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ON CLOSEDNESS OF SOME PERMUTATIVE POSEMIGROUP IDENTITIES

R. ALAM, W. ASHRAF * AND N. MOHAMMAD KHAN

Abstract. As we know that all non-trivial permutation identities are not preserved under epimorphisms of partially ordered semigroups. In this paper towards this open problem, first we show that certain non-trivial identities in conjunction with the permutation identity $z_1z_2\cdots z_n = z_{i_1}z_{i_2}\cdots z_{i_n}$ $(n \geq 2)$ with $i_n \neq n$ $[i_1 \neq 1]$ are preserved under epimorphisms of partially ordered semigroups. Further, we extend a result of Ahanger and Shah which showed that the center of a partially ordered semigroup S is closed in S and show that the normalizer of any element of a partially ordered semigroup S is closed in S.

1. Introduction and Preliminaries

A partially ordered semigroup, briefly a posemigroup is a pair $(S, \leq)$ comprising a semigroup S and a partial order $\leq$ on S that is compatible with its binary operation, i.e. for all $s_1, s_2, t_1, t_2 \in S, s_1 \leq t_1$ and $s_2 \leq t_2$ implies that $s_1s_2 \leq t_1t_2$. If S is a monoid, we call $(S, \leq)$ a partially ordered monoid, shortly a pomonoid. Further, we call $(U, \leq_U)$ a subposemigroup of a posemigroup $(S, \leq_S)$ if U is subsemigroup of the semigroup S and $\leq_U = \leq_S \cap (U \times U)$. The corresponding notion of a subpomonoid is defined analogously.

A posemigroup morphism $f : (S, \leq_S) \to (T, \leq_T)$ is a monotone map i.e. $(x \leq_S y \implies f(x) \leq_T f(y))$ which is also a semigroup morphism

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*Corresponding author .

of underlying semigroups.

We shall denote, in the sequel, posemigroups (pomonoids) by S, T etc. whenever no explicit mention of the order relation is required.

A class of posemigroups is called a variety of posemigroups if it is closed under taking the products (endowed with componentwise operation and order), morphic images and subposemigroups. A variety of pomonoids may be defined similarly. It is also possible to describe posemigroup (pomonoid) varieties alternatively with the help of inequalities using a Birkhoff type characterization; we refer to [2] for details. Because every term equality in an algebraic variety can be replaced by two (term) inequalities, see [2], in a usual way, a class of posemigroups (pomonoids) is a variety if the class of underlying semigroups (monoids) is a variety of semigroups (monoids). Also, every variety (whether algebraic or order theoretic) naturally gives rise to a category.

Let S and T be posemigroups and $f : S \to T$ be a posemigroup morphism. Then f is said to be an epimorphism (epi for short) if for any posemigroup W and any posemigroup morphisms $\alpha, \beta : T \to W$, $\alpha \circ f = \beta \circ f$ implies $\alpha = \beta$. We observe that $f : S \to T$ is necessarily a posemigroup epimorphism if $f : S \to T$ is a semigroup epimorphism, where in the latter case we disregard the orders (and hence the monotonocity) and treat S and T as semigroups.

Let U be a subposemigroup of a posemigroup S and $d \in S$. We say that U dominates d if for all $\alpha, \beta : S \to T$ posemigroup morphisms, such that $\alpha(u) = \beta(u)$ for all $u \in U$, one has $\alpha(d) = \beta(d)$. The set of all elements of S that are dominated by U is called the posemigroup dominion of U in S and is denoted by $\widehat{Dom}(U, S)$. One can easily verify that $\widehat{Dom}(U, S)$ is a subposemigroup of S containing U. A posemigroup U is said to be saturated if $\widehat{Dom}(U, S) \neq S$ for every posemigroup S containing U properly as a subposemigroup. A variety of posemigroups is saturated if each member of the variety is saturated. Also, it can be easily verified that a posemigroup morphism $f : S \to T$ is an epi if and only if the inclusion $i : f(S) \to T$ is epi and the inclusion $i : U \to S$ is epi if and only if $\widehat{Dom}(U, S) = S$.

An identity $u = v$ is said to be preserved under posemigroup epis if for all posemigroups U and S with U as a subposemigroup of S and such that $\widehat{Dom}(U,S) = S$, U satisfies $u = v$ implies, S also satisfies $u = v$. A variety $\mathcal{U}$ of posemigroups is said to be epimorphically closed if for all $U \in \mathcal{U}$ and for any posemigroup S containing U properly as a subposemigroup such that $\widehat{Dom}(U,S) = S$ implies, $S \in \mathcal{U}$.

The semigroup theoretic notations and conventions of Howie [4] will be used throughout without explicit mention.

The following result is known as the Zigzag Theorem for posemigroups provided by Sohail [6] and will frequently be used in what follows.

Theorem 1.1. *([6], Theorem 5) Let U be a subposemigroup of a posemigroup S. Then we have $d \in \widehat{Dom}(U,S)$ if and only if $d \in U$ or*

$$\begin{aligned}
d &\le x_1u_0, & u_0 &\le u_1y_1, \\
x_iu_{2i-1} &\le x_{i+1}u_{2i}, & u_{2i}y_i &\le u_{2i+1}y_{i+1}\ (1 \le i \le m-1), \\
x_mu_{2m-1} &\le u_{2m}, & u_{2m}y_m &\le d;
\end{aligned} \tag{1.1}$$

$$\begin{aligned}
v_0 &\le s_1v_1, & d &\le v_0t_1, \\
s_jv_{2j} &\le s_{j+1}v_{2j+1}, & v_{2j-1}t_j &\le v_{2j}t_{j+1}\ (1 \le j \le m'-1), \\
s_{m'}v_{2m'} &\le d, & v_{2m'-1}t_{m'} &\le v_{2m'};
\end{aligned} \tag{1.2}$$

where $u_0, v_0, \ldots, u_{2m}, v_{2m'} \in U, x_1, y_1, \ldots, x_m, y_m, s_1, t_1, \ldots, s_{m'}, t_{m'} \in S$.

Let us call the above inequalities posemigroup zigzag inequalities in S over U with value d and length (m, m') and we say that it is of minimal length (m, m') if m and m' are the least positive integers. Also, the first half (1.1) and the second half (1.2) of the above zigzag inequalities will be called, in whatever follows, as the upper half and the lower half of the zigag inequalities respectively. The upper half (1.1) of the zigzag inequalities gives:

$$d \le x_1u_0 \le x_1u_1y_1 \le x_2u_2y_1 \le \cdots \le x_mu_{2m-1}y_m \le u_{2m}y_m \le d.$$

This gives

$$d = x_1u_0 = x_1u_1y_1 = x_2u_2y_1 = \cdots = x_mu_{2m-1}y_m = u_{2m}y_m. \tag{1.3}$$

Similarly, the lower half (1.2) of the zigzag inequalities gives:

$$d \le v_0t_1 \le s_1v_1t_1 \le s_1v_2t_2 \le \cdots \le s_{m'}v_{2m'-1}t_{m'} \le s_{m'}v_{2m'} \le d.$$

This gives

$$d = v_0t_1 = s_1v_1t_1 = s_1v_2t_2 = \cdots = s_{m'}v_{2m'-1}t_{m'} = s_{m'}v_{2m'}. \qquad (1.4)$$

The next following theorems are from [1] and are very important for our investigations.

Theorem 1.2. *([1], Lemma 3.2) Let* $d \in \widehat{Dom}(U,S)\setminus U$ *and (1.1) and (1.2) be the zigzag inequalities for* d *of minimal length* (m, m'). *Then* $x_i, y_i \in S\setminus U$ *for* $i = 1, 2, \ldots, m$ *and* $s_j, t_j \in S\setminus U$ *for all* $j = 1, 2 \ldots, m'$.

Theorem 1.3. *([1], Lemma 3.3) If* U *is a subposemigroup of a posemigroup* S *such that* $\widehat{Dom}(U,S) = S$, *then for any* $d \in S \setminus U$ *and for any positive integers* k *and* k' *there exist* $u_1, u_2, \ldots, u_k, v_1, v_2 \ldots, v_{k'} \in U$ *and* $d_k, d_{k'} \in S \setminus U$ *such that* $d = u_1u_2\cdots u_kd_k = d_{k'}v_{k'}v_{k'-1}\cdots v_2v_1$.

Theorem 1.4. *([7], Lemma 3.10) If* U *is a subposemigroup of a posemigroup* S *such that* $\widehat{Dom}(U,S) = S$ *then for* $x \in S \setminus U$ *and* $y \in U$, $(xy)^k = x^ky^k$ *for all positive integers* k.

Bracketed statements whenever used shall mean the dual to the other statements.

2. Variety of Permutative Posemigroups

A semigroup S is said to be permutative if S satisfies a permutation identity

$$z_1z_2\cdots z_n = z_{i_1}z_{i_2}\cdots z_{i_n}, (n \geq 2) \qquad (2.1)$$

where i is a non trivial permutation of the set $\{1, 2, \ldots, n\}$ and $i_1, i_2, \ldots, i_n$ are the images of $1, 2, \ldots, n$ under the permutation i respectively. A posemigroup S is said to be a permutative if it is so as a semigroup.

We call a posemigroup S a permutative posemigroup if it is such as a semigroup. In [1], the authors have shown that if U is a commutative posemigroup then for any containing posemigroup S, $\widehat{Dom}(U,S)$ is also a commutative posemigroup. In particular, it shows that commutativity is preserved under epimorphism in the category of posemigroups. The determination of all identities which are preserved under epis in conjunction with the general permutation identity (2.1) is an open problem in the category of all semigroups and therefore in the category of all posemigroups. However, in ([8], Theorem 4.7), Khan showed that some identities were preserved under epis in conjunction with the general permutation identity (2.1). In the next theorem, we find certain posemigroup identities which are preserved under epis in conjunction with the permutation identity (2.1) with $i_n \neq n$ $[i_1 \neq 1]$.

Throughout the paper, by a permutative posemigroup, we shall mean a posemigroup S satisfying any permutation identity of the form (2.1) and by a permutative variety $\mathcal{V}$, we shall mean a variety of posemigroups defined by any permutation identity of the type (2.1).

Theorem 2.1. *All non trivial identities of the form* $z_1^{p_1} z_2^{p_2} \cdots z_r^{p_r} = z_1'^{q_1} z_2'^{q_2} \cdots z_{r'}'^{q_{r'}}$*, where* $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_{r'} > 0$*, are preserved under epis of posemigroups in conjunction with the permutation identity (2.1) with* $i_n \neq n$ $[i_1 \neq 1]$.

Proof. Let U be a subposemigroup of a posemigroup S such that $\widehat{Dom}(U, S)$ $=S$ and let assume that U satisfies (2.1). Thus

$$z_1^{p_1} z_2^{p_2} \cdots z_r^{p_r} = z_1'^{q_1} z_2'^{q_2} \cdots z_{r'}'^{q_{r'}} \tag{2.2}$$

holds for all $z_1, z_2, \ldots, z_r, z_1', z_2', \ldots, z_{r'}' \in U$.
To prove that S satisfies (2.2), we first show that

$$z_1^{p_1} z_2^{p_2} \cdots z_r^{p_r} = w_1'^{q_1} w_2'^{q_2} \cdots w_{r'}'^{q_{r'}} \tag{2.3}$$

for all $z_1, z_2, \ldots, z_r \in S$ and any $w_1', w_2', \ldots, w_{r'}' \in U$. So, take any $z_1, z_2, \ldots, z_r \in S$ and $w_1', w_2', \ldots, w_{r'}' \in U$. We prove it by induction on k $(1 \leq k \leq r)$ assuming that $z_1, z_2, \ldots, z_k \in S$ and $z_{k+1}, \ldots, z_r \in U$. For $k = 1$, we need not consider the case when $z_1 \in U$. So assume that $z_1 \in S \setminus U$ and let (1.1) be the upper half of zigzag inequalities for z_1 of minimal length. Then

$$\begin{aligned}
&z_1^{p_1} z_2^{p_2} \ldots z_r^{p_r} \\
&\quad \leq (x_1 u_0)^{p_1} z_2^{p_2} \ldots z_r^{p_r} \text{ (by zigzag inequalities (1.1))} \\
&\quad = x_1^{p_1} u_0^{p_1} z_2^{p_2} \ldots z_r^{p_r} \text{ (by Theorem 1.4)} \\
&\quad = x_1^{p_1} u_1^{p_1} z_2^{p_2} \ldots z_r^{p_r} \text{ (as } U \text{ satisfies (2.2))} \\
&\quad = (x_1 u_1)^{p_1} z_2^{p_2} \ldots z_r^{p_r} \text{ (by Theorem 1.4)}
\end{aligned}$$

$\leq (x_2u_2)^{p_1}z_2^{p_2}\dots z_r^{p_r}$ (by zigzag inequalities (1.1))
$= x_2^{p_1}u_2^{p_1}z_2^{p_2}\dots z_r^{p_r}$ (by Theorem 1.4)
$= x_2^{p_1}u_3^{p_1}z_2^{p_2}\dots z_r^{p_r}$ (as U satisfies (2.2))
$\leq x_i^{p_1}u_{2i-1}^{p_1}z_2^{p_2}\dots z_r^{p_r}$ (for $1\leq i\leq m$)
$= x_m^{p_1}u_{2m-1}^{p_1}z_2^{p_2}\dots z_r^{p_r}$ (for $i=m$)
$= (x_mu_{2m-1})^{p_1}z_2^{p_2}\dots z_r^{p_r}$ (by Theorem 1.4)
$\leq u_{2m}^{p_1}z_2^{p_2}\dots z_r^{p_r}$ (by zigzag inequalities (1.1))
$= w_1'^{q_1}w_2'^{q_2}\cdots w_{r'}'^{q_{r'}}$ (for any $w_1',w_2',\dots,w_{r'}'\in U$ as U satisfies (2.2)).

This implies

$$z_1^{p_1}z_2^{p_2}\cdots z_r^{p_r}\leq w_1'^{q_1}w_2'^{q_2}\cdots w_{r'}'^{q_{r'}}. \tag{2.4}$$

On the similar lines, by using the lower half (1.2) of zigzag inequalities, we may show that

$$z_1^{p_1}z_2^{p_2}\cdots z_r^{p_r}\geq w_1'^{q_1}w_2'^{q_2}\cdots w_{r'}'^{q_{r'}} \tag{2.5}$$

for any $w_1',w_2',\dots,w_{r'}'\in U$.
By combining equations (2.4) and (2.5), we get

$$z_1^{p_1}z_2^{p_2}\cdots z_r^{p_r}= w_1'^{q_1}w_2'^{q_2}\cdots w_{r'}'^{q_{r'}}.$$

Now, suppose inductively that (2.4) holds for all $1\leq k<r$; i.e. for all $z_1,z_2,\dots,z_k\in S$ and $z_{k+1},z_{k+2},\dots,z_r\in U$, we have

$$z_1^{p_1}z_2^{p_2}\dots z_l^{p_l}z_{l+1}^{p_{l+1}}\dots z_r^{p_r}= w_1'^{q_1}w_2'^{q_2}\cdots w_{r'}'^{q_{r'}}$$

for any $w_1',w_2',\dots,w_{r'}'\in U$.
From this, we need to show that (2.4) holds for all $z_1,z_2,\dots,z_k,z_{k+1}\in S$, $z_{k+2},z_{k+3},\dots,z_r\in U$ and for any $w_1',w_2',\dots,w_{r'}'\in U$. So, take any $z_1,z_2,\dots,z_l,z_{l+1}\in S$ and $z_{l+2},z_{l+3},\dots,z_r,w_1',w_2',\dots,w_{r'}'\in U$. If $z_{k+1}\in U$, then (2.4) holds by inductive hypothesis. So, assume that $z_{k+1}\in S\setminus U$ and let (1.1) be the upper half of zigzag inequalities of minimal length.
We shall use phrases, in whatever follows, '*expanding*' and '*collapsing*' $x_i^{p_{k+1}}$ $(1\leq i\leq m)$, by Theorems 1.3 and 1.4, to mean $x_i^{p_{k+1}}= x_i^{(i)p_{k+1}}b_1^{(i)p_{k+1}}b_2^{(i)p_{k+1}}\cdots b_k^{(i)p_{k+1}}$ for some $b_1^{(i)},b_2^{(i)},\dots,b_k^{(i)}\in U$ and $x_i^{(i)}\in S\setminus U$ respectively.

Now

$$
\begin{aligned}
&z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} z_{k+1}^{p_{k+1}} \cdots z_r^{p_r} \\
&\quad \leq z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} (x_1 u_0)^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \\
&\qquad\qquad \text{(by upper part (1.1) of zigzag inequalities)} \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_1^{p_{k+1}} u_0^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \text{ (by Theorem 1.4)} \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_1^{(1)p_{k+1}} b_1^{(1)p_{k+1}} b_2^{(1)p_{k+1}} \cdots b_k^{(1)p_{k+1}} u_0^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \\
&\qquad\qquad \text{(by expanding } x_1^{p_{k+1}}\text{)} \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_1^{(1)p_{k+1}} b_1^{(1)p_{k+1}} b_2^{(1)p_{k+1}} \cdots b_k^{(1)p_{k+1}} u_1^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \\
&\qquad\qquad \text{(as } U \text{ satisfies (2.2))} \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} (x_1 u_1)^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \\
&\qquad\qquad \text{(by collapsing } x_1^{p_{k+1}} \text{ and Theorem 1.4)} \\
&\quad \leq z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_2^{p_{k+1}} u_2^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \\
&\qquad\qquad \text{(by zigzag inequalities (1.1) and Theorem 1.4)} \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_2^{(2)p_{k+1}} b_1^{(2)p_{k+1}} b_2^{(2)p_{k+1}} \cdots b_k^{(2)p_{k+1}} u_2^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \\
&\qquad\qquad \text{(by expanding } x_2^{p_{k+1}}\text{)} \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_2^{(2)p_{k+1}} b_1^{(2)p_{k+1}} b_2^{(2)p_{k+1}} \cdots b_k^{(2)p_{k+1}} u_3^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \\
&\qquad\qquad \text{(as } U \text{ satisfies (2.2))} \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_2^{p_{k+1}} u_3^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \quad \text{(by collapsing } x_2^{p_{k+1}}\text{)} \\
&\quad \vdots \\
&\quad \leq z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_i^{p_{k+1}} u_{2i-1}^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \quad \text{(for } 1 \leq i \leq m\text{)} \\
&\quad \vdots \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_m^{p_{k+1}} u_{2m-1}^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \quad \text{(for } i = m\text{)} \\
&\quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} (x_m u_{2m-1})^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \quad \text{(by Theorem 1.4)} \\
&\quad \leq z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} u_{2m}^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_r^{p_r} \quad \text{(by zigzag inequalities (1.1))} \\
&\quad = w_1'^{q_1} w_2'^{q_2} \cdots w_{r'}'^{q_{r'}} \\
&\qquad\qquad \text{(by inductive hypothesis as } u_{2m}, z_{l+2}, \ldots, z_r \in U\text{).}
\end{aligned}
$$

Therefore,

$$z_1^{p_1} z_2^{p_2} \cdots z_r^{p_r} \leq w_1'^{q_1} w_2'^{q_2} \cdots w_{r'}'^{q_{r'}}. \tag{2.6}$$

Similarly, by using the lower half (1.2) of zigzag inequalities, we may get

$$z_1^{p_1} z_2^{p_2} \cdots z_r^{p_r} \geq w_1'^{q_1} w_2'^{q_2} \cdots w_{r'}'^{q_{r'}}. \tag{2.7}$$

By combining equations (2.6) and (2.7), we get

$$z_1^{p_1} z_2^{p_2} \cdots z_r^{p_r} = w_1'^{q_1} w_2'^{q_2} \cdots w_{r'}'^{q_{r'}}.$$

Therefore, (2.4) is true for $k+1$. Hence, by induction, (2.4) holds.
By the similar token, we may show that

$$z_1'^{q_1} z_2'^{q_2} \cdots z_{r'}'^{q_{r'}} = w_1^{p_1} w_2^{p_2} \cdots w_r^{p_r},$$

for any $z_1', z_2', \ldots, z_{r'}' \in S$ and $w_1, w_2, \ldots w_r \in U$.
As U satifies (2.3), we have $w_1'^{q_1} w_2'^{q_2} \cdots w_{r'}'^{q_{r'}} = w_1^{p_1} w_2^{p_2} \cdots w_r^{p_r}$ and so

$$z_1^{p_1} z_2^{p_2} \cdots z_r^{p_r} = z_1'^{q_1} z_2'^{q_2} \cdots z_r'^{q_r}$$

as required. □

Theorem 2.2. *Following type of non trivial identities are preserved under epis of posemigroups in conjunction with any permutation identity (2.1) with* $i_n \neq n$ $[i_1 \neq 1]$

$$z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} = 0, \quad \textit{where} \quad p_1, p_2, \ldots, p_n > 0 \tag{2.8}$$

(for any non-empty word u, we regard $u = 0$ as an identity which is the conjunction of two identities $uy = u = yu$, where y is a variable not occurring in the word u).

Proof. Let U be a subposemigroup of a posemigroup S such that $\widehat{Dom}(U, S)$ $=S$ and let U satisfies (2.1) with $i_n \neq n$ $[i_1 \neq 1]$. Therefore

$$z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} = 0$$

holds for all $z_1, z_2, \ldots, z_n \in U$.
We will prove it by induction on k assumming that $z_1, z_2, \ldots, z_k \in S$ and
$z_{k+1}, z_{k+2}, \ldots, z_n \in U$. For $k = 1$, $z_1 \in S$ and $z_2, z_3, \ldots, z_n \in U$. We need not consider the case when $z_1 \in U$. So $z_1 \in S \setminus U$ and let (1.1) be the upper half of zigzag inequalities for z_1 of minimal length. Now

$$\begin{aligned} z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} &= (x_1 u_0)^{p_1} z_2^{p_2} \cdots z_n^{p_n} \text{ (by zigzag inequalities (1.1))} \\ &= x_1^{p_1} u_0^{p_1} z_2^{p_2} \cdots z_n^{p_n} \text{ (by Theorem 1.4)} \\ &= x_1^{p_1} u_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} \text{ (as } U \text{ satisfies (2.2))} \end{aligned}$$

$= (x_1u_1)^{p_1} z_2^{p_2} \cdots z_n^{p_n}$ (by Theorem 1.4)
$\leq x_2^{p_1} u_2^{p_1} z_2^{p_2} \cdots z_n^{p_n}$
(by zigzag inequalities (1.1) and Theorem 1.4)
$= x_2^{p_1} u_3^{p_1} z_2^{p_2} \cdots z_n^{p_n}$ (as U satisfies (2.2))
$\vdots$
$\leq x_i^{p_1} u_{2i-1}^{p_1} z_2^{p_2} \cdots z_n^{p_n}$ (for $1 \leq i \leq m$)
$\vdots$
$= x_m^{p_1} u_{2m-1}^{p_1} z_2^{p_2} \cdots z_n^{p_n}$ (for $i = m$)
$= (x_m u_{2m-1})^{p_1} z_2^{p_2} \cdots z_n^{p_n}$ (by Theorem 1.4)
$\leq u_{2m}^{p_1} z_2^{p_2} \cdots z_n^{p_n}$ (by zigzag inequalities (1.1))
$= 0$ (as U satisfies Theorem 2.2).

This implies

$$z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n} \leq 0. \tag{2.9}$$

Similarly, by using the lower half (1.2) of zigzag inequalities, we may show that

$$z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n} \geq 0. \tag{2.10}$$

By equations (2.9) and (2.10), we get

$$z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n} = 0.$$

Let assume next that the result is true for all $z_1, z_2, \ldots, z_k \in S \setminus U$ and $z_{k+1}, \ldots, z_n \in U$; i.e.

$$z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} z_{k+1}^{p_{k+1}} \cdots z_n^{p_n} = 0. \tag{2.11}$$

Now, we show that the result is true for all $z_1, z_2, \ldots, z_k, z_{k+1} \in S \setminus U$ and $z_{k+2}, \ldots, z_n \in U$. So, take any $z_1, z_2, \ldots, z_k, z_{k+1} \in S \setminus U$ and $z_{k+2}, \ldots, z_n \in U$. By inductive hypothesis, we need not consider the case when $z_{k+1} \in U$. Then

$$
\begin{aligned}
& z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} z_{k+1}^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \\
& \quad \leq z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} (x_1 u_0)^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \\
& \qquad\qquad\qquad\qquad \text{(by upper half (1.1) of zigzag inequalities)} \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_1^{p_{k+1}} u_0^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \quad \text{(by Theorem 1.4)} \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_1^{(1)p_{k+1}} b_1^{(1)p_{k+1}} b_2^{(1)p_{k+1}} \cdots b_k^{(1)p_{k+1}} u_0^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \\
& \qquad \text{(by expanding } x_1^{p_{k+1}} \text{ and using Theorems 1.3 and 1.4)} \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_1^{(1)p_{k+1}} b_1^{(1)p_{k+1}} b_2^{(1)p_{k+1}} \cdots b_k^{(1)p_{k+1}} u_1^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{(as U satisfies (2.2))} \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_1^{p_{k+1}} u_1^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \\
& \qquad\qquad\qquad\qquad \text{(by collasping } x_1^{p_{k+1}} \text{ and Theorem 1.3)} \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} (x_1 u_1)^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \qquad \text{(by Theorem 1.4)} \\
& \quad \leq z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_2^{p_{k+1}} u_2^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \\
& \qquad\qquad\qquad\qquad \text{(by zigzag inequalities (1.1) and Theorem 1.4)} \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_2^{(2)p_{k+1}} b_1^{(2)p_{k+1}} b_2^{(2)p_{k+1}} \cdots b_k^{(2)p_{k+1}} u_2^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \\
& \qquad\qquad \text{(by expanding } x_2^{p_{k+1}} \text{ and using Theorems 1.3 and 1.4)} \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_2^{(2)p_{k+1}} b_1^{(2)p_{k+1}} b_2^{(2)p_{k+1}} \cdots b_k^{(2)p_{k+1}} \\
& \qquad u_3^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \qquad \text{(as } U \text{ satisfies (2.2))} \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_2^{p_{k+1}} u_3^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \quad \text{(by Theorem 1.3)} \\
& \quad \vdots \\
& \quad \leq z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_i^{p_{k+1}} u_{2i-1}^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \quad \text{(for } 1 \leq i \leq m) \\
& \quad \vdots \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} x_m^{p_{k+1}} u_{2m-1}^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \qquad \text{(for } i = m) \\
& \quad = z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} (x_m u_{2m-1})^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \quad \text{(by Theorem 1.4)} \\
& \quad \leq z_1^{p_1} z_2^{p_2} \cdots z_k^{p_k} u_{2m}^{p_{k+1}} z_{k+2}^{p_{k+2}} \cdots z_n^{p_n} \qquad \text{(by zigzag inequalities (1.1))} \\
& \quad \leq 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{(by inductive hypothesis).}
\end{aligned}
$$

Thus, we have shown that

$$z_1^{p_1} z_2^{p_2} \ldots z_k^{p_k} z_{k+1}^{p_{k+1}} z_{k+2}^{p_{k+2}} \ldots z_n^{p_n} \leq 0. \tag{2.12}$$

Similarly, by using the lower half (1.2) of zigzag inequalities, we may show that

$$z_1^{p_1} z_2^{p_2} \dots z_k^{p_k} z_{k+1}^{p_{k+1}} z_{k+2}^{p_{k+2}} \dots z_n^{p_n} \geq 0. \tag{2.13}$$

On combining equations (2.12) and (2.13), we get

$$z_1^{p_1} z_2^{p_2} \dots z_k^{p_k} z_{k+1}^{p_{k+1}} \dots z_n^{p_n} = 0.$$

This shows that the result is true for $k+1$. Therefore, by induction, the result follows. □

3. closed Varieties of Posemigroup

Let S be a posemigroup. Then an element $s \in S$ is said to be centralizer of a in S if $as = sa$. For any $a \in S$, the set $N(a)$ of all such elements of S is called normalizer of $a \in S$. In fact, it is easy to verify that $N(a)$ (a always belongs to $N(a)$) is a subposemigroup of S. In [1], Ahanger and Shah proved that the center of a posemigroup S is closed in S. Now we extend it to the normalizer $N(a)$ of any element $a \in S$ of a posemigroup S.

Theorem 3.1. *Let S be any posemigroup and $a \in S$. Then $N(a)$ is closed in S.*

Proof. To prove the theorem, we have to essentially show, for all $d \in \widehat{Dom}(N(a), S) \setminus N(a)$, $da = ad$. So take any $d \in \widehat{Dom}(N(a), S) \setminus N(a)$ and let (1.1) be the upper half of zigzag inequalities for d of minimal length. Then, by the definition of $N(a)$ and the upper half of the zigzag inequalities (1.1), we have

$$\begin{aligned}
da &\leq x_1 u_0 a \text{ (by zigzag inequalities (1.1))} \\
&= x_1 a u_0 \text{ (by the definition of } N(a)) \\
&\leq x_1 a u_1 y_1 \text{ (by zigzag inequalities (1.1))} \\
&= x_1 u_1 a y_1 \text{ (by the definition of } N(a)) \\
&\leq x_2 u_2 a y_1 \text{ (by zigzag inequalities (1.1))} \\
&= x_2 a u_2 y_1 \text{ (by the definition of } N(a)) \\
&\leq x_2 a u_3 y_2 \text{ (by zigzag inequalities (1.1))} \\
&= x_2 u_3 a y_2 \text{ (by the definition of } N(a)) \\
&\vdots \\
&\leq x_i u_{2i-1} a y_i
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
&= x_m u_{2m-1} a y_m \\
&\leq u_{2m} a y_m \text{ (by zigzag inequalities (1.1))} \\
&= a u_{2m} y_m \text{ (by the definition of } N(a)) \\
&\leq ad \text{ (by zigzag inequalities (1.1))}.
\end{aligned}$$

By the similar way, using the lower half (1.2) of the zigzag inequalities, we may show that $ad \leq da$.
Thus, $ad = da$. Hence, $N(a)$ is closed in S, as required. □

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Rizwan Alam
Department of Mathematics, Aligarh Muslim University, Aligarh, India
Email: rizwanamuba@gmail.com

Wajih Ashraf
Department of Mathematics, Aligarh Muslim University, Aligarh, India
Email: swashraf81@gmail.com

Noor Mohammad Khan
Department of Mathematics, Aligarh Muslim University, Aligarh, India
Email: nm_khan123@yahoo.co.in