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A k-IDEAL-BASED GRAPH OF COMMUTATIVE SEMIRINGS

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ABSTRACT. Let R be a commutative semiring and I be a k-ideal of R. In this paper, we introduce the k-ideal-based graph of R, denoted by $\Gamma_{I^*}(R)$. The basic properties and possible structures of the graph are studied.

1. INTRODUCTION

Throughout this paper, all semirings are commutative with nonzero identity. The semirings appear naturally in diverse mathematics areas such as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, and optimization theory. From an algebraic point of view, semirings give the most natural common generalization of the theories of rings and bounded distributive lattices. The techniques used in analyzing them are taken from both areas. There is much research on various graphs associated with algebraic structures (see [1],[3],[5],[9]-[14] and [18]). The paper aims to investigate the interaction between a semiring R's semiring properties and its k-ideal-based graphs' properties. Let R be a commutative semiring and let I be an ideal of R with $I^* = I \setminus \{0\}$. Here, we introduce a new class of graphs associated to a semiring R, denoted by $G = \Gamma_{I^*}(R)$, as the (undirected) simple graph with vertices $V(G) = \{v \in R \setminus I : v + v' \in I^* for\ some\ v \neq v' \in R \setminus I\}$, where distinct vertices v and v' are adjacent if and only if $v + v' \in I^*$. Let $r \in V(G)$. Then $N_G(r) = \{r' \in V(G) : r' \neq r, r + r' \in I^*\}$ is the

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neighborhood of vertex r in G consisting of all vertices adjacent to r in G. The graph for modules over commutative rings has been defined by Abbasi and Jahromi in [1]. The corresponding results are obtained by modification, and here, we present a complete image of this graph. In the next section, we give all the necessary definitions. Section 3 of this paper investigates some basic properties of the graph $\Gamma_{I^*}(R)$. In sections 4 and 5, we determine the diameter and the girth of $\Gamma_{I^*}(R)$ when I is a k-ideal and a Q_R-ideal of R, respectively. We also explore the structure of $\Gamma_{I^*}(R)$, when I is a Q_R-ideal of R (see Theorem 5.3 and Theorem 5.5)

2. Preliminaries

Now, we recall some definitions and notations on graphs and semimodules theory. Let Γ be a simple graph. The vertex set of Γ is denoted by $V(\Gamma)$. We recall that a graph is connected if a path connects two distinct vertices. The distance $\mathrm{d}(a,b)$ is the length of the shortest path from a to b; if such a path does not exist, then $d(a,b)=\infty$. The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a,b) : a,b \in V(\Gamma)\}$. A graph is complete if it is connected with a diameter less than or equal to one. The girth of a graph Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle; otherwise, $\mathrm{gr}(\Gamma)=\infty$, in this case Γ is called an acyclic graph. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (respectively, Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (respectively, Γ_2). We denote the complete bipartite graph on m and n vertices by $K^{m,n}$. A component (connected component) of graph Γ is a subgraph in which any two vertices are connected to each other by paths and which is connected to no additional vertices in the graph Γ. We say that u is a universal vertex of Γ if u is adjacent to all other vertices of Γ. A vertex v in an undirected connected graph G is a cut-point (cut vertex) of G if removing it (and edges through it) disconnects the graph.

Now, we recall various notions from semimodule theory, which will be used in the sequel. For the definitions of semirings theory, we refer [2],[4]-[8], [15]-[17], and [19]. An ideal I of R is a k-ideal if $x, x+y \in I$ implies that $y \in I$ (so $\{0_R\}$ is a k-ideal of R). An element s of R is a zero-sum in R if $s+t=0$ for some $t \in R$. We use $S(R)$ to denote the set of all zero-sum elements of R. Similarly, if K is a subset of R, then $S(K)=\{r \in K : r+s=0, for\ some\ s \in K\}$. An ideal I of a semiring R is called a Q_R-ideal if there exists a subset

Q_R of R such that $R = \cup\{q + I : q \in Q_R\}$ and if $q_1, q_2 \in Q_R$ then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$. Let I be a Q_R-ideal of R and let $R/I = \{q + I : q \in Q_R\}$. Then R/I forms a semiring under the operations $\oplus$ and $\odot$ defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_3 \in Q_R$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$ and $(q_4 + I) \odot (q_1 + I) = q_5 + I$, where $q_5 \in Q_R$ is the unique element such that $q_4 q_1 + I \subseteq q_5 + I$. The semiring R/I is the quotient semiring of R by I (see [2]). There exists a unique element $q_0 \in Q_R$ such that $q_0 + I = I$. Thus, $q_0 + I$ is the zero element of R/I. It is shown that every Q_R-ideal is a k-ideal of R by [16, 8.23]. Throughout this paper, R is a commutative semiring, and I is an ideal of R with $I^* = I \setminus \{0\}$.

3. Some properties of $\Gamma_{I^*}(R)$

We devote this section to some properties of $\Gamma_{I^*}(R)$ graph. The following lemma contains several results, which we will use throughout this paper.

Lemma 3.1. *Let I be a k-ideal of semiring R. Then the following hold:*
(1) If r and s are adjacent for some $r, s \in \Gamma_{I^*}(R)$, then $2r + s \notin I$.
(2) If $s \in V(\Gamma_{I^*}(R))$ and $s + s' = 0$ for some $s' \in R$, then $s' \notin I$.
(3) Let $r, s \in V(\Gamma_{I^*}(R))$ with $r \neq s$ and $N_G(r) \cap N_G(s) \neq \emptyset$. If $s + s' = 0$ for some $s' \in R$, then $r + s' \in I^*$.
(4) If $r, s \in V(\Gamma_{I^*}(R))$ with $r + r' = 0$ and $s + s' = 0$ for some $r', s' \in R$, then $r + s \in I^*$ if and only if $r' + s' \in I^*$.

Proof. (1) Let $2r + s \in I$. Then $2r + s = r + (r + s) \in I$. This implies that $r \in I$, since $r + s \in I$ and I is a k-ideal of R, contradicts our assumption.
(2) It is clear since I is a k-ideal of R.
(3) Assume that $u \in N_G(r) \cap N_G(s)$. So $r + u, s + u \in I^*$. Let $r + s' = a$ for some $a \in R$. Then $r + u = r + s' + s + u = a + s + u$. So $a \in I$ since I is a k-ideal of R. If $r + s' = 0$, then $r = r + s' + s = s$ is a contradiction. Hence $r + s' \in I^*$.
(4) It is clear that $r + s = 0$ if and only if $r' + s' = 0$. Now, let $r + s \in I^*$. Then $r + s = a$ for some $0 \neq a \in I$. Therefore $a + r' + s' = r + s + r' + s' = 0 \in I$, thus $r' + s' \in I$ since I is a k-ideal of R. Similarly, the other side holds. $\square$

Remark 3.2. Let I be an ideal of semiring R and $I' = \{r \in R : r + s \in I,\ for\ some\ s \in R \setminus I\ where\ s \neq r\}$. If I is a k-ideal, then $I' \subseteq R \setminus I$.

In the following proposition, we consider the conditions under which $V(\Gamma_{I^*}(R)) \neq \emptyset$.

Proposition 3.3. *Let I be a k-ideal of R with $|I^*| \geq 2$. If $S(R) \cap I' \neq \emptyset$, then $V(\Gamma_{I^*}(R)) \neq \emptyset$.*

Proof. Let $r \in S(R) \cap I'$. Then $r + s = 0$ for some $s \in R \setminus I$ and $s \neq r$. If $r = s + a$ for every $0 \neq a \in I$, then $r = s + a = s + b$ for some distinct nonzero elements $a, b \in I$. So we have $a = r + s + a = r + s + b = b$, which is a contradiction. Thus there exists $0 \neq t \in I$ such that $r \neq s + t$. Hence $r + s + t = t \in I^*$ and so $r, s + t \in V(\Gamma_{I^*}(R))$. □

The next result is used to identify the adjacency between the vertices of the $\Gamma_{I^*}(R)$.

Theorem 3.4. *Let I be a k-ideal of R. If $r, s \in V(\Gamma_{I^*}(R))$ are distinct vertices connected by a path of length 3 and $r + s \neq 0$, then r and s are adjacent.*

Proof. Let $r - t_1 - t_2 - s$ be a path of length 3 between r and s for distinct vertices $r, t_1, t_2, s \in V(\Gamma_{I*}(R))$. So $r + t_1, t_1 + t_2, t_2 + s \in I^*$. Therefore we have $(r+s)+(2(t_1+t_2)) = (r+t_1)+(t_1+t_2)+(t_2+s) \in I$. Then $r + s \in I^*$ because $t_1 + t_2 \in I$ and I is a k-ideal. Thus, r and s are adjacent. □

Theorem 3.5. *Let I be a k-ideal of R. If $r, s \in V(\Gamma_{I^*}(R))$ are distinct vertices connected by a path of length 4, then there exists a path of length 2 between them. In particular, $N_G(r) \cap N_G(s) \neq \emptyset$.*

Proof. Let $r - t_1 - t_2 - t_3 - s$ be a path of length 4 between r and s for distinct vertices $r, t_1, t_2, t_3, s \in V(\Gamma_{I*}(R))$. If either $r + t_3 \neq 0$ or $t_1 + s \neq 0$, then r and t_3 or t_1 and s are adjacent by Theorem 3.4, as we desired. So we may assume that $r + t_3 = 0$ and $t_1 + s = 0$. If $r = t_1 + t_2 + t_3$, then $r + s = t_1 + t_2 + t_3 + s = t_2 + t_3 \in I^*$. This implies that r and s are adjacent, a contradiction. Similarly, if $s = t_1 + t_2 + t_3$, then $r + s = r + t_1 + t_2 + t_3 = t_1 + t_2 \in I^*$ which is also a contradiction. Therefore $r - (t_1 + t_2 + t_3) - s$ is a path of length 2 between r and s. □

Corollary 3.6. *Let I be a k-ideal of R. If P is a path of length 4 in $\Gamma_{I^*}(R)$, then $\Gamma_{I^*}(R)$ has a cycle.*

Proof. Let $r - t_1 - t_2 - t_3 - s$ be a path of length 4 between r and s for distinct vertices $r, t_1, t_2, t_3, s \in V(\Gamma_{I*}(R))$. If either $r + t_3 \neq 0$ or $t_1 + s \neq 0$, then r and t_3 or t_1 and s are adjacent by Theorem 3.4, as we desired. So we may assume that $r + t_3 = 0$ and $t_1 + s = 0$. Then $r - (t_1 + t_2 + t_3) - s$ is a path of length 2 between r and s by Theorem 3.5. If $t_1 + t_2 + t_3 = t_1$, then $s + t_1 + t_2 + t_3 = s + t_1$ and $t_2 + t_3 = 0$, a contradiction. Thus $t_1 + t_2 + t_3 \neq t_1$. Similarly, $t_1 + t_2 + t_3 \neq t_3$. If $t_1 + t_2 + t_3 = t_2$, then $r - t_1 - t_2 - r$ and $t_2 - t_3 - s - t_2$ are two cycles of length 3 in $\Gamma_{I^*}(R)$.

If $t_1+t_2+t_3 \neq t_2$, then $r-t_1-t_2-t_3-s-(t_1+t_2+t_3)-r$ is a cycle of length 6 in $\Gamma_{I^*}(R)$, as we desired. □

4. The diameter and the girth of $\Gamma_{I^*}(R)$

In this section, we compute the diameter and the girth of graph $\Gamma_{I^*}(R)$. In the next example, we introduce an ideal such that $\Gamma_{I^*}(R)$ is a union of two disjoint complete bipartite subgraphs.

Example 4.1. Let $R = Z^+ \cup \{0\}$. Assume that $I = 5Z^+ \cup \{0\}$, so I is a k-ideal of R. Assume that $r \in V(\Gamma_{I^*}(R))$, then $r \neq 5k$. Therefore either $r = 5k+1, r = 5k+2, r = 5k+3$ or $5k+4$ for some nonnegative integer k. Let $V_1 = \{x \in V(\Gamma_{I^*}(R)) : r = 5k+1 \ or \ r = 5k+4 \ for\ some\ nonnegative\ integer\ k\}$ and $V_2 = \{r \in V(\Gamma_{I^*}(R)) : r = 5k+2 \ or \ r = 5k+3 \ for\ some\ nonnegative\ integer\ k\}$. It is easy to see that the induced subgraphs of $\Gamma_{I^*}(R)$ with vertex set V_1 and V_2 are complete bipartite graphs and these two subgraphs are disjoint. So $gr(\Gamma_{I^*}(R)) = 4$ and $diam(\Gamma_{I^*}(R)) = \infty$.

Theorem 4.2. *Let I be a k-ideal of R. Then $diam(\Gamma_{I^*}(R)) \leq 3$.*

Proof. We can reduce every path of length greater than 3 to a path of length at most 3 by Theorem 3.5. □

Theorem 4.3. *Let I be a k-ideal of a semiring R and $\Gamma_{I^*}(R)$ be a connected graph. If $2u = 0$ for every $u \in V(\Gamma_{I*}(R))$, then $\Gamma_{I^*}(R)$ is a complete graph.*

Proof. By Theorem 4.2, we have $diam(\Gamma_{I^*}(R)) \leq 3$. First suppose that $r, s \in V(\Gamma_{I^*}(R))$ with $r \neq s$ and $d(r,s) = 2$, so we have $r-u-s$ in $\Gamma_{I^*}(R)$, that this is a path in $\Gamma_{I^*}(R)$. Hence $r+s = r+s+2u = (r+u)+(s+u) \in I$. If $r+s = 0$, then $s = 2r+s = r$, which is a contradiction. So $r+s \neq 0$ and r and s are adjacent. If $d(r,s) = 3$, then r and s are adjacent by Theorem 3.4. □

In the following example we introduce an ideal such that $\Gamma_{I^*}(R)$ is a complete graph but $2u \neq 0$ for every $u \in V(\Gamma_{I*}(R))$. So, the converse of Theorem 4.3 is not always true.

Example 4.4. Let $R = Z^+ \cup \{0\}$ be the semiring of all non-negative integers. Then $I = 2Z^+ \cup \{0\}$ is a k-ideal of R. Assume that $r \in V(\Gamma_{I^*}(R))$ so $r \neq 2k$ and so $r = 2k+1$ for some nonnegative integer k. Let $r, s \in V(\Gamma_{I^*}(R))$. Then $r = 2k+1$ and $s = 2k'+1$ for some nonnegative integers k and k'. So $r+s = 2k+1+2k'+1 = 2(k+k'+1) \in I^*$, thus $\Gamma_{I^*}(R)$ is a complete graph and $diam(\Gamma_{I^*}(R)) = 1$. Also $gr(\Gamma_{I^*}(R)) = 3$. It is clear that $2r \neq 0$ for every $r \in V(\Gamma_{I*}(R))$.

Our next goal is to determine the girth of $\Gamma_{I^*}(R)$.

Theorem 4.5. *Let I be a k-ideal of R and $\Gamma_{I^*}(R)$ has a cycle. Then $gr(\Gamma_{I^*}(r)) \leq 6$.*

Proof. Let $r_1 - r_2 - r_3 - r_4 - r_5 - r_6 - r_7 - r_1$ be a cycle of length 7 in $\Gamma_{I^*}(R)$. Then there exist a path $r_1 - m - r_5$ of length 2 between r_1 and r_5 by Theorem 3.5. If $m \notin \{r_2, r_3, r_4\}$, then $r_1 - m - r_5 - r_6 - r_7 - r_1$ is a cycle of length 5 in $\Gamma_{I^*}(R)$. Now suppose that $m \in \{r_2, r_3, r_4\}$. Then, we have a cycle with a length of less than 6. □

Theorem 4.6. *Let I be a k-ideal of R. If $|S(R) \setminus I| \geq 4$ and $2r \in I^*$ for every $0 \neq r \in R$, then $gr(\Gamma_{I^*}(R)) \leq 4$.*

Proof. Let $0 \neq a \in S(R) \setminus I$. Then there exists $b \in S(R)$ such that $a + b = 0$. It is clear that $b \notin I^*$ since I is a k-ideal. If $a = b$, then $2a = 0$, which contradicts our assumption. Now let $c \in S(R) \setminus I$ and $c \neq a, b$. Similarly $c + d = 0$ for some $d \in S(R) \setminus I$ and $d \notin \{a, b, c\}$.
Case 1. If $a + c \in I$, then $b + c = a + c + 2b \in I$ since $2b \in I$. If $b + c = 0$, then $b = b + c + d = d$, a contradiction. This implies that $b + c \in I^*$. By a similar argument, we can show that $b + d, d + a \in I^*$. So in this case we have a cycle $a - d - b - c - a$ of length 4 in $\Gamma_{I^*}(R)$.
Case2. Assume that $a+c \notin I$. If $a+d \in I$, then $a+c = a+d+2c \in I$ by assumption, which is a contradiction. So we have $a + d \notin I$. Similarly, we can show that $b + d, b + c \notin I$. Now we show that $(a + c) - (a + d) - (b + d) - (b + c) - (a + c)$ is a cycle of length 4 in $\Gamma_{I^*}(R)$. It suffices to show that $a + c \neq b + d$ and $a + d \neq b + c$. Suppose that $a+c = b+d$. Then $2a = 2a+c+d = a+c+a+d = b+d+a+d = 2d$. This implies that $2(a + c) = 2a + 2c = 2d + 2c = 2(d + c) = 0$ which contradicts our assumption. So $a + c \neq b + d$. If $a + d = b + c$, then $2a = 2a + c + d = a + d + a + c = b + c + a + c = 2c$ and we have $2(a + d) = 2a + 2d = 2c + 2d = 0$, a contradiction. So we have a cycle $(a + c) - (a + d) - (b + d) - (b + c) - (a + c)$ of length 4 in $\Gamma_{I^*}(R)$. □

Theorem 4.7. *Let I be a k-ideal of R and $|I^* \setminus S(R)| \geq 2$. Then $\Gamma_{I^*}(R)$ is an acyclic graph if and only if it is a distinct union of some star components.*

Proof. Let $\Gamma_{I^*}(R)$ be an acyclic graph. If $\Gamma_{I^*}(R)$ has no star components, then there exists a path $r - t_1 - t_2 - s$ of length 3 in $\Gamma_{I^*}(R)$. If $r + s \neq 0$, then r and s are adjacent by Theorem 3.4. So we have a cycle in $\Gamma_{I^*}(R)$, which is a contradiction. Then, we may assume that $r + s = 0$.
Let $a, b \in I^* \setminus S(R)$ and $a \neq b$. If $r+a = t_1$, then $s+t_1 = s+r+a = a \in I^*$. So we have a cycle $t_1 - t_2 - s - t_1$ in $\Gamma_{I^*}(R)$, a contradiction. Now assume that $r+a = s$, then $2r+a = r+s = 0$. This implies that $a \in S(R)$,

which contradicts the assumption. Therefore $r + a \notin \{r, t_1, s\}$. By a similar argument we have $r + b \notin \{r, t_1, s\}$ and $s + a \notin \{s, r, t_2\}$. If either $r + a \neq t_2$ or $s + a \neq t_1$, then $r - t_1 - t_2 - s - (r + a)$ or $(s+a) - r - t_1 - t_2 - s$ is a path of length 4 in $\Gamma_{I^*}(R)$. Thus we have a cycle in $\Gamma_{I^*}(R)$ by Corollary 3.6. So we may assume that $r+a = t_2$ and $s + a = t_1$. If $r + b = t_2 = r + a$, then we have $a = b$, a contradiction. Then $r + b \neq t_2$ and so $r - t_1 - t_2 - s - (r + b)$ is a path of length 4 in $\Gamma_{I^*}(R)$. So we have a cycle in $\Gamma_{I^*}(R)$ by Corollary 3.6, which is a contradiction. □

Lemma 4.8. *Let I be a k-ideal of R with $2t \in I^*$ for all $t \in V(\Gamma_{I^*}(R))$. Then:*
(1) If r and s are adjacent with $r \neq s$ and $r + r' = 0$ for some $r' \in R$, then $s + r' \in I^*$.
(2) If $t - r - v$ is the shortest path from t to v and $2r + t = 2t + r = r$, then $t + v = 0$ and $2r + v = 2v + r = r$.
(3) If $v - r - w$ is the shortest path from v to w, then $v + w = 0$.

Proof. (1) It is clear that $s+r' \neq 0$, since $s \neq r$ by assumption. Suppose that $s + r' = w$ for some $w \in R$, then $s + r = s + r' + 2r = w + 2r$. So $s + r' \in I$, since I is a k-ideal of R and $s + r, 2r \in I$.
(2) Assume that $2r + t = r$, so $2r + t + v = r + v \in I$. This implies that $t + v \in I$, since I is a k-ideal of R and $2r \in I$. Therefore $t + v = 0$ since $d(t, v) = 2$. Also $r + t = 2t + v + r = v + r$. Therefore we have $2r + v = 2v + r = r$.
(3) Let $v - r - w$ be the shortest path from v to w. Then $2r + v + w = v + r + r + w \in I$ and so $v + w \in I$ since $2r \in I$ and I is a k-ideal. Therefore, $v + w = 0$ since v and w are not adjacent. □

We end this section with the following theorem.

Theorem 4.9. *Let I be a k-ideal of R. If $\Gamma_{I^*}(R)$ is a connected graph with $|R \setminus I^*| \geq 4$ and $2u \in I^*$ for all $u \in R$, then $\Gamma_{I^*}(R)$ has no cut-points.*

Proof. Suppose that r is a cut-point of $\Gamma_{I^*}(R)$. So there exist vertices $v, w \in V(\Gamma_{I^*}(R))$ such that $r \neq v, w$ and r lies on every path from v to w. Then the shortest path from v to w is of length 2 or 3 by Theorem 4.2.
Case 1. Suppose that $v-r-s-w$ is a path of the shortest length from v to w. So $v+w = 0$ by Theorem 3.4. Also, $2r+v, 2s+w \notin I$ by Lemma 3.1. First suppose that $r \notin S(R)$ and $s \notin S(R)$. If $2r + v = r$, then $r + w = 2r + v + w = 2r \in I^*$. This implies that r and w are adjacent, which is contradictory. Similarly, $2s + w \neq r$. If $2r + v = 2r + w$,

then $v - 2r + v - w$ is a path of length 2 which is a contradiction. Hence $v - (2r + v) - (2s + w) - w$ is a path between v and w which also contradicts our assumption. Now, suppose that either $r \in S(R)$ or $s \in S(R)$. If $r \in S(R)$, then $v - r' - w$ is a path from v to w by Lemma 3.1 and Lemma 4.8, where $r + r' = 0$ which is a contradiction. The case $s \in S(R)$ is similar.
Case 2. Assume that $v - r - w$ is a path of the shortest length from v to w. Then $v + w = 0$ by Lemma 4.8. First, suppose that $r \in S(R)$. So $r + r' = 0$ for some $r' \in R \setminus I$ by Lemma 3.1. Therefore, $v - r' - w$ is a path from v to w by Lemma 4.8, which contradicts our assumption. Now assume that $r \notin S(R)$. If either $2r + v \neq r$ or $r + 2v \neq r$, then $v - (2r + v) - w$ or $v - (2v + r) - w$ is a path from v to w, which is a contradiction. So, we may assume that $2r + v = 2v + r = r$. Since $|R \setminus I^*| \geq 4$ and $\Gamma_{I^*}(R)$ is a connected graph, there exists $a \in R \setminus \{v, r, w\}$ such that $a \notin I^*$ and a is adjacent to one vertex of path $v - r - w$. First, suppose that a and r are not adjacent and $a - v - r - w$ is a path from a to w. If $a - v - r - w$ is the shortest path from a to w, then by Case.1, we have a path P from a to w which $r \notin V(P)$, so $P' = P \cup \{v, a\}$ is a path from v to w which is a contradiction. Now, assume that $a - t - w$ is the shortest path from a to w where $t \neq r$. Thus $v - a - t - w$ is a path from v to w, a contradiction. If $v - r - w - a$ is a path, the proof is similar. So we may assume that a and r are adjacent and $a + r \in I^*$. This implies that $2r + v + a = r + a \in I$, thus $v + a \in I$. If $v + a = 0$, then $a = v + a + w = w$, a contradiction. Thus $v + a \in I^*$. Now we show that $w + a \in I^*$. We have $w + 2r + a = w + r + r + a \in I$. So $w + a \in I$ since $2r \in I$ and I is a k-ideal. If $w + a = 0$, then $v = v + w + a = a$, a contradiction. Therefore, $v - a - w$ is a path from v to w, which is also a contradiction. $\square$

5. The case when I is a Q_R-ideal of R

In this section, we assume that I is a Q_R-ideal of R, and we shall describe the $\Gamma_{I^*}(R)$ graph with its structure, girth, and diameter. First, we begin with the following lemma.

Lemma 5.1. *Let I be a Q_R-ideal of R with $Q'_R = Q_R \setminus \{q_0\}$. Then the following hold:*
(1) $Q'_R \subseteq R \setminus I$.
(2) If $q \in Q'_R$ and $q + q' = 0$ for some $q' \in R$, then $q' \in R \setminus I$.
(3) Let $q \in Q'_R \cap S(R)$ and $a + q$ and $b + q$ are adjacent in $\Gamma_{I^*}(R)$ for some $a, b \in I$. Then $2q = 0$.

Proof. (1) Let $q \in Q'_R$. If $q \in I$, then $q \in I \cap Q_R$. So $q + q_0 \in (q + I) \cap (q_0 + I)$ and $(q + I) \cap (q_0 + I) \neq \emptyset$. This implies that $q = q_0$,

which is a contradiction.
(2) It is clear by part (1), and Since I is a k-ideal of R by [16, 8.23].
(3) Let $a+q$ and $b+q$ be adjacent. Then $2q+a+b=n$ for some $n \in I^*$. Since $q \in S(R)$, so $q+p=0$ for some $p \in R \setminus I$. We have $p \in q'+I$ for some $q' \in Q'_R$. Then $p=q'+i$ for some $i \in I$. Hence $q+q'+i=0$ and $q+a+b=q+q'+i+q+a+b=q'+2q+a+b+i=q'+n+i \in (q+I)\cap(q'+I)$. Then $(q+I)\cap(q'+I)\neq\emptyset$ and so $q=q'$. Thus, we have $2q=0$. □

In the following lemma, we see that if $2 \in I$, then the adjacent vertices of $\Gamma_{I^*}(R)$ are in the same coset of Q_R-ideal I of semiring R.

Lemma 5.2. *Let I is a Q_R-ideal of R with $2 \in I$ and $Q'_R = Q_R \setminus \{q_0\}$. Then the following hold:*
(1) If r and s are adjacent in $\Gamma_{I^*}(R)$, then there exists $q \in Q_R \setminus \{q_0\}$ such that $r, s \in q+I$.
(2) If $r \in V(\Gamma_{I^*}(R))$, then $N_G(r) \subseteq q+I$ for some $q \in Q_R \setminus \{q_0\}$.
(3) If $q \in Q'_R \cap S(R)$ and $|I^*| \geq 1$, then $2q+i=0$ for some $i \in I$.

Proof. (1) Let $r \in q_1+I$ and $s \in q_2+I$ for some $q_1, q_2 \in Q_R \setminus \{q_0\}$. We show that $q_1=q_2$ and so r and s are in a same coset. Assume that $r=q_1+a$ and $s=q_2+b$ for some $a,b \in I$. Therefore we have $q_1+q_2+a+b=r+s \in I^*$ since r and s are adjacent. Hence $q_1+q_2 \in I$ since $a+b \in I$ and I is a k-ideal ideal by [16, 8.23]. So $q_2+2q_1=q_1+(q_1+q_2) \in q_1+I$. Likewise, $q_2+2q_1 \in q_2+I$ since $2 \in I$. So $q_2+2q_1 \in (q_1+I)\cap(q_2+I)$; hence $q_1=q_2$.
(2) It is clear by part (1).
(3) Let $q+p=0$ for some $p \in R\setminus I$. So $p \in q'+I$ for some $q' \in Q'_R$. Then $p=q'+i$ for some $i \in I$. Hence $q+q'+i=0$ and $q+q'+i+r=r \in I^*$ for every $r \in I^*$. So q and $q'+i+r$ are adjacent vertices and there exists $q'' \in Q_R \setminus \{q_0\}$ such that $q, q'+i+r \in q''+I$ by Part (1). This implies that $q \in (q+I)\cap(q''+I)$ and $(q+I)\cap(q''+I)\neq\emptyset$. Thus $q=q''$. On the other hand, $q'+i+r \in (q'+I)\cap(q''+I)$ and then $(q+I)\cap(q'+I)\neq\emptyset$. So we have $q'=q''$ and $2q+i=q+q'i=0$. □

We can now prove the following theorem that provides a characterization of $\Gamma_{I^*}(R)$, when $2 \in I$.

Theorem 5.3. *Let I be a Q_R-ideal of R with $2 \in I$ and $Q'_R = Q_R \setminus \{q_0\}$. If $|I^*| \geq 1$, $\alpha = |Q'_R \setminus S(R)|$ and $\beta = |Q'_R \cap S(R)|$, then $\Gamma_{I^*}(R)$ is a union of disjoint α complete subgraphs and β connected subgraphs with a universal vertex.*

Proof. Let $q \in Q'_R$. First suppose that $q \notin S(R)$, then $q+n+q+n' = 2q+n+n' \in I^*$ for every $n, n' \in I$. So the induced subgraph of $\Gamma_{I^*}(R)$

with vertices set $q+I$ is complete. So we have α disjoint complete subgraphs by Lemma 5.2. Now, assume that $q \in S(R)$. So $2q+i=0$ for some $i \in I$ by Lemma 5.2. Then $q+q+i+r = 2q+i+r = r \in I^*$ for every $r \in I^*$. Then the induced subgraph of $\Gamma_{I^*}(R)$ with vertices set $q+I$ is connected and $q+i$ is a universal vertex of this subgraph. Also, these subgraphs are disjoint by Lemma 5.2. $\square$

Proposition 5.4. *Let I be a Q_R-ideal of R with $Q'_R = Q_R \setminus \{q_0\}$ and $|I^*| \geq 1$. If $q+q' \in I$ for some $q, q' \in Q'_R$, then the induced subgraphs of $\Gamma_{I^*}(R)$ with vertices set $V = (q+I) \cup (q'+I)$ are connected subgraphs.*

Proof. First, suppose that $q+q'=0$. Then $q+(q'+r) = (q+r)+q' = r \in I^*$ for every $r \in I^*$. This implies that every element of $q+I$ is adjacent to q' in $q'+I$, and also, every element of $q'+I$ is adjacent to q in $q+I$. Therefore the induced subgraph with vertices set $(q+I)\cup(q'+I)$ is a connected subgraph. Now suppose that $q+q' \neq 0$. We split the proof into two following cases:
Case.1. If $S(I) = I$, then $q+q'+r = 0$ for some $r \in I$. Now let $0 \neq u \in I$. Then $q+u+q'+r = q+u+q'+r = u \in I^*$. It means that every element of $q+I$ is adjacent to $q'+r$ in $q'+I$, and also, every element of $q'+I$ is adjacent to $q+r$ in $q+I$. Therefore the induced subgraph with vertices set $(q+I)\cup(q'+I)$ is a connected subgraph.
Case.2. Assume that $S(I) \neq I$, then there exists $t \in I$ such that $t+u' \neq 0$ for every $u' \in I$. This implies that $q+t+q'+u' = q+u'+q'+t \in I^*$. Hence, every element of $q+I$ is adjacent to $q'+t$ in $q'+I$, and also, every element of $q'+I$ is adjacent to $q+t$ in $q+I$, as required. $\square$

Theorem 5.5. *Let I be a Q_R-ideal of R. Then $diam(\Gamma_{I^*}(R)) = \{1, 2, 3, \infty\}$ and $gr(\Gamma_{I^*}(R)) = \{3, 4, \infty\}$.*

Proof. Let $2 \notin I$ and $q \in Q'_R = Q_R \setminus \{q_0\}$. If either $q \in S(R)$ or $N_G(q) \neq \emptyset$, then $q+q' \in I$ for some $q' \in Q_R$. So the induced subgraphs of $\Gamma_{I^*}(R)$ with vertices set $V = (q+I)\cup(q'+I)$ is a connected subgraph by Proposition 5.4. Now assume that $q \notin S(R)$ and $N_G(q) = \emptyset$. If $2q \in I$, then $q+u+q+u' = 2q+u+u' \in I^*$ and the induced subgraph with vertices set $q+I$ is a complete graph. Now, we may assume that $2q \notin I$, then $q+a+q+b = 2q+a+b \notin I$ since I is a k-ideal. Therefore, the induced subgraph with vertices set $q+I$ is a totally disconnected subgraph. So the proof is complete by Theorem 5.3. $\square$

We end the paper with the following example.

Example 5.6. (i) Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$. Set $I = \{0, 2, 4, 6, 8, ...\}$ and $Q_R = \{0, 1\}$. Then I is a Q_R-ideal of R. It is clear that $2 \in I$. Since $|R/I| = 2$, so $\Gamma_{I^*}(R)$ is a complete graph by Theorem 5.3 with

$diam(\Gamma_{I^*}(R)) = 1$ and $gr(\Gamma_{I^*}(R)) = 3$.
(ii) Let $R = \mathbb{Z}^* = \mathbb{Z}^+ \cup \{0\}$. If $I = 3R = \{0, 3, 6, 9, ...\}$ and $Q_R = \{0, 1, 2\}$, then I is a Q_R-ideal of R and $2 \notin I$. Then $\Gamma_{I^*}(R)$ is a graph with vertices set $(1 + I) \cup (2 + I)$. It is easy to see that this graph is bipartite with $diam(\Gamma_{I^*}(R)) = 2$ and $gr(\Gamma_{I^*}(R)) = 4$.

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