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ON COHOMOLOGY FOR MODULE BUNDLES OVER ASSOCIATIVE ALGEBRA BUNDLES

H. M. PRASAD, R. RAJENDRA* AND B. S. KIRANAGI

Abstract. We define cohomology of module bundles over associative algebra bundles. We establish a one to one correspondence between the first cohomology classes and the extensions of module bundles. Using this correspondence we give sufficient condition in terms of cohomology for an algebra bundle to be semisimple.

1. Introduction

David S. Trushin in his paper [10], introduced and studied cohomology for modules over associative algebra bundles. We follow his techniques in this paper to develop the theory of cohomology for module bundles over associative algebra bundles.

The notions of algebra bundles and module bundles are defined and studied in [3, 4, 5]. Some recent developments in the theory of algebra bundles can be found in [1, 6, 7]. We shall recall few necessary definitions:

A vector bundle $\xi = (\xi, p, X)$ is called a weak algebra bundle if there is a morphism $\theta : \xi \oplus \xi \to \xi$ which induces an algebra structure on each fiber $\xi_x, x \in X$ [3].

An algebra bundle is a vector bundle $\xi = (\xi, p, X)$ in which each fibre ξ_x is an algebra and for each x in X, there is an open neighbourhood U of x, an algebra A and a homeomorphism $\phi : U \times A \to p^{-1}(U)$ such that for each y in U, $\phi_y : A \to p^{-1}(y)$ is an algebra isomorphism [4].

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*Corresponding author .

A vector bundle $\mathscr{M} = (\mathscr{M}, q, X)$ is a left ξ-module bundle or simply a ξ-module bundle if there exists a morphism $\theta : \xi \oplus \mathscr{M} \to \mathscr{M}$ which induces a right ξ_x-module structure on $\mathscr{M}_x$ for each $x \in X$ [3].

A vector bundle $\mathscr{M} = (\mathscr{M}, q, X)$ is a ξ-bimodule bundle if there exist morphisms $\theta_1 : \mathscr{M} \oplus \xi \to \mathscr{M}$ and $\theta_2 : \xi \oplus \mathscr{M} \to \mathscr{M}$ which induce ξ_x-bimodule structure on $\mathscr{M}_x$ for each $x \in X$ [3].

A vector bundle morphism $\phi : \mathscr{M} \to \mathscr{N}$ of a ξ-bimodule bundle $\mathscr{M}$ into a ξ-bimodule bundle $\mathscr{N}$ is called a ξ-bimodule bundle morphism or simply ξ-morphism if for each $x \in X, \phi_x : \mathscr{M}_x \to \mathscr{N}_x$ is a ξ_x-bimodule homomorphism. If ϕ is a homeomorphism then it is called ξ-isomorphism.

All underlying vector spaces are real and finite dimensional and all algebras considered in the paper are finite dimensional associative algebras. All module bundles, algebra bundles and submodule bundles have same base space X which is compact Hausdorff. Throughout this paper $\xi = (\xi, p, X)$ denotes an associative algebra bundle. By a module bundle we mean a ξ- module bundle unless otherwise specified. By an algebra bundle we mean associative algebra bundle.

2. Cohomology of Module Bundles over associative algebra bundles

Let $\mathscr{M}, \mathscr{N}$ be ξ-module bundles. Let $\rho_1 : \xi \oplus \mathscr{M} \to \mathscr{M}$ and $\rho_2 : \xi \oplus \mathscr{N} \to \mathscr{N}$ be the morphisms which induce module structure on fibers of $\mathscr{M}$ and $\mathscr{N}$, respectively. We use the notation $S^n(\mathscr{N}, \mathscr{M})$ to denote the finitely generated projective $C(X)$-module $Hom(\xi^{\otimes n} \otimes \mathscr{N}, \mathscr{M})$ of all multilinear morphisms of $\xi^{\otimes n} \otimes \mathscr{N}$ into $\mathscr{M}$, where $C(X)$ is the ring of all continuous real valued functions on X.

Define $\delta^n : S^n(\mathscr{N}, \mathscr{M}) \to S^{n+1}(\mathscr{N}, \mathscr{M})$ as follows:

Given $f \in S^n(\mathscr{N}, \mathscr{M})$, we define

$$g : \xi^{\otimes(n+1)} \otimes \mathscr{N} \to \mathscr{M}$$

by

$$\begin{aligned} g(a_1 \otimes \cdots \otimes a_{n+1} \otimes q) = & \rho_1\big(a_1, f(a_2 \otimes \cdots \otimes a_{n+1} \otimes n)\big) \\ & + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ & + (-1)^{n+1} f\big(a_1 \otimes \cdots \otimes a_n \otimes \rho_2(a_{n+1}, q)\big). \end{aligned}$$

Then g is continuous being a combination of continuous maps ρ_1, f and ρ_2. Also, g is multilinear. Hence $g \in S^{n+1}(\mathscr{N}, \mathscr{M})$. We define

$$\delta^n(f) := g.$$

It is clear that $f \mapsto \delta^n(f)$ is $C(X)$-linear. The $C(X)$-module $S^n(\mathscr{N}, \mathscr{M})$ is called the space of n-cochains of $\mathscr{N}$ into $\mathscr{M}$ with coefficients in ξ. The maps $\{\delta^n\}_{n\geq 0}$ now can be considered as a coboundary operator on the collection of cochain spaces $\{S^n(\mathscr{N}, \mathscr{M})\}$. By necessary computations we can show that $\delta^{n+1} \circ \delta^n = 0$.

Definition 2.1. Let $\mathscr{N}$ and $\mathscr{M}$ be ξ-module bundles. Let $Z^n(\mathscr{N}, \mathscr{M}) = \ker \delta^n$ and $B^n(\mathscr{N}, \mathscr{M}) = Im\ \delta^{n-1}$. Since $\delta^n \circ \delta^{n-1} = 0$, $B^n(\mathscr{N}, \mathscr{M}) \subseteq Z^n(\mathscr{N}, \mathscr{M})$. The quotient $H^n(\mathscr{N}, \mathscr{M}) = Z^n(\mathscr{N}, \mathscr{M})/B^n(\mathscr{N}, \mathscr{M})$ is called the n^{th} cohomology module of $\mathscr{N}$ into $\mathscr{M}$ over $C(X)$ with coefficients in ξ. The elements of $Z^n(\mathscr{N}, \mathscr{M}), B^n(\mathscr{N}, \mathscr{M})$ and $H^n(\mathscr{N}, \mathscr{M})$ are called n-cocycles, n-coboundaries and n-cohomology classes respectively. For $f \in Z^1(\mathscr{N}, \mathscr{M})$, the corresponding cohomology class in $H^1(\mathscr{N}, \mathscr{M})$ is denoted by $\overline{f}$.

Remark 2.2. Let $\mathscr{N}, \mathscr{M}_1$ and $\mathscr{M}_2$ be ξ-module bundles. If the map $\phi : \mathscr{M}_1 \to \mathscr{M}_2$ is a ξ-morphism, then we can define

$$\phi^{(n)} : S^n(\mathscr{N}, \mathscr{M}_1) \to S^n(\mathscr{N}, \mathscr{M}_2)$$

by

$$\phi^{(n)}(f)(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes q) = \phi f(a_1 \otimes a_2 \otimes \cdots a_n \otimes q).$$

Proposition 2.3. *Let $\mathscr{N}$, $\mathscr{M}_1$ and $\mathscr{M}_2$ be ξ-module bundles. If $\phi : \mathscr{M}_1 \to \mathscr{M}_2$ is a ξ-morphism, then $\phi^{(n)}$ maps cocycles to cocycles and coboundaries to coboundaries.*

Proof. Let $\rho_{11} : \xi \oplus \mathscr{M}_1 \to \mathscr{M}_1$, $\rho_{12} : \xi \oplus \mathscr{M}_2 \to \mathscr{M}_2$ and $\rho_2 : \xi \oplus \mathscr{N} \to \mathscr{N}$ be vector bundle morphisms which induce ξ-module structures on $\mathscr{M}_1$, $\mathscr{M}_2$ and $\mathscr{N}$, respectively. Consider the following diagram :

$$\begin{array}{ccc}
S^n(\mathscr{N}, \mathscr{M}_1) & \xrightarrow{\phi^{(n)}} & S^{n+1}(\mathscr{N}, \mathscr{M}_2) \\
\Big\downarrow {\scriptstyle \delta_1{}^n} & & \Big\downarrow {\scriptstyle \delta_2^n} \\
S^{n+1}(\mathscr{N}, \mathscr{M}_2) & \xrightarrow{\phi^{(n+1)}} & S^n(\mathscr{N}, \mathscr{M}_2)
\end{array}$$

First we show that this diagram commutes i.e., $\delta_2^n \phi^{(n)} = \phi^{(n+1)} \delta_1^n$. Let $f \in S^n(\mathscr{N}, \mathscr{M}_1)$ and $a_1 \otimes a_2 \otimes \cdots a_n \otimes q \in \xi^{\otimes n} \otimes \mathscr{N}$. Then

$$\begin{aligned}
&\delta_2^n \phi^{(n)}(f)(a_1 \otimes \cdots \otimes a_{n+1} \otimes q) \\
&\qquad = \delta_2^n \phi f(a_1 \otimes \cdots \otimes a_{n+1} \otimes q) \\
&\qquad = \rho_{12}(a_1, \phi f(a_2 \otimes \cdots a_{n+1} \otimes q))
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{n}(-1)^i \phi f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\
&\quad + (-1)^{n+1}\phi f\big(a_1 \otimes \cdots a_n \otimes \rho_2(a_{n+1}, q)\big) \\
&= \phi\rho_{11}(a_1, f\big(a_2 \otimes \cdots \otimes a_n \otimes q)\big) \\
&\quad + \phi\Big(\sum_{i=1}^{n}(-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})\Big) \\
&\quad + \phi\big((-1)^{n+1} f\big(a_1 \otimes \cdots a_n \otimes \rho_2(a_{n+1}, q)\big) \\
&= \phi\delta_1^n(f)(a_1 \otimes a_2 \otimes \cdots a_{n+1} \otimes q) \\
&= \phi^{(n+1)}\delta_1^n(f). \qquad (2.1)
\end{aligned}
$$

Let $f \in Z^n(\mathscr{N}, \mathscr{M}_1)$. Then $\delta_1^n(f) = 0$ so that $\phi^{(n+1)}\delta_1^n(f) = 0$. By (2.1), $\delta_2^n\phi^{(n)}(f) = 0$ which implies $\phi^{(n)}(f) \in Z^n(\mathscr{N}, \mathscr{M}_2)$. Hence $\phi^{(n)}$ maps cocycles to cocycles.

Suppose $f \in B^n(\mathscr{N}, \mathscr{M}_1)$ then $f = \delta_1^{n-1}(g)$ for some $g \in S^{n-1}(\mathscr{N}, \mathscr{M})$. But then $\phi^{(n)}(f) = \phi^{(n)}\delta_1^{n-1}(g) = \delta_2^{(n-1)}\phi^{(n-1)}(g)$ so that $\phi^{(n)}(f) \in B^n(\mathscr{N}, \mathscr{M}_2)$. Hence coboundaries are mapped to coboundaries under $\phi^{(n)}$. □

Definition 2.4. Let $\mathscr{N}$ be any ξ-module bundle. If there is an exact sequence

$$0 \to \mathscr{M} \to \mathscr{K} \to \mathscr{N} \to 0$$

of ξ-module bundles, then we say that $\mathscr{K}$ is an extension of $\mathscr{N}$ by $\mathscr{M}$. If $\mathscr{K} \cong \mathscr{M} \oplus \mathscr{N}$, then we say that $\mathscr{K}$ is a split extension of $\mathscr{N}$ by $\mathscr{M}$.

We establish a correspondence between extensions and first cohomology module in next two theorems.

Theorem 2.5. *Let $\mathscr{N}$ and $\mathscr{M}$ be ξ-module bundles. For every element in $H^1(\mathscr{N}, \mathscr{M})$ there exists a unique (up to isomorphism) extension of $\mathscr{N}$ by $\mathscr{M}$.*

Proof. Let $\rho_1 : \xi \oplus \mathscr{M} \to \mathscr{M}$ and $\rho_2 : \xi \oplus \mathscr{N} \to \mathscr{N}$ be the morphisms which induce module structure on the fibers of $\mathscr{M}$ and $\mathscr{N}$, respectively. Consider any $f \in Z^1(\mathscr{N}, \mathscr{M})$. Let $\mathscr{K}_f = \mathscr{N} \oplus \mathscr{M}$ as vector bundles. Define

$$\rho : \xi \oplus \mathscr{K}_f \to \mathscr{K}_f$$

by

$$\rho(a, (q, m)) = (\rho_2(a, q), \rho_1(a, m) + f(a \otimes q)) \ \forall a \in \xi, q \in \mathscr{N}, m \in \mathscr{M}.$$

Since ρ_2, ρ_1 and f are continuous, ρ is also continuous. Since $f \in Z^1(\mathscr{N}, \mathscr{M})$, $\delta f = 0$ which implies

$$f(a_1 a_2 \otimes q) = \rho_1(a_1, f(a_2, q)) + f(a_1 \otimes \rho_2(a_2, q)) \ , a_1, a_2 \in \xi, q \in \mathscr{N}.$$

Hence

$$\begin{aligned}\rho(a_1 a_2, (q, m)) &= \big(\rho_2(a_1 a_2, q), \rho_1(a_1 a_2, m) + f(a_1 a_2 \otimes q)\big) \\ &= \big(\rho_2(a_1 a_2, q), \rho_1(a_1 a_2, m) + \rho_1(a_1, f(a_2, q)) \\ &\quad + f(a_1 \otimes \rho_2(a_2, q))\big) \\ &= \rho\big(a_1, \rho(a_2, (q, m))\big).\end{aligned}$$

This shows that ρ induces module structure on the fibers of $\mathscr{K}_f$. Hence $\mathscr{K}_f$ is a ξ-module bundle. Let $v : \mathscr{K}_f \to \mathscr{N}$ be the projection $(q, m) \mapsto m$ and $u : \mathscr{M} \to \mathscr{K}_f$ be the inclusion $m \mapsto (0, m)$. Then clearly

$$0 \to \mathscr{M} \xrightarrow{u} \mathscr{K}_f \xrightarrow{v} \mathscr{N} \to 0$$

is an exact sequence of ξ-module bundles. Suppose f and f' are the elements of $Z^1(\mathscr{N}, \mathscr{M})$ which belong to the same cohomology class in $H^1(\mathscr{N}, \mathscr{M})$. i.e., $f - f' = \delta(g)$ for some $g \in C^0(\mathscr{N}, \mathscr{M}) = Hom(\mathscr{N}, \mathscr{M})$. We need to show that $\mathscr{K}_f \cong \mathscr{K}_{f'}$. Let ρ' be the morphism which induce module structure on fibers of $\mathscr{K}_{f'}$. Define

$$\phi : \mathscr{K}_f \to \mathscr{K}_{f'}$$

by

$$\phi_x(q, m) = (q, m + g_x(q)) \text{ for } q \in \mathscr{N}_x, \mathscr{M}_x, x \in X.$$

Since g is continuous, ϕ is also continuous. It is clear that each ϕ_x is vector space isomorphism and hence ϕ is a homeomorphism [2]. It remains to show that ϕ is a ξ-morphism. For, consider $x \in X$, $a \in \xi_x$ and $(q, m) \in \mathscr{K}_f$. Since $f - f' = \delta(g)$,

$$f(a \otimes q) - f'(a \otimes q) = \rho_1(a, g(q)) - g(\rho_2(a, q)).$$

Then

$$\begin{aligned}\phi\left(\rho(a, (q, m))\right) &= \phi\left(\rho_2(a, q), \rho_1(a, m) + f(a \otimes q)\right) \\ &= \left(\rho_2(a, q),\ \rho_1(a, m) + f(a \otimes q) + g(\rho_2(a, q))\right) \\ &= \left(\rho_2(a, q),\ \rho_1(a, m) + f'(a \otimes q) + \rho_1(a, g(q))\right) \\ &= \left(\rho_2(a, q),\ \rho_1(a, m + g(q)) + f'(a \otimes q)\right) \\ &= \rho'\left(a, (q, m + g(q))\right) \\ &= \rho'\left(a, \phi(q, m)\right).\end{aligned}$$

Thus, $\mathscr{K}_f$ and $\mathscr{K}_{f'}$ are isomorphic as ξ-module bundles. ∎

Remark 2.6. By Theorem 2.5, the extension determined by $f \in Z^1(\mathscr{N}, \mathscr{M})$ is completely determined by the cohomology class $\overline{f}$ in $H^1(\mathscr{N}, \mathscr{M})$, we denote this extension by $\mathscr{K}_{\overline{f}}$.

Theorem 2.7. *Let $\mathscr{M}$ and $\mathscr{N}$ be ξ-module bundles. Let $\mathscr{K}$ be any ξ-module bundle extension of $\mathscr{N}$ by $\mathscr{M}$. Then there exists a unique $\overline{f} \in H^1(\mathscr{N}, \mathscr{M})$ such that $\mathscr{K}$ and $\mathscr{K}_{\overline{f}}$ are isomorphic as ξ-module bundles.*

Proof. Let $\rho_1 : \xi \oplus \mathscr{M} \to \mathscr{M}$, $\rho_2 : \xi \oplus \mathscr{N} \to \mathscr{N}$ and $\rho : \xi \oplus \mathscr{K} \to \mathscr{K}$ be the morphisms which induce module structure on the fibers of $\mathscr{M}$, $\mathscr{N}$ and $\mathscr{K}$ respectively. Since $\mathscr{K}$ is an extension, there is an exact sequence $0 \to \mathscr{M} \xrightarrow{u} \mathscr{K} \xrightarrow{v} \mathscr{N} \to 0$ of ξ-module bundles. Then u is a monomorphism, v is an epimorphism and $Im\ u = \ker v$. Let $w : \mathscr{N} \to \mathscr{K}$ be any vector bundle morphism such that $v(w(q)) = q$, $\forall\ q \in \mathscr{N}$. Since $u : \mathscr{M} \to \mathscr{K}$ is a monomorphism, $Im\ u$ is a submodule bundle of $\mathscr{K}$ and $u : \mathscr{M} \to Im\ u$ is a ξ-morphism [2, Lemma 1.3.1]. Let $\tilde{u} : Im\ u \to \mathscr{M}$ denote the inverse of u. It is evident that $\tilde{u}$ is a ξ-morphism. Define

$$f_w : \xi \otimes \mathscr{N} \to \mathscr{M}$$

by

$$f_w(a \otimes q) = \tilde{u}\left(\rho(a, w(q)) - w(\rho_2(a, q))\right), a \in \xi_x, q \in \mathscr{N}_x, x \in X.$$

We have

$$\begin{aligned} v\left(\rho(a, w(q)) - w(\rho_2(a, q))\right) &= \rho_2\big(a, vw(q)\big) - vw\big(\rho_2\big(a, q)\big), \\ &\qquad\qquad \text{since } v \text{ is a } \xi\text{-morphism} \\ &= \rho_2(a, q) - \rho_2(a, q) \\ &= 0. \end{aligned}$$

Hence $\rho(a, w(q)) - w(\rho_2(a, q)) \in \ker\ v = Im\ u$. This shows that f_w is well defined. It is clear that f_w is continuous being combination of continuous maps and is a vector bundle morphism of $\xi \otimes \mathscr{N}$ into $\mathscr{M}$. Hence $f_w \in S^1(\mathscr{N}, \mathscr{M})$. We show $f_w \in Z^1(\mathscr{N}, \mathscr{M})$. Let $a_1, a_2 \in \xi_x$ and $q \in \mathscr{N}_x$. Then

$$\begin{aligned} \delta f_w(a_1 \otimes a_2 \otimes q) &= \rho_1\big(a_1, f_w(a_2 \otimes q)\big) - f_w(a_1 a_2 \otimes q) \\ &\quad + f_w(a_1 \otimes \rho_2(a_2, q)) \\ &= \rho_1\left(a_1, \tilde{u}\big(\rho(a_2, w(q)) - w(\rho_2(a_2, q))\big)\right) \\ &\quad - \tilde{u}\left(\rho\big(\rho(a_1 a_2, w(q)\big) + w\big(\rho_2(a_1 a_2, q)\big)\right) \\ &\quad + \tilde{u}\left(\rho\big(a_1, w(\rho_2(a_2, q))\big) - w\big(a_1, \rho_2(a_2, q)\big)\right) \end{aligned}$$

$$
\begin{aligned}
&= \tilde{u}\Big(\rho\big(a_1, \rho(a_2, w(q))\big) - w\big(\rho_2(a_2, q)\big)\\
&\quad - \rho\big(a_1a_2, w(q)\big) + w\big(\rho_2(a_1a_2, q)\big)\\
&\quad + \rho\big(a_1, \rho_2(a_2, w(q))\big) - w\rho_2(a_1a_2, q))\Big)\,,\\
&\qquad\qquad\qquad\qquad \text{since } \tilde{u}, w \text{ are } \xi\text{-morphisms.}\\
&= \tilde{u}(0) = 0.
\end{aligned}
$$

Hence f_w determines a cohomology class in $H^1(\mathscr{N}, \mathscr{M})$. Suppose that $w_1 : \mathscr{N} \to \mathscr{K}$ is any vector bundle morphism such that $vw_1(q) = q,\ \forall q \in \mathscr{N}$. Then

$$
\begin{aligned}
(f_w - f_{w_1})(a \otimes q) &= \tilde{u}\left(\rho(a, w(q)) - w\rho_2(a, q) - \rho(a, w_1(q) - w_1\rho_2(a, q)\right)\\
&= \tilde{u}\Big(\rho\big(a, (w - w_1)(q)\big) - (w - w_1)\rho_2(a, q)\Big)\\
&= \rho_1\big(a, \tilde{u}(w - w_1)(q)\big) - \tilde{u}(w - w_1)\rho_2(a, q)\\
&= \delta\tilde{u}(w - w_1)(a \otimes q).
\end{aligned}
$$

Hence $f_w - f_{w_1} = \delta\tilde{u}(w - w_1)$ and so $\bar{f}_w = \bar{f}_{w_1}$. Let $\bar{\rho}$ denote the morphism which induce module structure on fibers of $\mathscr{K}_{\bar{f}_w}$. We define $\phi : \mathscr{K} \to \mathscr{K}_{\bar{f}_w}$ by $\phi(k) = \big(v(k), \tilde{u}(k - wv(k))\big)$ for $k \in \mathscr{K}$. Clearly ϕ is one-one. Suppose $(q, m) \in \mathscr{K}_{\bar{f}_w}$, $q \in \mathscr{N}_x, m \in \mathscr{M}_x, x \in X$. Since v is onto, there is a $k \in \mathscr{K}_x$ such that $v(k) = q$. Let $k' = m + wv(k)$. Since $vu = 0$, it can be shown that $\phi(k') = (q, m)$. Hence ϕ is onto. Evidently ϕ is a vector bundle morphism. It remains to show that ϕ is a ξ-morphism.

For the sake of simplicity in computation the following usage of notations is followed: $\rho(a, k) = ak, \rho_1(a, m) = am, \rho_2(a, q) = aq$ and $\bar{\rho}(a, (m, q)) = a(m, q)$, for $a \in \xi_x, k \in \mathscr{K}_x$ and $q \in \mathscr{N}_x$, $x \in X$. Now,

$$
\begin{aligned}
\phi(\rho(a, k)) = \phi(ak) &= \big(v(ak),\ \tilde{u}\big(ak, -wv(ak)\big)\big)\\
&= \big(av(k),\ \tilde{u}\big(ak - wav(k)\big)\big)\\
&= (av(k),\ \tilde{u}(ak) - \tilde{u}(awv(k)) + f_w(a \otimes v(k)))\\
&= \big(av(k),\ a\tilde{u}\big(k - wv(k)\big) + f_w(a \otimes v(k))\big)\\
&= a\left(v(k),\ \tilde{u}\big(k - wv(k)\big)\right)\\
&= a\phi(k)\\
&= \bar{\rho}(a, \phi(k)).
\end{aligned}
$$

Hence the proof. □

From the Theorem 2.7, it follows that there is a one to one correspondence between the elements of $H^1(\mathscr{N}, \mathscr{M})$ and the ξ-module bundle extensions of $\mathscr{N}$ by $\mathscr{M}$.

Corollary 2.8. *Let $\mathscr{M}, \mathscr{N}$ be ξ-module bundles. An extension $\mathscr{K}$ of $\mathscr{N}$ by $\mathscr{M}$ is split if and only if $\mathscr{K}$ corresponds to the 0 class of $H^1(\mathscr{N}, \mathscr{M})$.*

Proof. Let $\mathscr{K}_0 = \mathscr{N} \oplus \mathscr{M}$ be the direct Whitney sum of $\mathscr{N}$ and $\mathscr{M}$. Let $v : \mathscr{K}_0 \to \mathscr{N}$ be the projection of $\mathscr{K}_0$ onto $\mathscr{N}$ and $w : \mathscr{N} \to \mathscr{K}_0$ be given by $q \mapsto (q, 0)$. It is clear that v, w are ξ-morphisms with $vw(q) = q \; \forall \; q \in \mathscr{N}$. Now $f_w(a \otimes q) = \tilde{u}\,(\rho(a, w(q)) - w(\rho_2(a, q))) = \tilde{u}(0) = 0$. Hence the extension K_0 corresponds to the 0 cohomology class in $H^1(\mathscr{N}, \mathscr{M})$. If an extension $\mathscr{K}$ splits then it is isomorphic to $\mathscr{K}_0$ and hence corresponds to the 0 cohomology class. The converse is straight forward. □

We conclude this paper by giving an application in next theorem.

Theorem 2.9. *If ξ is semisimple, then $H^1(\mathscr{N}, \mathscr{M}) = 0$ for every pair of ξ-module bundles $\mathscr{M}$ and $\mathscr{N}$.*

Proof. Let $\mathscr{K}$ be a ξ-module bundle extension of $\mathscr{N}$ by $\mathscr{M}$. Then there is an exact sequence $0 \to \mathscr{M} \xrightarrow{u} \mathscr{K} \xrightarrow{v} \mathscr{N} \to 0$ of ξ-module bundles. Since u is monomorphism, $Im\ u$ is a module subbundle of $\mathscr{K}$. By [8, Theorem 5] for an associative algebra bundle, there is a module subbundle $\mathscr{K}_1$ of $\mathscr{K}$ such that $\mathscr{K} = Im\ u \oplus \mathscr{K}_1$. Clearly $Im\ u$ is isomorphic to $\mathscr{M}$ and $\mathscr{K}_1$ is isomorphic to $\mathscr{N}$ as ξ-module bundles. Hence $\mathscr{K}$ is isomorphic to $\mathscr{N} \oplus \mathscr{M}$. This shows that every extension of $\mathscr{N}$ by $\mathscr{M}$ splits. By the correspondence discussed above and the Corollary 2.8, it follows that $H^1(\mathscr{N}, \mathscr{M}) = 0$. □

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H. M. Prasad
Department of Mathematics, Field Marshal K.M. Cariappa College, (A Constituent College of Mangalore University), Madikeri - 571201, India
Email: prasadhm2011@gmail.com

R. Rajendra
Department of Mathematics, Field Marshal K.M. Cariappa College, (A Constituent College of Mangalore University), Madikeri - 571201, India
Email: rrajendrar@gmail.com

B. S. Kiranagi
Department of Mathematics, Mangalore University, Mangalagangothri - 574199 and Department of Mathematics, University of Mysore, Manasagangotri - 570006, India
Email: bskiranagi@gmail.com