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GENERALIZED LOCAL COHOMOLOGY AND SERRE COHOMOLOGICAL DIMENSION

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ABSTRACT. Let R be a commutative Noetherian ring, I, J be two ideals of R, and M, N be two R-modules. Let S be a Serre subcategory of the category of R-modules. We introduce Serre cohomological dimension of N, M with respect to (I, J), as $\mathrm{cd}_S(I, J, N, M) = \sup\{i \in \mathbb{N}_0 : H^i_{I,J}(N, M) \notin S\}$. We study some properties of $\mathrm{cd}_S(I, J, N, M)$, and we get some formulas and upper bounds for it.

1. INTRODUCTION

Throughout this paper, R is a commutative Noetherian ring with a non-zero identity, I, J are two ideals of R, and M, N are two R-modules. For notations and terminologies not given in this paper, the reader is referred to [10], [11] and [23] if necessary.

The local cohomology theory has been a significant tool in commutative Algebra and Algebraic Geometry. There are some generalizations of this theory. For any non-negative integer i, Herzog [17] introduced the i-th generalized local cohomology functor $H^i_I(-,-)$ as $H^i_I(N, M) = \varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R(N/I^tN, M)$ for all R-modules N, M. It is clear that if $N = R$, then $H^i_I(N, -)$ is just the ordinary local cohomology functor $H^i_I(-)$.

Takahashi et al. [23] defined another generalization of the local cohomology theory. To be more precise, let $\Gamma_{I,J}(M) = \{x \in M : \exists\ t \in \mathbb{N}, I^tx \subseteq Jx\}$. It is easy to see that $\Gamma_{I,J}(M)$ is a submodule of M, and

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$\Gamma_{I,J}(-)$ is a covariant, R-linear functor from the category of R-modules to itself. For any non-negative integer i, the local cohomology functor $H^i_{I,J}(-)$ with respect to (I,J), is defined to be the i-th right derived functor of $\Gamma_{I,J}(-)$. If $J=0$, then $H^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$.

Let $\tilde{\mathrm{W}}(I,J)=\{\mathfrak{a}\leq R : I^t\subseteq J+\mathfrak{a}$ for some positive integer $t\}$. One can see that $x\in\Gamma_{I,J}(M)$ if and only if $\mathrm{Ann}(x)\in\tilde{\mathrm{W}}(I,J)$. Let $\mathrm{W}(I,J)=\{\mathfrak{p}\in\mathrm{Spec}(R) : I^t\subseteq J+\mathfrak{p}$ for some positive integer $t\}$. It is shown in [23, Corollary 1.8] that $x\in\Gamma_{I,J}(M)$ if and only if $\mathrm{Supp}(Rx)\subseteq\mathrm{W}(I,J)$.

Nam et al. [21] introduced a common generalization of the abovementioned notions as follows. The module $\Gamma_{I,J}(\mathrm{Hom}_R(N,M))$ is denoted by $\Gamma_{I,J}(N,M)$. It is easy to see that $\Gamma_{I,J}(N,-)$ is a left exact, covariant functor from the category of R-modules to itself. For a non-negative integer i, the i-th generalized local cohomology functor with respect to (I,J), denoted by $H^i_{I,J}(N,-)$, is defined as the i-th right derived functor of $\Gamma_{I,J}(N,-)$. Also $H^i_{I,J}(N,M)$ is called the i-th generalized local cohomology module of N, M with respect to (I,J). If $N=R$, then $H^i_{I,J}(N,M)\cong H^i_{I,J}(M)$. It is easy to see that $H^i_{I,J}(N,M)\cong\varinjlim_{\mathfrak{a}\in\tilde{\mathrm{W}}(I,J)}\mathrm{Ext}^i_R(N/\mathfrak{a}N,M)$. It follows that if $J=0$, then $H^i_{I,J}(N,M)\cong H^i_I(N,M)$. When N is a finitely generated R-module, then $H^i_{I,J}(N,M)$ is just the i-th generalized local cohomology of N,M relative to (I,J), which is defined by Zamani [27]. Note that the generalized local cohomology with respect to a pair of ideals, is a special case of the generalized local cohomology with respect to a system of ideals, which was introduced by Bijan-Zadeh [8].

The cohomological dimension of M with respect to I, is an important invariant linked to the local cohomology modules $H^i_I(M)$. This notion, denoted by $\mathrm{cd}(I,M)$, is defined as the supremum of all non-negative integers i for which $H^i_I(M)\neq 0$. Amjadi and Naghipour [4] introduced the cohomological dimension of two modules N, M with respect to an ideal I as $\mathrm{cd}(I,N,M)=\sup\{i\in\mathbb{N}_0 : H^i_I(N,M)\neq 0\}$. It is clear that if $N=R$, then $\mathrm{cd}(I,N,M)=\mathrm{cd}(I,M)$. As another generalization of this concept, Chu and Wang [13] introduced the cohomological dimension of M with respect to a pair of ideals (I,J), equal to the supremum of all non-negative integers i for which $H^i_{I,J}(M)\neq 0$. It is obvious that if $J=0$, then $\mathrm{cd}(I,J,M)=\mathrm{cd}(I,M)$. Let S be a Serre subcategory of the category of R-modules. In this direction, Aghapournahr et al. [2] introduced Serre cohomological dimension of M with respect to (I,J), as $\mathrm{cd}_S(I, J, M) = \sup\{i \geq 0 : H^i_{I,J}(M) \notin S\}$. For more study, see [1], [12], [14], and [24].

In this paper, we generalize all of these notions, and we introduce Serre cohomological dimension of N, M with respect to (I, J) as

$$\mathrm{cd}_S(I, J, N, M) = \sup\{i \in \mathbb{N}_0 : H^i_{I,J}(N, M) \notin S\}.$$

It is clear that if $N = R$, then $\mathrm{cd}_S(I, J, N, M) = \mathrm{cd}_S(I, J, M)$, and if $S = \{0\}$ and $J = 0$, then $\mathrm{cd}_S(I, J, N, M) = \mathrm{cd}(I, N, M)$.

In section 2, we get some basic properties of $\mathrm{cd}_S(I, J, N, M)$. In section 3, the cohomological dimensions are studied on a special Serre subcategory, namely localizing subcategory. Recall that a localizing subcategory is a Serre subcategory that is closed under taking direct limits. In section 4, we study the cohomological dimension of N, M with respect to $(I + I', J)$ and $(I, J \cap J')$, where I', J' are two ideals of R. More precisely, in Corollary 4.4, we show that the inequalities

$$\mathrm{cd}_S(I + I', J, N, M) \leq \mathrm{cd}_S(I, J, N, M) + \mathrm{cd}_S(I', J, M),$$
$$\mathrm{cd}_S(I, J \cap J', N, M) \leq \mathrm{cd}_S(I, J, N, M) + \mathrm{cd}_S(I, J', M),$$

are true, where S is a localizing subcategory of the category of R-modules, M is a finitely generated R-module, and N is a finitely generated R-module of finite projective dimension. Finally, in section 5, we investigate the relations between different types of the cohomological dimensions, and we get some upper bounds for $\mathrm{cd}_S(I, J, N, M)$. It is shown, in Theorem 5.3, that if either S is a localizing subcategory of the category of R-modules or R is a local ring, then $\mathrm{cd}_S(I, J, N, M) \leq \mathrm{pd}(N) + \mathrm{cd}_S(I, J, M)$, whenever N is a finitely generated R-module of finite projective dimension. As a consequence, it follows that if M is a finitely generated R-module, and N is a finitely generated R-module of finite projective dimension, then $\mathrm{cd}(I, J, N, M) \leq \mathrm{pd}(N) + \dim(M/JM) + 1$; see Proposition 5.4.

2. Basic properties of Serre cohomological dimensions

Recall that R is a Noetherian ring, I, J are two ideals of R, and M, N are two R-modules.

Definition 2.1. A full subcategory of the category of R-modules is said to be a Serre subcategory, if it is closed under taking submodules, quotients, and extensions.

Example 2.2. [3, Example 2.4] The following classes of modules are Serre subcategories of the category of R-modules.

(a) The class of zero modules.
(b) The class of Artinian R-modules.

(c) The class of finitely generated R-modules.
(d) The class of all R-modules M with $\dim(M) \leq k$, where k is an integer. When $k = -1$, we get the part (a).
(e) Let $\Phi \subseteq \operatorname{Spec}(R)$ be a closed set under specialization, that is, if $\mathfrak{q} \supseteq \mathfrak{p} \in \Phi$, then $\mathfrak{q} \in \Phi$. The class of all R-modules M with $\operatorname{Ass}(M) \subseteq \Phi$ (equivalently $\operatorname{Supp}(M) \subseteq \Phi$). For example, Φ could be a closed set $\Phi = \mathrm{V}(\mathfrak{c})$, for a given ideal $\mathfrak{c}$ of R. If we take $\Phi = \{\mathfrak{p} \in \operatorname{Spec}(R) : \dim(R/\mathfrak{p}) \leq k\}$ where k is an integer, then we recover part (d).

In the rest of the paper, S denotes a Serre subcategory of the category of R-modules.

Definition 2.3. Serre cohomological dimension of N, M with respect to (I, J), denoted by $\operatorname{cd}_S(I, J, N, M)$, is defined as

$$\operatorname{cd}_S(I, J, N, M) = \sup\{i \in \mathbb{N}_0 : H^i_{I,J}(N, M) \notin S\}.$$

If $S = \{0\}$, then $\operatorname{cd}_S(I, J, N, M) = \operatorname{cd}(I, J, N, M)$, which is just the supremum of all non-negative integers i for which $H^i_{I,J}(N, M) \neq 0$. If $N = R$, then $\operatorname{cd}_S(I, J, N, M) = \operatorname{cd}_S(I, J, M)$, which was defined in [2]. If $J = 0$, then $\operatorname{cd}_S(I, J, N, M) = \operatorname{cd}_S(I, N, M)$, which was defined in [14] (see also [1]).

Now, we get some basic properties of this invariant. The proof of the next result is trivial.

Proposition 2.4. *Suppose that S_1 and S_2 are two Serre subcategories such that $S_1 \subseteq S_2$. Then $\operatorname{cd}_{S_2}(I, J, N, M) \leq \operatorname{cd}_{S_1}(I, J, N, M)$. In particular, $\operatorname{cd}_S(I, J, N, M) \leq \operatorname{cd}(I, J, N, M)$ for every Serre subcategory S.*

The following Lemma is key for the next result.

Lemma 2.5. *Let I', J' be two ideals of R. Then the following statements are valid for all $i \in \mathbb{N}_0$.*
(i) $H^i_{\sqrt{I},J}(N, M) = H^i_{I,J}(N, M) = H^i_{I,\sqrt{J}}(N, M)$.
(ii) $H^i_{II',J}(N, M) = H^i_{I\cap I',J}(N, M)$.
(iii) $H^i_{I,JJ'}(N, M) = H^i_{I,J\cap J'}(N, M)$.
(iv) *If $J' \subseteq J$, then $H^i_{I+J',J}(N, M) = H^i_{I,J}(N, M)$. In particular, $H^i_{I+J,J}(N, M) = H^i_{I,J}(N, M)$.*

Proof. The parts (i), (ii) and (iii) follow by [19, Lemma 2.5] and [8, Lemma 2.1]. If $J' \subseteq J$, it is easy to see that $\tilde{\mathrm{W}}(I + J', J) = \tilde{\mathrm{W}}(I, J)$. Now, the part (iv) follows by [8, Lemma 2.1]. □

The following result collects some basic properties of $\operatorname{cd}_S(I, J, N, M)$, and it is a direct consequence of Lemma 2.5.

Corollary 2.6. *Let I', J' be two ideals of R.*

(i) $\mathrm{cd}_S(I, J, N, M) = \mathrm{cd}_S(\sqrt{I}, J, N, M) = \mathrm{cd}_S(I, \sqrt{J}, N, M)$.
(ii) $\mathrm{cd}_S(II', J, N, M) = \mathrm{cd}_S(I \cap I', J, N, M)$.
(iii) $\mathrm{cd}_S(I, JJ', N, M) = \mathrm{cd}_S(I, J \cap J', N, M)$.
(iv) *If $J' \subseteq J$, then $\mathrm{cd}_S(I + J', J, N, M) = \mathrm{cd}_S(I, J, N, M)$. In particular,*

$$\mathrm{cd}_S(I + J, J, N, M) = \mathrm{cd}_S(I, J, N, M).$$

In the following result, the behavior of $\mathrm{cd}_S(I, J, N, M)$ along exact sequences is studied.

Proposition 2.7. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. Then*

$$\mathrm{cd}_S(I, J, N, M) \leq \max\{\mathrm{cd}_S(I, J, N, M'), \mathrm{cd}_S(I, J, N, M'')\}.$$

Proof. The claim follows from the long exact sequence

$$\cdots \longrightarrow H^i_{I,J}(N, M') \longrightarrow H^i_{I,J}(N, M) \longrightarrow H^i_{I,J}(N, M'') \longrightarrow \cdots .$$

□

In the next two Lemmas, the relations on the generalized local cohomology modules and their supports from the upper bound, are investigated.

Lemma 2.8. *Let M be a finitely generated R-module, and t be an integer. If $H^i_{I,J}(N, R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$ and all $i > t$, then $H^i_{I,J}(N, M) \in S$ for all $i > t$.*

Proof. Let $i > t$ be an integer. There is a filtration $0 = M_k \subseteq M_{k-1} \subseteq \cdots \subseteq M_0 = M$ of submodules of M, such that for each j with $1 \leq j \leq k$, $M_{j-1}/M_j \cong R/\mathfrak{q}_j$ where $\mathfrak{q}_j \in \mathrm{Supp}(M)$. Now, for each j with $1 \leq j \leq k$, the exact sequence $0 \longrightarrow M_j \longrightarrow M_{j-1} \longrightarrow R/\mathfrak{q}_j \longrightarrow 0$ yields the exact sequence $H^i_{I,J}(N, M_j) \longrightarrow H^i_{I,J}(N, M_{j-1}) \longrightarrow H^i_{I,J}(N, R/\mathfrak{q}_j)$. It follows by these sequences that $H^i_{I,J}(N, M) \in S$. □

The following result is a direct consequence of Lemma 2.8.

Corollary 2.9. *Let M be a finitely generated R-module. Then*

$$\mathrm{cd}_S(I, J, N, M) \leq \sup\{\mathrm{cd}_S(I, J, N, R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Supp}(M)\}.$$

Lemma 2.10. *Let N be an R-module of finite projective dimension, M be a finitely generated R-module, and t be an integer. If $H^i_{I,J}(N, M) \in S$ for all $i > t$, then $H^i_{I,J}(N, R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$ and all $i > t$.*

Proof. We prove the claim by the descending induction on t. It follows by [18, Lemma 3.1] that, for any sufficiently large t, the claim is valid. So assume, inductively, that the result has been proved for $i > t+1$. It is enough to show that $H^{t+1}_{I,J}(N, R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$. We suppose that $\mathfrak{q}$ is maximal of those $\mathfrak{p} \in \mathrm{Supp}(M)$ such that $H^{t+1}_{I,J}(N, R/\mathfrak{p}) \notin S$, and look for a contradiction. It follows by [9, Chapter II, Section 4, Proposition 20] that there is a non-zero R-homomorphism $f : M \longrightarrow R/\mathfrak{q}$. Let $\mathfrak{b}$ be the ideal of R such that $\mathfrak{q} \subsetneq \mathfrak{b}$ and $\mathrm{Im}\, f = \mathfrak{b}/\mathfrak{q}$. Since $\mathrm{Supp}(\mathrm{Ker}\, f) \subseteq \mathrm{Supp}(M)$, it follows by the inductive hypothesis that $H^{i}_{I,J}(N, R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ker}\, f)$ and all $i > t+1$, and by Lemma 2.8, $H^{t+2}_{I,J}(N, \mathrm{Ker}\, f) \in S$. Now, the exact sequence $0 \longrightarrow \mathrm{Ker}\, f \longrightarrow M \longrightarrow \mathrm{Im}\, f \longrightarrow 0$ induces the exact sequence $H^{t+1}_{I,J}(N, M) \longrightarrow H^{t+1}_{I,J}(N, \mathrm{Im}\, f) \longrightarrow H^{t+2}_{I,J}(N, \mathrm{Ker}\, f)$, and it follows that $H^{t+1}_{I,J}(N, \mathrm{Im}\, f) \in S$. There is a filtration $0 = L_k \subseteq L_{k-1} \subseteq \cdots \subseteq L_0 = R/\mathfrak{b}$ of submodules of $R/\mathfrak{b}$, such that for each i with $1 \le i \le k$, $L_{i-1}/L_i \cong R/\mathfrak{q}_i$ where $\mathfrak{q}_i \in \mathrm{V}(\mathfrak{b})$. It follows by the maximality of $\mathfrak{q}$ that $H^{t+1}_{I,J}(N, R/\mathfrak{q}_i) \in S$ for each i with $1 \le i \le k$. Now, for each i with $1 \le i \le k$, the exact sequence $0 \longrightarrow L_i \longrightarrow L_{i-1} \longrightarrow R/\mathfrak{q}_i \longrightarrow 0$ yields the exact sequence $H^{t+1}_{I,J}(N, L_i) \longrightarrow H^{t+1}_{I,J}(N, L_{i-1}) \longrightarrow H^{t+1}_{I,J}(N, R/\mathfrak{q}_i)$. Hence $H^{t+1}_{I,J}(N, R/\mathfrak{b}) \in S$. Next the exact sequence $0 \longrightarrow \mathrm{Im}\, f \longrightarrow R/\mathfrak{q} \longrightarrow R/\mathfrak{b} \longrightarrow 0$ induces the exact sequence $H^{t+1}_{I,J}(N, \mathrm{Im}\, f) \longrightarrow H^{t+1}_{I,J}(N, R/\mathfrak{q}) \longrightarrow H^{t+1}_{I,J}(N, R/\mathfrak{b})$. It follows that $H^{t+1}_{I,J}(N, R/\mathfrak{q}) \in S$, which is a contradiction. □

The next Proposition provides some formulas which are useful in the computation of Serre cohomological dimensions.

Proposition 2.11. *Let N be an R-module of finite projective dimension, and M, M', and M'' be finitely generated R-modules.*

(i) *If $\mathrm{Supp}(M) \subseteq \mathrm{Supp}(M')$, then $\mathrm{cd}_S(I, J, N, M) \le \mathrm{cd}_S(I, J, N, M')$.*

(ii) *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence. Then*

$$\mathrm{cd}_S(I, J, N, M) = \max\{\mathrm{cd}_S(I, J, N, M'), \mathrm{cd}_S(I, J, N, M'')\}.$$

(iii)

$$\begin{aligned} \mathrm{cd}_S(I, J, N, M) &= \sup\{\mathrm{cd}_S(I, J, N, R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Supp}(M)\} \\ &= \sup\{\mathrm{cd}_S(I, J, N, R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Min}(M)\}. \end{aligned}$$

Proof. (i) Set $t = \mathrm{cd}_S(I, J, N, M')$. Then $H^{i}_{I,J}(N, M') \in S$ for all $i > t$, and it follows by Lemma 2.10 that $H^{i}_{I,J}(N, R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in \mathrm{Supp}(M')$ and all $i > t$. Therefore $H^{i}_{I,J}(N, R/\mathfrak{p}) \in S$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$ and all $i > t$. Now, it follows by Lemma 2.8 that $H^{i}_{I,J}(N, M) \in S$ for all $i > t$, and so $\mathrm{cd}_S(I, J, N, M) \le t$.

(ii) The inequality $\geq$ follows from (i), and the other inequality follows from Proposition 2.7.

(iii) The first equality follows by Corollary 2.9 and Lemma 2.10. For the second, we show that

$$\begin{aligned} & \sup\{\mathrm{cd}_S(I,J,N,R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Supp}(M)\} \\ = & \sup\{\mathrm{cd}_S(I,J,N,R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Min}(M)\}. \end{aligned}$$

Let $\mathfrak{p} \in \mathrm{Supp}(M)$. There is $\mathfrak{q} \in \mathrm{Min}(M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$, and it follows by part (i) that $\mathrm{cd}_S(I,J,N,R/\mathfrak{p}) \leq \mathrm{cd}_S(I,J,N,R/\mathfrak{q})$. Therefore

$$\begin{aligned} & \sup\{\mathrm{cd}_S(I,J,N,R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Supp}(M)\} \\ \leq & \sup\{\mathrm{cd}_S(I,J,N,R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Min}(M)\}. \end{aligned}$$

The inequality $\geq$ is trivial. □

The following result is a direct consequence of Proposition 2.11(i), and is used in the next results.

Corollary 2.12. *Let N be an R-module of finite projective dimension, and M be a finitely generated R-module. Then $\mathrm{cd}_S(I,J,N,M) = \mathrm{cd}_S(I,J,N,R/\mathrm{Ann}(M))$.*

Now, we characterize the membership of $H^i_{I,J}(N,R)$ in a Serre subcategory from above.

Lemma 2.13. *Let N be an R-module of finite projective dimension, and t be an integer. The following are equivalent.*

(i) *$H^i_{I,J}(N,R) \in S$ for all $i > t$.*
(ii) *$H^i_{I,J}(N,M) \in S$ for all $i > t$ and for every R-module M.*

Proof. The only non-trivial part is (i)⇒(ii). We prove the claim by the descending induction on t. It follows by [18, Lemma 3.1] that, for any sufficiently large t, the claim is valid. So assume, inductively, that the result has been proved for $i > t+1$, and for every R-module M. It is enough to show that $H^{t+1}_{I,J}(N,M) \in S$. There is an epimorphism $f : F \longrightarrow M$ where F is a free R-module. Set $K = \mathrm{Ker}\, f$. The exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ induces the exact sequence $H^{t+1}_{I,J}(N,F) \longrightarrow H^{t+1}_{I,J}(N,M) \longrightarrow H^{t+2}_{I,J}(N,K)$. Since $H^i_{I,J}(N,-)$ commutes with direct sums, it follows by the hypothesis that $H^{t+1}_{I,J}(N,F) \in S$. Also, by the inductive hypothesis, $H^{t+2}_{I,J}(N,K) \in S$, and the claim follows. □

The following result is obtained directly from Lemma 2.13.

Corollary 2.14. *Let N be an R-module of finite projective dimension. Then*

$$\begin{aligned}\operatorname{cd}_S(I,J,N,R) &= \sup\{\operatorname{cd}_S(I,J,N,M) : M \textit{ is an } R\textit{-module}\}\\ &= \sup\{i \in \mathbb{N}_0 : H^i_{I,J}(N,M) \notin S \textit{ for some } R\textit{-module } M\}.\end{aligned}$$

3. Cohomological dimensions on localizing subcategories

In this section, we study the cohomological dimension of two modules with respect to a pair of ideals on *localizing subcategories*. Recall that a Serre subcategory S is localizing, if it is closed under direct limits. It is shown in [16] that there is a bijection between the set of localizing subcategories of the category of R-modules, and the set of specialization closed subsets of $\operatorname{Spec}(R)$:

$$\{\text{Localizing subcategories}\} \underset{\text{Supp}^{-1}}{\overset{\text{Supp}}{\rightleftarrows}}$$
$$\{\text{Specialization closed subsets of } \operatorname{Spec}(R)\}$$

where for a localizing subcategory S, $\operatorname{Supp}(S) = \bigcup_{M\in S}\operatorname{Supp}(M)$, and for a specialization closed subset Φ of $\operatorname{Spec}(R)$, $\operatorname{Supp}^{-1}(\Phi)$ denotes the class of all R-modules M with $\operatorname{Supp}(M) \subseteq \Phi$. Therefore, a localizing subcategory is the same as the class of R-modules, which is mentioned in Example 2.2(e).

The following Lemma is key for the next results.

Lemma 3.1. *Let S be a localizing subcategory of the category of R-modules. Then*

$$\operatorname{cd}_S(I,J,N,M) \le$$
$$\sup\{\operatorname{cd}_S(I,J,N,L) : L \textit{ is a finitely generated submodule of } M\}.$$

Proof. The claim follows from these facts that the R-module M is equal to the direct limit of its finitely generated submodules, and the functor $H^i_{I,J}(N,-)$ commutes with direct limits. □

The next result improves Corollary 2.9.

Proposition 3.2. *Let S be a localizing subcategory of the category of R-modules.*

(i) $\operatorname{cd}_S(I,J,N,M) \le \sup\{\operatorname{cd}_S(I,J,N,R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Supp}(M)\}$.

(ii) *If N is an R-module of finite projective dimension, then*

$$\operatorname{cd}_S(I,J,N,M) \le \sup\{\operatorname{cd}_S(I,J,N,R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Min}(M)\}.$$

Proof. (i) It follows by Lemma 3.1 that there is a finitely generated submodule L of M such that $\mathrm{cd}_S(I, J, N, M) \leq \mathrm{cd}_S(I, J, N, L)$. Therefore we can assume that M is finitely generated. Now the claim follows by Corollary 2.9.

(ii) By a similar method to that of Proposition 2.11(iii), one can show that

$$\begin{aligned} & \sup\{\mathrm{cd}_S(I, J, N, R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Supp}(M)\} \\ = \; & \sup\{\mathrm{cd}_S(I, J, N, R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Min}(M)\}. \end{aligned}$$

Now the claim follows by part (i). □

The following Theorem is essential in this section.

Theorem 3.3. *Let S be a localizing subcategory of the category of R-modules. Let N be an R-module of finite projective dimension, and let M' be a finitely generated R-module such that* $\mathrm{Supp}(M) \subseteq \mathrm{Supp}(M')$. *Then* $\mathrm{cd}_S(I, J, N, M) \leq \mathrm{cd}_S(I, J, N, M')$.

Proof. It follows by Lemma 3.1 that there is a finitely generated submodule L of M with $\mathrm{cd}_S(I, J, N, M) \leq \mathrm{cd}_S(I, J, N, L)$. Now, we have $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(M')$, and it follows from Proposition 2.11(i) that $\mathrm{cd}_S(I, J, N, L) \leq \mathrm{cd}_S(I, J, N, M')$. □

The next result is a direct consequence of Theorem 3.3.

Corollary 3.4. *Let S be a localizing subcategory of the category of R-modules. Let N be an R-module of finite projective dimension, and $\mathfrak{b}$ be an ideal of R with $\mathfrak{b} \subseteq \mathrm{Ann}(M)$. Then* $\mathrm{cd}_S(I, J, N, M) \leq \mathrm{cd}_S(I, J, N, R/\mathfrak{b})$.

The following result improves Proposition 2.11(ii).

Corollary 3.5. *Let S be a localizing subcategory of the category of R-modules. Let N be an R-module of finite projective dimension, M be a finitely generated R-module, and let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. Then*

$$\mathrm{cd}_S(I, J, N, M) = \max\{\mathrm{cd}_S(I, J, N, M'), \mathrm{cd}_S(I, J, N, M'')\}.$$

Proof. It follows by Proposition 2.7, Theorem 3.3, and Corollary 3.4 that

$$\begin{aligned} \mathrm{cd}_S(I, J, N, M) &\leq \max\{\mathrm{cd}_S(I, J, N, M'), \mathrm{cd}_S(I, J, N, M'')\} \\ &\leq \max\{\mathrm{cd}_S(I, J, N, R/\mathrm{Ann}(M')), \mathrm{cd}_S(I, J, N, R/\mathrm{Ann}(M''))\} \\ &\leq \mathrm{cd}_S(I, J, N, R/\mathrm{Ann}(M)). \end{aligned}$$

Finally, the extreme terms are equal according to Corollary 2.12. □

The next result is a generalization of [4, Theorem C].

Proposition 3.6. *Let either S be a localizing subcategory of the category of R-modules or M be a finitely generated R-module. Let N be a finitely generated R-module of finite projective dimension, and M' be a finitely generated R-module such that* $\mathrm{Supp}(M/\Gamma_{I,J}(M)) \subseteq \mathrm{Supp}(M'/\Gamma_{I,J}(M'))$*. Then*

$$\mathrm{cd}(I, J, N, M) \leq \max\{\mathrm{cd}(I, J, N, M'), \mathrm{pd}(N)\}.$$

Proof. Let i be an integer such that $i > \max\{\mathrm{cd}(I, J, N, M'), \mathrm{pd}(N)\}$. It is enough to show that $H^i_{I,J}(N, M) = 0$. The exact sequence

$$0 \longrightarrow \Gamma_{I,J}(M) \longrightarrow M \longrightarrow M/\Gamma_{I,J}(M) \longrightarrow 0$$

induces the following long exact sequence

$$\cdots \to H^i_{I,J}(N, \Gamma_{I,J}(M)) \to H^i_{I,J}(N, M) \to H^i_{I,J}(N, M/\Gamma_{I,J}(M)) \to \cdots .$$

Since $\mathrm{Supp}(M/\Gamma_{I,J}(M)) \subseteq \mathrm{Supp}(M'/\Gamma_{I,J}(M')) \subseteq \mathrm{Supp}(M')$, it follows by either Proposition 2.11(i) or Theorem 3.3 that

$$\mathrm{cd}_S(I, J, N, M/\Gamma_{I,J}(M)) \leq \mathrm{cd}_S(I, J, N, M') < i,$$

and therefore $H^i_{I,J}(N, M/\Gamma_{I,J}(M)) = 0$. Also, by [21, Proposition 2.6], $H^i_{I,J}(N, \Gamma_{I,J}(M)) \cong \mathrm{Ext}^i_R(N, \Gamma_{I,J}(M))$, and so $H^i_{I,J}(N, \Gamma_{I,J}(M)) = 0$. Now, it follows by the above long exact sequence that $H^i_{I,J}(N, M) = 0$. □

4. Cohomological dimensions with respect to sum and intersection of ideals

In this section, we study the cohomological dimension of N, M with respect to $(I + I', J)$ and $(I, J \cap J')$, where I', J' are two ideals of R. The following Theorem is essential in this direction.

Theorem 4.1. *Let I', J' be two ideals of R, N be a finitely generated R-module, and let t be a non-negative integer.*

(i) *If $H^{t-i}_{I,J}(N, H^i_{I',J}(M)) \in S$ for all $0 \leq i \leq t$, then $H^t_{I+I',J}(N, M) \in S$.*
(ii) *If $H^{t-i}_{I,J}(N, H^i_{I,J'}(M)) \in S$ for all $0 \leq i \leq t$, then $H^t_{I,J\cap J'}(N, M) \in S$.*

Proof. (i) Let $F(-) = \Gamma_{I,J}(N, -)$ and $G(-) = \Gamma_{I',J}(-)$ be functors from the category of R-modules to itself. It follows by [21, Proposition 2.2] and [23, Proposition 1.4] that

$$\begin{aligned} FG(M) &= \Gamma_{I,J}(N, \Gamma_{I',J}(M)) = \Gamma_{I,J}(\mathrm{Hom}_R(N, \Gamma_{I',J}(M))) \\ &= \Gamma_{I,J}(\Gamma_{I',J}(\mathrm{Hom}_R(N, M))) = \Gamma_{I+I',J}(\mathrm{Hom}_R(N, M)) \\ &= \Gamma_{I+I',J}(N, M). \end{aligned}$$

Let E be an injective R-module. It follows by [5, Lemma 4.4] that $G(E) = \Gamma_{I',J}(E)$ is an injective R-module. So $H^i_{I,J}(N, \Gamma_{I',J}(N, E)) = 0$ for all $i > 0$. Now, by [22, Theorem 10.47], there is the Grothendieck third quadrant spectral sequence

$$E_2^{p,q} := H^p_{I,J}(N, H^q_{I',J}(M)) \underset{p}{\Longrightarrow} H^{p+q}_{I+I',J}(N, M).$$

Hence there is a bounded filtration

$$0 = \phi^{p+q+1}H^{p+q} \subseteq \phi^{p+q}H^{p+q} \subseteq \cdots \subseteq \phi^0 H^{p+q} = H^{p+q}_{I+I',J}(N, M)$$

such that $E_\infty^{p+q-i,i} \cong \phi^{p+q-i}H^{p+q}/\phi^{p+q+1-i}H^{p+q}$ for all $0 \leq i \leq p+q$. Also we have $E^{p,q}_{p+q+2} = E^{p,q}_{p+q+3} = \cdots = E^{p,q}_\infty$ for all $p \geq 0$ and $q \geq 0$.

We have to show that $H^t_{I+I',J}(N, M) = \phi^0 H^t \in S$. Since $E_j^{t-i,i}$ is a subquotient of $E_2^{t-i,i}$ for all $j \geq 2$ and $0 \leq i \leq t$, it follows by the hypothesis that $E_j^{t-i,i} \in S$ for all $j \geq 2$ and $0 \leq i \leq t$, and therefore $E_\infty^{t-i,i} \in S$ for all $0 \leq i \leq t$. Now the exact sequences $0 \longrightarrow \phi^{t-i+1}H^t \longrightarrow \phi^{t-i}H^t \longrightarrow E_\infty^{t-i,i} \longrightarrow 0$ for all $0 \leq i \leq t$, imply that $\phi^0 H^t \in S$.

(ii) Let $F(-) = \Gamma_{I,J}(N, -)$ and $G(-) = \Gamma_{I,J'}(-)$ be functors from the category of R-modules to itself. The proof is quite similar to the proof of part (i). □

Proposition 4.2. *Let I', J' be two ideals of R, and N be a finitely generated R-module of finite projective dimension.*

(i) $\operatorname{cd}_S(I + I', J, N, M) \leq \operatorname{cd}_S(I, J, N, R) + \operatorname{cd}_S(I', J, M)$.
(ii) $\operatorname{cd}_S(I, J \cap J', N, M) \leq \operatorname{cd}_S(I, J, N, R) + \operatorname{cd}_S(I, J', M)$.

Proof. (i) Let t be an integer such that $t > \operatorname{cd}_S(I, J, N, R) + \operatorname{cd}_S(I', J, M)$. In view of Theorem 4.1(i), it is enough to show that $H^{t-i}_{I,J}(N, H^i_{I',J}(M)) \in S$ for all $0 \leq i \leq t$. If $i > \operatorname{cd}_S(I', J, M)$, then $H^i_{I',J}(M) \in S$ and so $H^{t-i}_{I,J}(N, H^i_{I',J}(M)) \in S$. Otherwise, $i \leq \operatorname{cd}_S(I', J, M)$ and so $t - i > \operatorname{cd}_S(I, J, N, R)$. By Corollary 2.14, we have

$$\operatorname{cd}_S(I, J, N, H^i_{I,J}(M)) \leq \operatorname{cd}_S(I, J, N, R) < t - i,$$

and so $H^{t-i}_{I,J}(N, H^i_{I',J}(M)) \in S$ as desired.

(ii) The proof is similar to that of (i). □

Proposition 4.3. *Let S be a localizing subcategory of the category of R-modules. Let I', J' be two ideals of R, and N be a finitely generated R-module of finite projective dimension.*

(i) $\operatorname{cd}_S(I + I', J, N, M) \leq \operatorname{cd}_S(I, J, N, R/\operatorname{Ann}(M)) + \operatorname{cd}_S(I', J, M)$.
(ii) $\operatorname{cd}_S(I, J \cap J', N, M) \leq \operatorname{cd}_S(I, J, N, R/\operatorname{Ann}(M)) + \operatorname{cd}_S(I, J', M)$.

Proof. (i) It follows by Theorem 4.1(i) that

$$\mathrm{cd}_S(I+I',J,N,M) \le \max\{i+j : H^i_{I,J}(N, H^j_{I',J}(M)) \notin S\}.$$

By [21, Proposition 2.6] and [6, Lemma 2.1], we have

$$\begin{aligned} &\max\{i+j : H^i_{I,J}(N, H^j_{I',J}(M)) \notin S\} \\ \le\ & \mathrm{cd}_S(I,J,N,H^j_{I',J}(M)) + \mathrm{cd}_S(I',J,M). \end{aligned}$$

It follows by Corollary 3.4 that

$$\mathrm{cd}_S(I,J,N,H^j_{I',J}(M)) \le \mathrm{cd}_S(I,J,N,R/\mathrm{Ann}(H^j_{I',J}(M))).$$

Moreover, since the functor $H^i_{I',J}(-)$ is R-linear, therefore $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(H^j_{I',J}(M))$. So, by Proposition 2.11(i),

$$\mathrm{cd}_S(I,J,N,R/\mathrm{Ann}(H^j_{I',J}(M))) \le \mathrm{cd}_S(I,J,N,R/\mathrm{Ann}(M)),$$

and the claim follows.

(ii) The proof is similar to that of (i). □

The following result is a generalization of [25, Theorem 2.1 and Corollary 2.1].

Corollary 4.4. *Let S be a localizing subcategory of the category of R-modules. Let I', J' be two ideals of R, M be a finitely generated R-module, and let N be a finitely generated R-module of finite projective dimension.*

(i) $\mathrm{cd}_S(I+I',J,N,M) \le \mathrm{cd}_S(I,J,N,M) + \mathrm{cd}_S(I',J,M)$.
(ii) $\mathrm{cd}_S(I,J\cap J',N,M) \le \mathrm{cd}_S(I,J,N,M) + \mathrm{cd}_S(I,J',M)$.

Proof. The claim follows by Corollary 2.12 and Proposition 4.3. □

The next result is obtained directly from Corollary 4.4 by assuming $N = R$.

Corollary 4.5. *Let S be a localizing subcategory of the category of R-modules. Let I', J' be two ideals of R, and M be a finitely generated R-module.*

(i) $\mathrm{cd}_S(I+I',J,M) \le \mathrm{cd}_S(I,J,M) + \mathrm{cd}_S(I',J,M)$.
(ii) $\mathrm{cd}_S(I,J\cap J',M) \le \mathrm{cd}_S(I,J,M) + \mathrm{cd}_S(I,J',M)$.

The following result is a generalization of [15, Corollary 2.2(iii)].

Corollary 4.6. *Let S be a localizing subcategory of the category of R-modules. Let M be a finitely generated R-module, and N be a finitely generated R-module of finite projective dimension. If $\mathfrak{q} \in \mathrm{Min}(J)$, then*

$$\mathrm{cd}_S(I,J,N,M) \le \mathrm{cd}_S(I,\mathfrak{q},N,M) + \sum_{\mathfrak{p}\in \mathrm{Min}(J)-\{\mathfrak{q}\}} \mathrm{cd}_S(I,\mathfrak{p},M).$$

Proof. It follows by Corollary 2.6(i), Corollary 4.4(ii), and Corollary 4.5(ii) that

$$\begin{aligned} \operatorname{cd}_S(I,J,N,M) &= \operatorname{cd}_S(I,\sqrt{J},N,M) = \operatorname{cd}_S(I,\bigcap_{\mathfrak{p}\in\operatorname{Min}(J)}\mathfrak{p},N,M) \\ &\leq \operatorname{cd}_S(I,\mathfrak{q},N,M) + \operatorname{cd}_S(I,\bigcap_{\mathfrak{p}\in\operatorname{Min}(J)-\{\mathfrak{q}\}}\mathfrak{p},M) \\ &\leq \operatorname{cd}_S(I,\mathfrak{q},N,M) + \sum_{\mathfrak{p}\in\operatorname{Min}(J)-\{\mathfrak{q}\}}\operatorname{cd}_S(I,\mathfrak{p},M). \end{aligned}$$

□

5. Some relations between different cohomological dimensions

In this section, we investigate the relations between different types of cohomological dimensions. In the first, the relations between the cohomological dimension of two modules with respect to a pair of ideals, and the cohomological dimension of a module with respect to a pair of ideals, are studied. The following two Lemmas are key for the next Theorem.

Lemma 5.1. *Let S be a localizing subcategory of the category of R-modules, and let N be a projective R-module. Then $\operatorname{cd}_S(I,J,N,M) = \operatorname{cd}_S(I,J,M)$.*

Proof. It follows by [20, Theorem 2.5] that any projective R-module is locally free, i.e., its localization at every prime ideal is free over the corresponding localization of the ring. Let $\mathfrak{p}$ be a prime ideal of R, and Λ be a set such that $N_{\mathfrak{p}} \cong \oplus_{\lambda\in\Lambda}R_{\mathfrak{p}}$. Now, it follows by [8, Lemma 2.1] that

$$\begin{aligned} (H^i_{I,J}(N,M))_{\mathfrak{p}} &\cong (\varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}}(N,M))_{\mathfrak{p}} \cong \varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}},M_{\mathfrak{p}}) \\ &\cong \varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}R_{\mathfrak{p}}}(\bigoplus_{\lambda\in\Lambda}R_{\mathfrak{p}},M_{\mathfrak{p}}) \cong \prod_{\lambda\in\Lambda}\varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}},M_{\mathfrak{p}}) \\ &\cong \prod_{\lambda\in\Lambda}\varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}},M_{\mathfrak{p}}) \cong \prod_{\lambda\in\Lambda}\varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \\ &\cong \prod_{\lambda\in\Lambda}(\varinjlim_{\mathfrak{a}\in\tilde{W}(I,J)} H^i_{\mathfrak{a}}(M))_{\mathfrak{p}} \cong \prod_{\lambda\in\Lambda}(H^i_{I,J}(M))_{\mathfrak{p}}. \end{aligned}$$

Therefore $\operatorname{Supp}(H^i_{I,J}(N,M)) = \operatorname{Supp}(H^i_{I,J}(M))$ for any non-negative integer i, and the claim follows. □

Lemma 5.2. *Let R be a local ring, and let N be a finitely generated and projective R-module. Then $\mathrm{cd}_S(I,J,N,M) = \mathrm{cd}_S(I,J,M)$.*

Proof. It follows by [20, Theorem 2.5] that N is a finitely generated free R-module. Let k be a positive integer such that $N \cong \bigoplus_{j=1}^{k} R$. Then

$$H^i_{I,J}(N,M) \cong \varinjlim_{\mathfrak{a}\in\tilde{\mathrm{W}}(I,J)} H^i_{\mathfrak{a}}(N,M) \cong \varinjlim_{\mathfrak{a}\in\tilde{\mathrm{W}}(I,J)} (\bigoplus_{j=1}^{k} H^i_{\mathfrak{a}}(M)) \cong \bigoplus_{j=1}^{k} H^i_{I,J}(M).$$

It follows that $\mathrm{cd}_S(I,J,N,M) = \mathrm{cd}_S(I,J,M)$. □

The following Theorem is the main result of this section.

Theorem 5.3. *Let either S be a localizing subcategory of the category of R-modules or R be a local ring. Let N be a finitely generated R-module of finite projective dimension. Then $\mathrm{cd}_S(I,J,N,M) \leq \mathrm{pd}(N) + \mathrm{cd}_S(I,J,M)$.*

Proof. We use induction on $\mathrm{pd}(N)$. If $\mathrm{pd}(N) = 0$, then the claim follows by Lemma 5.1 and Lemma 5.2. Now assume inductively that $\mathrm{pd}(N) > 0$, and the result has been proved for all R-modules with projective dimensions less than $\mathrm{pd}(N)$. Let $t > \mathrm{pd}(N) + \mathrm{cd}_S(I,J,M)$ be fixed. We have to show that $H^t_{I,J}(N,M) \in S$. There is the exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$, whenever F is a finitely generated free R-module, and K is a finitely generated R-module with $\mathrm{pd}(K) = \mathrm{pd}(N) - 1$. Therefore, there exists the exact sequence

$$H^{t-1}_{I,J}(K,M) \longrightarrow H^t_{I,J}(N,M) \longrightarrow H^t_{I,J}(F,M).$$

It follows by the inductive hypothesis that $\mathrm{cd}_S(I,J,K,M) \leq \mathrm{pd}(K) + \mathrm{cd}_S(I,J,M)$, and so $H^i_{I,J}(K,M) \in S$ for all $i > \mathrm{pd}(K) + \mathrm{cd}_S(I,J,M)$. Hence $H^{t-1}_{I,J}(K,M) \in S$. Let k be a positive integer such that $F = \bigoplus_{j=1}^{k} R$. Then $H^t_{I,J}(F,M) \cong \bigoplus_{j=1}^{k} H^t_{I,J}(M) \in S$. Now, the claim follows by the above exact sequence. □

In the next result, we get some upper bounds for cohomological dimensions.

Proposition 5.4. *Let N be a finitely generated R-module of finite projective dimension.*

(i) $\mathrm{cd}(I,J,N,M) \leq \mathrm{pd}(N) + \mathrm{ara}(I\bar{R})$, *where* $\bar{R} = R/\sqrt{J + \mathrm{Ann}(M)}$.
(ii) $\mathrm{cd}(I,J,N,M) \leq \mathrm{pd}(N) + \mathrm{ara}(I)$.
(iii) $\mathrm{cd}(I,J,N,M) \leq \mathrm{pd}(N) + \dim(M)$.
(iv) *If M is a finitely generated R-module, then*

$$\mathrm{cd}(I,J,N,M) \leq \mathrm{pd}(N) + \dim(M/JM) + 1.$$

(v) *If M is a finitely generated R-module of finite Krull dimension, then*

$$\operatorname{cd}(I, J, N, M) \leq \operatorname{pd}(N) + \dim(M \otimes_R N).$$

Proof. (i) By [23, Proposition 4.11], we have $\operatorname{cd}(I, J, M) \leq \operatorname{ara}(I\bar{R})$. Now, the claim follows by Theorem 5.3 by assuming $S = \{0\}$.

(ii) It is easy to show that $\operatorname{ara}(I\bar{R}) \leq \operatorname{ara}(I)$. Now, the claim follows by part (i).

(iii) By [7, Part 2.7], we have $\operatorname{cd}(I, J, M) \leq \dim(M)$. Now, the claim follows by Theorem 5.3 by assuming $S = \{0\}$.

(iv) By [23, Theorem 4.7](2), we have $\operatorname{cd}(I, J, M) \leq \dim(M/JM)+1$. Now, the claim follows by Theorem 5.3 by assuming $S = \{0\}$.

(v) The claim follows by [26, Theorem 3.7]. □

The following result is a generalization of [15, Corollary 2.4].

Corollary 5.5. *Let I', J' be two ideals of R, and let N be a finitely generated R-module of finite projective dimension. Set $\bar{R} = R/\sqrt{J + \operatorname{Ann}(M)}$.*

(i) $\operatorname{cd}(I + I', J, N, M) \leq \operatorname{pd}(N) + \operatorname{ara}(I\bar{R}) + \operatorname{cd}(I', J, M)$.

(ii) $\operatorname{cd}(I, J \cap J', N, M) \leq \operatorname{pd}(N) + \operatorname{ara}(I\bar{R}) + \operatorname{cd}(I, J', M)$.

Proof. The claim follows by Proposition 4.3 and Proposition 5.4(i). □

Corollary 5.6. *Let J' be an ideal of R, and N be a finitely generated R-module of finite projective dimension. Then*

$$\operatorname{cd}(I, J\cap J', N, M) \leq \operatorname{pd}(N)+\dim(R/(J + \operatorname{Ann}(M)))+1+\operatorname{cd}(I, J', M).$$

Proof. The claim follows by Proposition 4.3(ii) and Proposition 5.4(iv). □

Proposition 5.7. *Let R be a local ring, and let N be a finitely generated R-module of finite projective dimension.*

(i) *If M is a finitely generated R-module, and $J \neq R$, then*

$$\operatorname{cd}(I, J, N, M) \leq \operatorname{pd}(N) + \dim(M/JM).$$

(ii) $\operatorname{cd}(I, J, N, M) \leq \operatorname{pd}(N) + \dim(R/J)$.

Proof. (i) By [23, Theorem 4.3], we have $\operatorname{cd}(I, J, M) \leq \dim(M/JM)$. Now, the claim follows by Theorem 5.3 by assuming $S = \{0\}$.

(ii) By [23, Corollary 4.4], we have $\operatorname{cd}(I, J, M) \leq \dim(R/J)$. Now, the claim follows by Theorem 5.3 by assuming $S = \{0\}$. □

Corollary 5.8. *Let R be a local ring, J' be an ideal of R, and let N be a finitely generated R-module of finite projective dimension. If $J \neq R$, then*

$$\operatorname{cd}(I, J \cap J', N, M) \leq \operatorname{pd}(N) + \dim(R/(J + \operatorname{Ann}(M))) + \operatorname{cd}(I, J', M).$$

Proof. The claim follows by Proposition 4.3(ii) and Proposition 5.7(i). □

Finally, we get some formulas on the connections between the cohomological dimension with respect to a pair of ideals and the cohomological dimension with respect to an ideal. The following Lemma is key for the next result.

Lemma 5.9. *For each non-negative integer i,*

$$H^i_{I,J}(N, M) \cong \varinjlim_{\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)} H^i_{\mathfrak{a}}(N/\mathfrak{a}N, M).$$

Proof. Note that, $H^i_{\mathfrak{a}}(N/\mathfrak{a}N, M) \cong \mathrm{Ext}^i_R(N/\mathfrak{a}N, M)$ for any $i \in \mathbb{N}_0$ and $\mathfrak{a} \in \tilde{\mathrm{W}}(I, J)$. □

Proposition 5.10. *Let S be a localizing subcategory of the category of R-modules.*

(i) $\mathrm{cd}_S(I, J, N, M) \leq \sup\{\mathrm{cd}_S(\mathfrak{a}, N, M) : \mathfrak{a} \in \tilde{\mathrm{W}}(I, J)\}$.
(ii) $\mathrm{cd}_S(I, J, N, M) \leq \sup\{\mathrm{cd}_S(\mathfrak{a}, N/\mathfrak{a}N, M) : \mathfrak{a} \in \tilde{\mathrm{W}}(I, J)\}$.
(iii) *If N is a projective R-module, then*

$$\mathrm{cd}_S(I, J, N, M) \leq \sup\{\mathrm{cd}_S(\mathfrak{a}, R/\mathfrak{a}, \mathrm{Hom}_R(N, M)) : \mathfrak{a} \in \tilde{\mathrm{W}}(I, J)\}.$$

(iv) $\mathrm{cd}_S(I, J, M) \leq \sup\{\mathrm{cd}_S(\mathfrak{a}, R/\mathfrak{a}, M) : \mathfrak{a} \in \tilde{\mathrm{W}}(I, J)\}$.

Proof. (i) The claim follows by [8, Lemma 2.1].
(ii) The claim follows by Lemma 5.9.
(iii) By [22, Corollary 10.65], we have

$$\begin{aligned} H^i_{\mathfrak{a}}(N/\mathfrak{a}N, M) &\cong \mathrm{Ext}^i_R(N/\mathfrak{a}N, M) \cong \mathrm{Ext}^i_R(R/\mathfrak{a} \otimes_R N, M) \\ &\cong \mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{Hom}_R(N, M)) \cong H^i_{\mathfrak{a}}(R/\mathfrak{a}, \mathrm{Hom}_R(N, M)). \end{aligned}$$

Now the claim follows by part (ii).
(iv) In part (iii), put $N = R$. □

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