles P_i containing it. For any $x \in B$,

$$\begin{aligned}[(\zeta_1)_x + \eta_x, (\zeta_2)_x + \eta_x] &= [(\zeta_1)_x, (\zeta_2)_x] + [\eta_x, (\zeta_2)_x] + [(\zeta_1)_x, \eta_x] + [\eta_x, \eta_x] \\ &\subseteq \cup_{x\in B}\, \eta_x.\end{aligned}$$

Therefore, $[\zeta_1 + \eta, \zeta_2 + \eta] = \cup_{x\in B}[(\zeta_1)_x + \eta_x, (\zeta_2)_x + \eta_x] \subseteq \cup_{x\in B}\eta_x = \eta$. η is contained in P_i for infinite number of prime ideal bundles P_i and hence one of $\zeta_1 + \eta$ and $\zeta_2 + \eta$ must be contained in infinite number of prime ideal bundles P_i. Without loss of generality we may assume that $\zeta_1 + \eta$ is contained in infinitely many P_i. $\zeta_1 + \eta$ contains η and is different from η for if $\zeta_1 + \eta = \eta$ then $\zeta_1 \subseteq \eta$ which is not possible. If P is any prime ideal bundle of ξ and $\zeta_1 + \eta \subseteq P \subseteq P_i$ then $P = P_i$ otherwise, $\eta \subseteq P \subseteq P_i$ which is a contradiction to the minimality of P_i. Thus P_i which contain $\zeta_1 + \eta$ are minimal prime ideal bundles containing $\zeta_1 + \eta$ and $\zeta_1 + \eta \supsetneq \eta$. Continuing the above same argument we arrive at a strictly increasing sequence of ideal bundles of ξ,

$$\eta \subsetneq \zeta_1 + \eta \subsetneq \cdots,$$

which is not possible from our assumption. This proves the theorem. □

The following theorem gives characterization of semiprime ideal bundles when $\xi \in Max - \triangleleft$.

Theorem 2.21. *Let $\xi \in Max - \triangleleft$ and Q be an ideal bundle of ξ. Then the following statements are equivalent.*

(1) *Q is a semiprime ideal bundle.*
(2) *$Q = r(Q)$.*
(3) *Q is a finite intersection of prime ideal bundles of ξ.*

Proof. (1) $\implies$ (2): $Rad(\xi/Q) = \cup_{x\in B} rad((\xi/Q)_x) = (0)$ since Q is semiprime. By Theorem 2.14, $rad(\xi/Q) = r(\xi/Q)$. Therefore $r(\xi/Q) = (0)$ and intersection of prime ideals of ξ_x containing Q_x is Q_x. Thus, $r(Q) = \cup_{x\in B} r(Q_x) = Q$.

(2) $\implies$ (3): From Theorem 2.20, $r(Q)$ is a finite intersection of prime ideal bundles containing Q and from (2) $Q = r(Q)$. Therefore (3) follows.

(3) $\implies$ (1): By Theorem 2.15, Q is semiprime. □

Lemma 2.22. *Every ideal bundle of a semisimple Lie algebra bundle ξ is a semiprime ideal bundle.*

Proof. Let Q be an ideal bundle of ξ. For any other ideal bundle A of ξ, $A = [A, A]$ since ξ is a semisimple Lie algebra bundle. Then $A^2 \subseteq Q$ implies $A \subseteq Q$. Thus Q is semiprime. □

Definition 2.23. An ideal bundle η of ξ which is different from ξ is said to be **strongly irreducible** if for any pair of ideal bundles ζ_1 and ζ_2 of ξ, $\zeta_1 \cap \zeta_2 \subseteq \eta$ implies $\zeta_1 \subseteq \eta$ or $\zeta_2 \subseteq \eta$.

It follows directly that every strongly irreducible ideal bundle of ξ is a irreducible.

Theorem 2.24. *Every prime ideal bundle of a Lie algebra bundle ξ is strongly irreducible.*

Proof. Let P be a prime ideal bundle of ξ. Suppose that ζ_1 and ζ_2 are two ideal bundles of ξ. If $\zeta_1 \cap \zeta_2 \subseteq P$, then

$$[\zeta_1, \zeta_2] = \cup_{x\in B}[(\zeta_1)_x, (\zeta_2)_x] \subseteq \cup_{x\in B}[(\zeta_1)_x \cap (\zeta_2)_x] = \zeta_1 \cap \zeta_2 \subseteq P.$$

Therefore, $\zeta_1 \subseteq P$ or $\zeta_2 \subseteq P$, which shows that P is strongly irreducible. □

Theorem 2.25. *Every strongly irreducible ideal bundle of a semisimple Lie algebra bundle ξ is prime.*

Proof. Let P be a strongly irreducible ideal bundle of ξ. By Lemma 2.22, P is semiprime. Let H, K be two ideal bundles of ξ satisfying $[H, K] \subseteq P$. Set

$$N = (H + P) \cap (K + P).$$

Then N is an ideal bundle of ξ and $N^2 \subseteq P$ which gives $N \subseteq P$, since ξ is semisimple. From our assumption, $H + P \subseteq P$ or $K + P \subseteq P$. Therefore $H \subseteq P$ or $K \subseteq P$. Hence P is prime. □

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M. V. Monica
Department of Mathematics, Field Marshal K. M. Cariappa College (A constituent college of Mangalore University), Madikeri- 571201, India
Email: monicavital8@gmail.com

R. Rajendra
Department of Mathematics, Field Marshal K.M. Cariappa College (A constituent college of Mangalore University), Madikeri- 571201, India
Email: rrajendrar@gmail.com