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Stochastic permanence and extinction of a hybrid predator-prey system with jumps

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Abstract. This paper concerns the dynamics of a stochastic Holling-type II predator-prey system with Markovian switching and Levy noise. First, the existence and uniqueness of global positive solution to the system with the given initial value is proved. Then, sufficient conditions for extinction and stochastic permanence of the system are obtained. Finally, an example and its numerical simulations are given to support the theoretical results.

Keywords: Stochastic permanence, Markov chain, Levy noise, predator-prey system.
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1 Introduction

Recently, stochastic population systems driven by white noise have been received great attention [2, 5, 7–9, 11, 13, 16, 17, 19, 24, 27]. The stochastic Holling-type II predator-prey system can be expressed as follows:

$$\begin{cases} \mathrm{d}x(t) = x(t)\left(r_1 - a_{11}x(t) - \frac{a_{12}y(t)}{1+x(t)}\right)\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B(t), \\ \mathrm{d}y(t) = y(t)\left(r_2 + \frac{a_{21}x(t)}{1+x(t)} - a_{22}y(t)\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B(t), \end{cases} \tag{1}$$

where r_i and a_{ij} are positive constants $(i, j = 1, 2)$. $B(t)$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, P)$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Lv et al. [18] studied stochastic boundedness, stochastic permanence and persistence in mean of system (1).

However, on the one hand, in the real world population systems often suffer sudden environmental perturbations which cannot be described by white noise: earthquakes, hurricanes, planting, harvesting [12,14,15,25,29]. Bao et al. [3,4] pointed out that introducing Levy jumps into the underlying population system may be a reasonable way to describe these phenomena.

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On the other hand, parameters in some population systems may suffer abrupt changes, for instance, some authors [17, 27] claimed that the growth rates of some species in summer will be much different from those in winter, and one can use a continuous-time Markov chain with a finite state space to describe these abrupt changes [14, 15, 28]. Especially, Takeuchi et al. [23] studied a predator-prey Lotka-Volterra system with regime switching and revealed the significant effect of environmental noise on the population system: both its subsystems develop periodically but switching between them makes them become neither permanent nor dissipative [10, 22, 23].

So, in this paper we study the dynamics of the following stochastic Holling type II predator-prey system with Markovian switching and Levy noise:

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\Big[\Big(r_1(\rho(t)) - a_{11}(\rho(t))x(t^-) - \frac{a_{12}(\rho(t))y(t^-)}{1+x(t^-)}\Big)\mathrm{d}t \\ \qquad +\sigma_1(\rho(t))\mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma_1(\mu,\rho(t))\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\Big], \\ \mathrm{d}y(t) = y(t^-)\Big[\Big(r_2(\rho(t)) + \frac{a_{21}(\rho(t))x(t^-)}{1+x(t^-)} - a_{22}(\rho(t))y(t^-)\Big)\mathrm{d}t \\ \qquad +\sigma_2(\rho(t))\mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma_2(\mu,\rho(t))\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\Big], \end{cases} \tag{2}$$

where $x(t^-)$ and $y(t^-)$ are the left limits of $x(t)$ and $y(t)$, respectively. The $\rho(t)$ is a right-continuous Markov chain with a finite state space $\mathbb{S} = \{1,\ldots,S\}$, N is a Poisson counting measure with characteristic measure λ on a subset $\mathbb{Z}$ of $[0,+\infty)$ with $\lambda(\mathbb{Z}) < +\infty$, and $\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu) = N(\mathrm{d}t,\mathrm{d}\mu) - \lambda(\mathrm{d}\mu)\mathrm{d}t$. $\gamma_i(\mu) > -1$ $(\mu \in \mathbb{Z})$ are bounded functions $(i = 1,2)$. System (2) is operated as follows: If $\rho(0) = i_0$, then system (2) obeys

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\Big[\Big(r_1(i) - a_{11}(i)x(t^-) - \frac{a_{12}(i)y(t^-)}{1+x(t^-)}\Big)\mathrm{d}t + \sigma_1(i)\mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma_1(\mu,i)\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\Big], \\ \mathrm{d}y(t) = y(t^-)\Big[\Big(r_2(i) + \frac{a_{21}(i)x(t^-)}{1+x(t^-)} - a_{22}(i)y(t^-)\Big)\mathrm{d}t + \sigma_2(i)\mathrm{d}B(t) + \int_{\mathbb{Z}} \gamma_2(\mu,i)\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\Big], \end{cases} \tag{3}$$

with $i = i_0$ until τ_1 when $\rho(t)$ jumps to i_1 from i_0; system (2) will then obey system (3) with $i = i_1$ from τ_1 until τ_2 when $\rho(t)$ jumps to i_2 from i_1. System (2) will go on switching as long as $\rho(t)$ jumps. Hence, system (2) can be regarded as system (3) switching from one to another in accordance with the law of Markov chain. The different systems (3) $(i \in \mathbb{S})$ are therefore referred to as the subsystems of system (2). If the switching between environmental regimes disappears, namely, $\rho(t)$ has only one state, then system (2) degenerates into system (3).

2 Existence and uniqueness of global positive solution

Throughout this paper, the generator $\Gamma = (\gamma_{ij})_{S\times S}$ of $\rho(t)$ is given by

$$P\{\rho(t+\varsigma) = j \,|\, \rho(t) = i\} = \begin{cases} \gamma_{ij}\varsigma + o(\varsigma), & i \neq j, \\ 1 + \gamma_{ij}\varsigma + o(\varsigma), & i = j, \end{cases} \tag{1}$$

where $\varsigma > 0$. γ_{ij} is the transition rate from i to j and $\gamma_{ij} \geq 0$ if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. Assume that $\rho(t)$, $B(t)$ and N are mutually independent and that $\rho(t)$ is irreducible. Hence, system (2) can switch from any regime to any other regime and $\rho(t)$ has a unique stationary probability distribution $\pi = (\pi_1, ..., \pi_S) \in \mathbb{R}^{1\times S}$ determined by solving $\pi\Gamma = 0$, subject to

$$\sum_{i=1}^{S} \pi_i = 1 \text{ and } \pi_i > 0,\ \forall i \in \mathbb{S}. \tag{2}$$

In this paper, we impose the following assumptions:

($\mathbf{H_1}$) For each $i \in \mathbb{S}$, $r_j(i) > 0$, $a_{jk}(i) > 0$, there exist $\gamma_j^*(i) \geq \gamma_{j*}(i) > -1$ such that $\gamma_{j*}(i) \leq \gamma_j(\mu, i) \leq \gamma_j^*(i)\, (\mu \in \mathbb{Z})$, $j,k = 1,2$.

($\mathbf{H_2}$) For some $j \in \mathbb{S}$, $\gamma_{ij} > 0$, $\forall i \neq j$.

($\mathbf{H_3}$) For some $i \in \mathbb{S}$, $r_j(i) > 0$, $a_{jk}(i) > 0$, there exist $\gamma_j^*(i) \geq \gamma_{j*}(i) > -1$ such that $\gamma_{j*}(i) \leq \gamma_j(\mu, i) \leq \gamma_j^*(i)\, (\mu \in \mathbb{Z})$, $j,k = 1,2$.

Also, for simplicity, denote

$$\begin{cases} X(t) = (x(t), y(t))^{\mathrm{T}},\ |X(t)| = \sqrt{x^2(t) + y^2(t)}, \\ \langle \nu(t) \rangle = t^{-1} \int_0^t \nu(s) \mathrm{d}s,\ \langle \nu(t) \rangle^* = \limsup_{t\to+\infty} \langle \nu(t) \rangle,\ \langle \nu(t) \rangle_* = \liminf_{t\to+\infty} \langle \nu(t) \rangle, \\ B_j(i) = r_j(i) - \frac{\sigma_j^2(i)}{2} - \int_{\mathbb{Z}} [\gamma_j(\mu,i) - \ln(1+\gamma_j(\mu,i))]\, \lambda(\mathrm{d}\mu), \\ B(i) = \min_{j=1,2} \left\{ B_j(i) \right\},\ \mathscr{B} = \sum_{i=1}^{S} \pi_i B(i),\ \mathscr{B}_j = \sum_{i=1}^{S} \pi_i B_j(i), \\ \overline{r_j} = \max_{i\in\mathbb{S}} \left\{ r_j(i) \right\},\ \underline{r_j} = \min_{i\in\mathbb{S}} \left\{ r_j(i) \right\},\ r = \max_{j=1,2} \left\{ \overline{r_j} \right\}, \\ \overline{a_{jk}} = \max_{i\in\mathbb{S}} \left\{ a_{jk}(i) \right\},\ \underline{a_{jk}} = \min_{i\in\mathbb{S}} \left\{ a_{jk}(i) \right\},\ \sigma = \max_{j=1,2} \max_{i\in\mathbb{S}} \left\{ \left| \sigma_j(i) \right| \right\}\ (j,k = 1,2). \end{cases}$$

Theorem 1. *Under* ($\mathbf{H_1}$)*, for any* $X(0) \in \mathbb{R}_+^2$*, system* (2) *has a unique global solution* $X(t) \in \mathbb{R}_+^2$ *on* $t \geq 0$ *a.s.*

Proof. Consider the following stochastic differential equation:

$$\begin{cases} \mathrm{d}u(t) = \left(B_1(\rho(t)) - a_{11}(\rho(t))\mathrm{e}^{u(t)} - \frac{a_{12}(\rho(t))\mathrm{e}^{v(t)}}{1+\mathrm{e}^{u(t)}} \right) \mathrm{d}t + \sigma_1(\rho(t))\mathrm{d}B(t) \\ \qquad\qquad + \int_{\mathbb{Z}} \ln[1 + \gamma_1(\mu, \rho(t))] \widetilde{N}(\mathrm{d}t, \mathrm{d}\mu), \\ \mathrm{d}v(t) = \left(B_2(\rho(t)) + \frac{a_{21}(\rho(t))\mathrm{e}^{u(t)}}{1+\mathrm{e}^{u(t)}} - a_{22}(\rho(t))\mathrm{e}^{v(t)} \right) \mathrm{d}t + \sigma_2(\rho(t))\mathrm{d}B(t) \\ \qquad\qquad + \int_{\mathbb{Z}} \ln[1 + \gamma_2(\mu, \rho(t))] \widetilde{N}(\mathrm{d}t, \mathrm{d}\mu), \\ u(0) = \ln x(0),\ v(0) = \ln y(0). \end{cases} \tag{3}$$

Since the coefficients of system (3) are locally Lipschitz continuous, from [21] and [1], system (3) admits a unique local solution $(u(t), v(t))^{\mathrm{T}}$ on $t \in [0, \tau_e)$ a.s., where τ_e is the explosion time. By Itô's formula, $X(t) = (\mathrm{e}^{u(t)}, \mathrm{e}^{v(t)})^{\mathrm{T}}$ is the unique local solution to system (2) with $X(0) \in \mathbb{R}_+^2$. The proof of its global solution is almost identical to that for systems with Markovian switching driven by white noise (see e.g. [16, 26, 28], and hence it is omitted. □

3 Extinction

Theorem 2. *Under* ($\mathbf{H_1}$), *let* $X(t)$ *be the solution to system* (2) *with* $X(0) \in \mathbb{R}^2_+$.
(1) If $\mathscr{B}_1 < 0$, *then* $\lim\limits_{t\to+\infty} x(t) = 0$ *a.s.*
Moreover, if $\mathscr{B}_2 < 0$, *then* $\lim\limits_{t\to+\infty} y(t) = 0$ *a.s.; if* $\mathscr{B}_2 > 0$, *then*

$$\frac{\mathscr{B}_2}{\overline{a_{22}}} \leq \langle y(t)\rangle_* \leq \langle y(t)\rangle^* \leq \frac{\mathscr{B}_2}{\underline{a_{22}}} \text{ a.s.} \tag{1}$$

(2) If $\mathscr{B}_2 < -\sum_{i=1}^S \pi_i a_{21}(i)$, *then* $\lim\limits_{t\to+\infty} y(t) = 0$ *a.s. Moreover, if* $\mathscr{B}_1 < 0$, *then* $\lim_{t\to+\infty} x(t) = 0$ *a.s.; if* $\mathscr{B}_1 > 0$, *then*

$$\frac{\mathscr{B}_1}{\overline{a_{11}}} \leq \langle x(t)\rangle_* \leq \langle x(t)\rangle^* \leq \frac{\mathscr{B}_1}{\underline{a_{11}}} \text{ a.s.} \tag{2}$$

Proof. We only prove (1), the proof of (2) is analogous. From system (3), we have

$$\begin{cases} \ln\left(\frac{x(t)}{x(0)}\right) = \int_0^t B_1(\rho(s))\mathrm{d}s - \int_0^t a_{11}(\rho(s))x(s)\mathrm{d}s - \int_0^t \frac{a_{12}(\rho(s))y(s)}{1+x(s)}\mathrm{d}s + \sum_{j=1}^2 M_{1j}(t), \\ \ln\left(\frac{y(t)}{y(0)}\right) = \int_0^t B_2(\rho(s))\mathrm{d}s + \int_0^t \frac{a_{21}(\rho(s))x(s)}{1+x(s)}\mathrm{d}s - \int_0^t a_{22}(\rho(s))y(s)\mathrm{d}s + \sum_{j=1}^2 M_{2j}(t), \end{cases} \tag{3}$$

where, for $j = 1,2$,

$$\begin{cases} M_{j1}(t) = \int_0^t \sigma_j(\rho(s))\mathrm{d}B(s),\ \langle M_{j1}\rangle(t) \leq \sigma^2 t, \\ M_{j2}(t) = \int_0^t \int_{\mathbb{Z}} \ln\left[1+\gamma_j(\mu,\rho(s))\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}\mu), \\ \langle M_{j2}\rangle(t) \leq \max\limits_{j=1,2}\max\limits_{i\in\mathbb{S}}\left\{\left[\ln\left(1+\gamma_{j*}(i)\right)\right]^2, \left[\ln\left(1+\gamma_j^*(i)\right)\right]^2\right\}\lambda(\mathbb{Z})t. \end{cases} \tag{4}$$

By Lemma 3.1 in [3], we obtain

$$\lim_{t\to+\infty} t^{-1}M_{ij}(t) = 0 \ \text{ a.s.} \ \ (i,j=1,2). \tag{5}$$

Combining (3) with (5) yields

$$\begin{cases} \limsup\limits_{t\to+\infty} t^{-1}\ln x(t) \leq \langle B_1(\rho(t))\rangle^* = \mathscr{B}_1, \\ \limsup\limits_{t\to+\infty} t^{-1}\ln y(t) \leq \langle B_2(\rho(t)) + a_{21}(\rho(t))\rangle^* \leq \mathscr{B}_2 + \sum\limits_{i=1}^S \pi_i a_{21}(i). \end{cases} \tag{6}$$

In view of (6), if $\mathscr{B}_1 < 0$, then $\lim_{t\to+\infty}\langle x(t)\rangle = 0$ a.s. Hence, $\forall \varepsilon \in (0,1)$ and sufficiently large t,

$$\begin{cases} \ln y(t) \leq (\mathscr{B}_2 + \varepsilon)t - \underline{a_{22}}\int_0^t y(s)\mathrm{d}s \ \text{ a.s.}, \\ \ln y(t) \geq (\mathscr{B}_2 - \varepsilon)t - \overline{a_{22}}\int_0^t y(s)\mathrm{d}s \ \text{ a.s.} \end{cases} \tag{7}$$

The desired assertion follows from Lemma 2 in [15] and the arbitrariness of ε. □

Corollary 1. *Under* $(\mathbf{H_3})$*, the solutions to system (3) satisfy:*
(1) If $B_1(i) < 0$*,* $B_2(i) < 0$*, then* $\lim_{t\to+\infty} X(t) = 0$ *a.s.*
(2) If $B_1(i) < 0$*,* $B_2(i) > 0$*, then* $\lim_{t\to+\infty} x(t) = 0$*,* $\lim_{t\to+\infty} \langle y(t)\rangle = \frac{B_2(i)}{a_{22}(i)}$ *a.s.*
(3) If $B_1(i) > 0$*,* $B_2(i) < -a_{21}(i)$*, then* $\lim_{t\to+\infty} \langle x(t)\rangle = \frac{B_1(i)}{a_{11}(i)}$*,* $\lim_{t\to+\infty} y(t) = 0$ *a.s.*

4 Stochastic permanence

Definition 1 (see e.g. [3, 10])**.** *System (2) is stochastically permanent, if for any* $\varepsilon \in (0,1)$*, there exist* $\delta_* = \delta_*(\varepsilon) > 0$ *and* $\delta^* = \delta^*(\varepsilon) > 0$ *such that*

$$\liminf_{t\to+\infty} P\{|X(t)| \ge \delta_*\} \ge 1-\varepsilon, \quad \liminf_{t\to+\infty} P\{|X(t)| \le \delta^*\} \ge 1-\varepsilon. \tag{1}$$

Lemma 1. *Under* $(\mathbf{H_1})$*, let* $X(t)$ *be the solution to system (2) with* $X(0) \in \mathbb{R}^2_+$*. Then for any* $\theta_1 > 0$*,* $\theta_2 > 0$*, there exists* $K(\theta_1,\theta_2) > 0$ *such that*

$$\limsup_{t\to+\infty} \mathbb{E}\left[x^{\theta_1}(t) + y^{\theta_2}(t)\right] \le K(\theta_1,\theta_2). \tag{2}$$

Proof. Define $W(x,y) = x^{\theta_1} + y^{\theta_2}$. By Itô's formula, we have

$$\begin{aligned}
\mathscr{L}[W(x,y)] &= \theta_1 x^{\theta_1}\left[r_1(\rho(t)) - a_{11}(\rho(t))x - \tfrac{a_{12}(\rho(t))y}{1+x}\right] + \tfrac{\theta_1(\theta_1-1)\sigma_1^2(\rho(t))}{2} x^{\theta_1} \\
&\quad + x^{\theta_1}\int_{\mathbb{Z}} \left\{[1+\gamma_1(\mu,\rho(t))]^{\theta_1} - 1 - \theta_1\gamma_1(\mu,\rho(t))\right\}\lambda(\mathrm{d}\mu) \\
&\quad + \theta_2 y^{\theta_2}\left[r_2(\rho(t)) + \tfrac{a_{21}(\rho(t))x}{1+x} - a_{22}(\rho(t))y\right] + \tfrac{\theta_2(\theta_2-1)\sigma_2^2(\rho(t))}{2} y^{\theta_2} \\
&\quad + y^{\theta_2}\int_{\mathbb{Z}} \left\{[1+\gamma_2(\mu,\rho(t))]^{\theta_2} - 1 - \theta_2\gamma_2(\mu,\rho(t))\right\}\lambda(\mathrm{d}\mu) \\
&\le \theta_1 x^{\theta_1}\left[\overline{r_1} - \underline{a_{11}}x\right] + \theta_2 y^{\theta_2}\left[\overline{r_2} + \overline{a_{21}} - \underline{a_{22}}y\right] + \tfrac{\sigma^2}{2}\left(\theta_1^2 x^{\theta_1} + \theta_2^2 y^{\theta_2}\right) \\
&\quad + x^{\theta_1}\int_{\mathbb{Z}} \max_{i\in\mathbb{S}}\left\{[1+\gamma_1^*(i)]^{\theta_1} - 1 - \theta_1\gamma_{1*}(i)\right\}\lambda(\mathrm{d}\mu) \\
&\quad + y^{\theta_2}\int_{\mathbb{Z}} \max_{i\in\mathbb{S}}\left\{[1+\gamma_2^*(i)]^{\theta_2} - 1 - \theta_2\gamma_{2*}(i)\right\}\lambda(\mathrm{d}\mu).
\end{aligned} \tag{3}$$

From (3), there exists $K(\theta_1,\theta_2) > 0$ such that

$$\mathscr{L}[W(x,y)] + W(x,y) \le K(\theta_1,\theta_2). \tag{4}$$

In view of Itô's formula and (4), we derive

$$\mathscr{L}[\mathrm{e}^t W(x,y)] \le \mathrm{e}^t K(\theta_1,\theta_2). \tag{5}$$

Based on (5), integrating $\mathrm{d}[\mathrm{e}^t W(x(t),y(t))]$ from 0 to t and then taking the expectations of both sides yield

$$\mathrm{e}^t\mathbb{E}\left[x^{\theta_1}(t) + y^{\theta_2}(t)\right] \le x^{\theta_1}(0) + y^{\theta_2}(0) + K(\theta_1,\theta_2)\left(\mathrm{e}^t - 1\right), \tag{6}$$

which implies the required assertion (2). □

Let C be a vector or matrix and by $C \gg 0$ we mean all elements of C are positive. Also, let $Y^{S\times S} = \{C = (c_{ij})_{S\times S} : c_{ij} \le 0, i \ne j\}$.

Lemma 2 (Lemma 5.3 in [21]). *If $C = (c_{ij})_{S\times S} \in Y^{S\times S}$ has all of its rows' sum positive, that is, for each $i \in \mathbb{S}$, $\sum_{j=1}^{S} c_{ij} > 0$, then $det(C) > 0$.*

Lemma 3 (Theorem 2.10 in [21]). *If $C = (c_{ij})_{S\times S} \in Y^{S\times S}$, then the following statements are equivalent:*
(1) C is a nonsingular M-matrix;
(2) All principal minors of C are positive; that is

$$\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & \dots & c_{kk} \end{vmatrix} > 0, \quad for\ every\ \ k = 1,2,...,S. \tag{7}$$

(3) C is semi-positive; that is, there exists $x \gg 0$ in $\mathbb{R}^{S\times 1}$ such that $Cx \gg 0$.

Lemma 4. *Under ($\mathbf{H_2}$), if $\mathscr{B} > 0$, then there is $\theta_0 > 0$ such that for any $\theta \in (0, \theta_0)$,*

$$G(\theta) = diag\,(\nu_1(\theta), ..., \nu_S(\theta)) - \Gamma \tag{8}$$

is a nonsingular M-matrix, where

$$\begin{cases} \nu_i(\theta) = B(i)\theta - \frac{\sigma^2}{2}\theta^2 - \int_{\mathbb{Z}} \left[\Gamma^\theta(i) - 1 - \theta \ln\Gamma(i)\right] \lambda(\mathrm{d}\mu), \\ \ln\Gamma(i) = \max\left\{\left|\ln\left(1 + \min_{j=1,2}\{\gamma_{j*}(i)\}\right)\right|, \left|\ln\left(1 + \max_{j=1,2}\{\gamma_j^*(i)\}\right)\right|\right\}. \end{cases} \tag{9}$$

Proof. Without loss of generality, let $j = S$, that is

$$\gamma_{iS} > 0,\ \forall\, 1 \le i \le S-1. \tag{10}$$

From Appendix A in [20], $\mathscr{B} > 0$ is equivalent to

$$\begin{vmatrix} B(1) & -\gamma_{12} & \dots & -\gamma_{1S} \\ B(2) & -\gamma_{22} & \dots & -\gamma_{2S} \\ \vdots & \vdots & & \vdots \\ B(S) & -\gamma_{S2} & \dots & -\gamma_{SS} \end{vmatrix} > 0. \tag{11}$$

Compute

$$\det G(\theta) = \begin{vmatrix} \nu_1(\theta) & -\gamma_{12} & \dots & -\gamma_{1S} \\ \nu_2(\theta) & \nu_2(\theta) - \gamma_{22} & \dots & -\gamma_{2S} \\ \vdots & \vdots & & \vdots \\ \nu_S(\theta) & -\gamma_{S2} & \dots & \nu_S(\theta) - \gamma_{SS} \end{vmatrix} = \sum_{i=1}^{S} \nu_i(\theta) M_i(\theta), \tag{12}$$

where $M_i(\theta)$ is the corresponding minor of $\nu_i(\theta)$ in the first column. By (9),

$$\nu_i(0) = 0,\ \frac{\mathrm{d}\nu_i(\theta)}{\mathrm{d}\theta}|_{\theta=0} = B(i). \tag{13}$$

In view of (12) and (13), we obtain

$$\frac{\mathrm{d}\,(\det G(\theta))}{\mathrm{d}\theta}|_{\theta=0}=\sum_{i=1}^{S}B(i)M_i(0)=\begin{vmatrix}B(1) & -\gamma_{12} & \dots & -\gamma_{1S}\\ B(2) & -\gamma_{22} & \dots & -\gamma_{2S}\\ \vdots & \vdots & & \vdots\\ B(S) & -\gamma_{S2} & \dots & -\gamma_{SS}\end{vmatrix}. \tag{14}$$

Combining (11) with (14) yields

$$\frac{\mathrm{d}\,(\det G(\theta))}{\mathrm{d}\theta}|_{\theta=0}>0. \tag{15}$$

Based on (12) and (13), $\det G(0)=0$. Thanks to (10), (13) and (15), there is $\theta_0\in(0,1)$ such that for any $\theta\in(0,\theta_0)$, $\det G(\theta)>0$ and

$$\nu_i(\theta)>-\gamma_{iS},\ 1\le i\le S-1. \tag{16}$$

For each $k=1,2,\dots,S-1$, consider the leading principal sub-matrix

$$G_k(\theta):=\begin{vmatrix}\nu_1(\theta)-\gamma_{11} & -\gamma_{12} & \dots & -\gamma_{1k}\\ -\gamma_{21} & \nu_2(\theta)-\gamma_{22} & \dots & -\gamma_{2k}\\ \vdots & \vdots & & \vdots\\ -\gamma_{k1} & -\gamma_{k2} & \dots & \nu_k(\theta)-\gamma_{kk}\end{vmatrix}, \tag{17}$$

of $G(\theta)$. Clearly, $G_k(\theta)\in Y^{k\times k}$. According to (16), we derive

$$\nu_i(\theta)-\sum_{j=1}^{k}\gamma_{ij}=\nu_i(\theta)+\sum_{j=k+1}^{S}\gamma_{ij}\ge\nu_i(\theta)+\gamma_{iS}>0,\ i=1,2,\dots,k. \tag{18}$$

By Lemma 2, $\det G_k(\theta)>0$. That is to say, all the leading principal minors of $G(\theta)$ are positive. Hence, the required assertion follows from Lemma 3. □

Lemma 5. *Let $X(t)$ be the solution to system (2) with $X(0)\in\mathbb{R}^2_+$. If there exists $\theta>0$ such that $G(\theta)$ is a nonsingular M-matrix, then there is $H(\theta)>0$ such that*

$$\limsup_{t\to+\infty}\mathbb{E}\left[(x(t)+y(t))^{-\theta}\right]\le H(\theta). \tag{19}$$

Proof. By part (3) of Lemma 3, there exists $(p_1,\dots,p_S)^{\mathrm{T}}\gg 0$ such that

$$\nu_i(\theta)p_i-\sum_{j=1}^{S}\gamma_{ij}p_j>0,\ 1\le i\le S. \tag{20}$$

In view of (20), there exists a constant $\kappa>0$ such that

$$\nu_i(\theta)p_i-\sum_{j=1}^{S}\gamma_{ij}p_j-\kappa p_i>0,\ 1\le i\le S. \tag{21}$$

Define $U=V^{-1}=(x+y)^{-1}$ and $\widetilde{U}=p_i\left(1+U\right)^{\theta}$. Compute

$$\begin{aligned}
\kappa\widetilde{U}+\mathscr{L}\left[\widetilde{U}\right]=&\kappa p_i\left(1+U\right)^{\theta}+\sum_{j=1}^{S}\gamma_{ij}p_j\left(1+U\right)^{\theta}-p_i\theta\left(1+U\right)^{\theta-1}U\tfrac{r_1(\rho(t))x+r_2(\rho(t))y}{V}\\
&+p_i\theta\left(1+U\right)^{\theta-1}\tfrac{a_{11}(\rho(t))x^2+\frac{(a_{12}(\rho(t))-a_{21}(\rho(t)))xy}{1+x}+a_{22}(\rho(t))y^2}{V^2}\\
&+p_i\theta\left(1+U\right)^{\theta-1}U\left(\tfrac{\sigma_1(\rho(t))x+\sigma_2(\rho(t))y}{V}\right)^2\\
&+p_i\theta\left(1+U\right)^{\theta-1}U\int_{\mathbb{Z}}\tfrac{\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}{V}\lambda(\mathrm{d}\mu)\\
&+p_i\tfrac{\theta(\theta-1)}{2}\left(1+U\right)^{\theta-2}U^2\left(\tfrac{\sigma_1(\rho(t))x+\sigma_2(\rho(t))y}{V}\right)^2\\
&+p_i\int_{\mathbb{Z}}\left[\left(1+\tfrac{1}{V+\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}\right)^{\theta}-\left(1+U\right)^{\theta}\right]\lambda(\mathrm{d}\mu)\\
=&\mathscr{O}\left(U^{\theta}\right)U^{\theta}+F\left(U\right),
\end{aligned}\tag{22}$$

where $\lim\limits_{U\to+\infty}\dfrac{F\left(U\right)}{U^{\theta}}=0$. By Jensen's inequality, we have

$$\begin{aligned}
\mathscr{O}\left(U^{\theta}\right)=&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_i\theta\tfrac{r_1(\rho(t))x+r_2(\rho(t))y}{V}+p_i\theta\left(\tfrac{\sigma_1(\rho(t))x+\sigma_2(\rho(t))y}{V}\right)^2\\
&+p_i\theta\int_{\mathbb{Z}}\tfrac{\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}{V}\lambda(\mathrm{d}\mu)+p_i\tfrac{\theta(\theta-1)}{2}\left(\tfrac{\sigma_1(\rho(t))x+\sigma_2(\rho(t))y}{V}\right)^2\\
&+p_i\int_{\mathbb{Z}}\left[\left(\tfrac{V}{V+\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}\right)^{\theta}-1\right]\lambda(\mathrm{d}\mu)\\
\leq&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_i\theta B_1(\rho(t))\tfrac{x}{V}-p_i\theta B_2(\rho(t))\tfrac{y}{V}\\
&+p_i\tfrac{\theta^2\sigma^2}{2}+p_i\theta\int_{\mathbb{Z}}\left[\tfrac{x}{V}\ln(1+\gamma_1(\mu,\rho(t)))+\tfrac{y}{V}\ln(1+\gamma_2(\mu,\rho(t)))\right]\lambda(\mathrm{d}\mu)\\
&+p_i\int_{\mathbb{Z}}\left[\left(1+\tfrac{x}{V}\gamma_1(\mu,\rho(t))+\tfrac{y}{V}\gamma_2(\mu,\rho(t))\right)^{-\theta}-1\right]\lambda(\mathrm{d}\mu)\\
\leq&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_i\theta\min_{j=1,2}\left\{B_j(\rho(t))\right\}+p_i\tfrac{\theta^2\sigma^2}{2}\\
&+p_i\int_{\mathbb{Z}}\left[\left(1+\tfrac{\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}{V}\right)^{-\theta}-1+\theta\ln\left(1+\tfrac{\gamma_1(\mu,\rho(t))x+\gamma_2(\mu,\rho(t))y}{V}\right)\right]\lambda(\mathrm{d}\mu)\\
\leq&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_iB(i)\theta+p_i\tfrac{\theta^2\sigma^2}{2}+p_i\int_{\mathbb{Z}}\left[\Gamma^{\theta}(i)-1-\theta\ln\Gamma(i)\right]\lambda(\mathrm{d}\mu)\\
=&\kappa p_i+\sum_{j=1}^{S}\gamma_{ij}p_j-p_i\nu_i(\theta).
\end{aligned}\tag{23}$$

In view of (21), (22) and (23), there is $\mathscr{H}(\theta) > 0$ such that

$$\mathscr{L}\left[\mathrm{e}^{\kappa t}\widetilde{U}\right] = \mathrm{e}^{\kappa t}\left\{\kappa\widetilde{U} + \mathscr{L}\left[\widetilde{U}\right]\right\} \leq \mathscr{H}(\theta)\mathrm{e}^{\kappa t}. \tag{24}$$

Integrating $\mathrm{d}\left[\mathrm{e}^{\kappa t}\widetilde{U}(t)\right]$ from 0 to t and then taking expectations yield

$$\mathbb{E}\left[p_i \mathrm{e}^{\kappa t}(1+U(t))^{\theta}\right] - p_i(1+U(0))^{\theta} \leq \frac{\mathscr{H}(\theta)}{\kappa}\left(\mathrm{e}^{\kappa t}-1\right). \tag{25}$$

Based on (25), we deduce

$$\mathbb{E}\left[(1+U(t))^{\theta}\right] \leq \frac{\mathscr{H}(\theta)}{\kappa \min_{i\in\mathbb{S}}\{p_i\}} + \left(1+\frac{1}{x(0)+y(0)}\right)^{\theta}\mathrm{e}^{-\kappa t}. \tag{26}$$

Define $H(\theta) = \frac{\mathscr{H}(\theta)}{\kappa \min_{i\in\mathbb{S}}\{p_i\}}$, from (26) we obtain the desired assertion (19). □

Theorem 3. *Under* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$*, if* $\mathscr{B} > 0$*, then system (2) is stochastically permanent.*

Proof. Noting that $|X(t)|^{-\theta} \leq 2^{\frac{\theta}{2}}U^{\theta}(t)$, from Lemma 5, we deduce

$$\limsup_{t\to+\infty}\mathbb{E}\left[|X(t)|^{-\theta}\right] \leq 2^{\frac{\theta}{2}}H(\theta). \tag{27}$$

By Chebyshev's inequality, $\forall\varepsilon\in(0,1)$, there is $\delta_* = \frac{\sqrt{2}}{2}\left(\frac{\varepsilon}{H(\theta)}\right)^{\frac{1}{\theta}} > 0$ such that

$$\limsup_{t\to+\infty}P\{|X(t)|<\delta_*\} \leq (\delta_*)^{\theta}\limsup_{t\to+\infty}\mathbb{E}\left[|X(t)|^{-\theta}\right] \leq \varepsilon. \tag{28}$$

In other words,

$$\liminf_{t\to+\infty}P\{|X(t)|\geq\delta_*\} \geq 1-\varepsilon. \tag{29}$$

The remaining part of (1) follows from combining Lemma 1 with Chebyshev's inequality. Hence, system (2) is stochastically permanent. □

Corollary 2. *Under* $(\mathbf{H_3})$*, if* $B(i) > 0$*, then system (3) is stochastically permanent.*

5 Asymptotic properties

Theorem 4. *Under* $(\mathbf{H_1})$*, the solution* $X(t)$ *to system (2) with* $X(0)\in\mathbb{R}^2_+$ *satisfies*

$$\limsup_{t\to+\infty}\frac{\ln[x(t)+y(t)]}{\ln t} \leq 1 \ \ a.s. \tag{1}$$

Proof. For simplicity, denote

$$H_0 = \max_{i\in\mathbb{S}}\{r_1(i), r_2(i)+a_{21}(i)\}, \ \ \gamma^{\star} = \max_{j=1,2}\max_{i\in\mathbb{S}}\left\{|\gamma_{j*}(i)|, |\gamma_j^*(i)|\right\}. \tag{2}$$

From system (2), for $u > t$, we obtain

$$\begin{aligned} V(u) - V(t) \leq & \int_t^u [r_1(\rho(s))x(s) + [r_2(\rho(s)) + a_{21}(\rho(s))] y(s)] \,\mathrm{d}s \\ & + \int_t^u [\sigma_1(\rho(s))x(s) + \sigma_2(\rho(s))y(s)] \,\mathrm{d}B(s) \\ & + \int_t^u \int_{\mathbb{Z}} [\gamma_1(\mu,\rho(s))x(s) + \gamma_2(\mu,\rho(s))y(s)] \widetilde{N}(\mathrm{d}s,\mathrm{d}\mu). \end{aligned} \tag{3}$$

Based on (3), we deduce

$$\begin{aligned} \mathbb{E}\left[\sup_{t\leq u\leq t+1} V(u)\right] \leq & \mathbb{E}[V(t)] + H_0 \int_t^{t+1} \mathbb{E}[V(s)] \,\mathrm{d}s \\ & + \mathbb{E}\left[\sup_{t\leq u\leq t+1} \int_t^u [\sigma_1(\rho(s))x(s) + \sigma_2(\rho(s))y(s)] \,\mathrm{d}B(s)\right] \\ & + \mathbb{E}\left[\sup_{t\leq u\leq t+1} \int_t^u \int_{\mathbb{Z}} [\gamma_1(\mu,\rho(s))x(s) + \gamma_2(\mu,\rho(s))y(s)] \widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right]. \end{aligned} \tag{4}$$

By Burkholder-Davis-Gundy inequality and Young inequality, we derive

$$\begin{aligned} & \mathbb{E}\left[\sup_{t\leq u\leq t+1} \int_t^u [\sigma_1(\rho(s))x(s) + \sigma_2(\rho(s))y(s)] \,\mathrm{d}B(s)\right] \\ & \leq J\mathbb{E}\left(\int_t^{t+1} [\sigma_1(\rho(s))x(s) + \sigma_2(\rho(s))y(s)]^2 \,\mathrm{d}s\right)^{0.5} \\ & \leq \frac{1}{2}\mathbb{E}\left(\sup_{t\leq u\leq t+1} V(u)\right) + \frac{\sigma^2 J^2}{2} \int_t^{t+1} \mathbb{E}[V(s)] \,\mathrm{d}s. \end{aligned} \tag{5}$$

And

$$\begin{aligned} & \mathbb{E}\left[\sup_{t\leq u\leq t+1} \int_t^u \int_{\mathbb{Z}} [\gamma_1(\mu,\rho(s))x(s) + \gamma_2(\mu,\rho(s))y(s)] \widetilde{N}(\mathrm{d}s,\mathrm{d}\mu)\right] \\ & \leq J\mathbb{E}\left(\int_t^{t+1} \int_{\mathbb{Z}} (\gamma_1(\mu,\rho(s))x(s) + \gamma_2(\mu,\rho(s))y(s))^2 N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5} \\ & \leq J\mathbb{E}\left(\int_t^{t+1} \int_{\mathbb{Z}} [\gamma^\star (x(s) + y(s))]^2 N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5} \\ & \leq J\left(\mathbb{E}\int_t^{t+1} \int_{\mathbb{Z}} [\gamma^\star (x(s) + y(s))]^2 N(\mathrm{d}s,\mathrm{d}\mu)\right)^{0.5} \\ & = J\left(\int_{\mathbb{Z}} [\gamma^\star]^2 \lambda(\mathrm{d}\mu)\right)^{0.5} \left(\mathbb{E}\int_t^{t+1} [x(s) + y(s)]^2 \,\mathrm{d}s\right)^{0.5}. \end{aligned} \tag{6}$$

Substituting (5) and (6) into (4) yields

$$\begin{aligned} \mathbb{E}\left(\sup_{t\leq u\leq t+1} V(u)\right) \leq & 2\mathbb{E}[V(t)] + 2H_0 \int_t^{t+1} \mathbb{E}[V(s)] \,\mathrm{d}s + \sigma^2 J^2 \int_t^{t+1} \mathbb{E}[V(s)] \,\mathrm{d}s \\ & + 2J\left(\int_{\mathbb{Z}} [\gamma^\star]^2 \lambda(\mathrm{d}\mu)\right)^{0.5} \left(2\int_t^{t+1} \mathbb{E}\left[x^2(s) + y^2(s)\right] \mathrm{d}s\right)^{0.5}. \end{aligned} \tag{7}$$

From Lemma 1, there is $K^*(\theta_1,\theta_2)>0$ such that $\sup_{t\geq 0}\mathbb{E}\left[x^{\theta_1}(t)+y^{\theta_2}(t)\right]\leq K^*(\theta_1,\theta_2)$. According to (7), we deduce

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq u\leq t+1}V(u)\right)&\leq 2K^*(1,1)+2H_0K^*(1,1)+\sigma^2J^2K^*(1,1)\\&\quad+2J\left(2K^*(2,2)\int_{\mathbb{Z}}[\gamma^*]^2\lambda(\mathrm{d}\mu)\right)^{0.5}=:\widetilde{K}.\end{aligned}\tag{8}$$

Thanks to (8), we obtain

$$\mathbb{E}\left(\sup_{k\leq u\leq k+1}V(u)\right)\leq\widetilde{K},\ k=1,2,\ldots\tag{9}$$

From Chebyshev's inequality and (9), $\forall\varepsilon\in(0,1)$,

$$P\left(\omega:\sup_{k\leq t\leq k+1}V(t)>k^{1+\varepsilon}\right)\leq\tfrac{\widetilde{K}}{k^{1+\varepsilon}},\ k=1,2,\ldots\tag{10}$$

By Borel-Cantelli's lemma, there is $\Omega_o\in\mathscr{F}$ with $P(\Omega_o)=1$ and an integer-valued random variable k_o such that for any $\omega\in\Omega_o$, $\sup_{k\leq t\leq k+1}V(t)\leq k^{1+\varepsilon}$ holds whenever $k\geq k_o(\omega)$. Thus, for almost all $\omega\in\Omega$, if $k\geq k_o$ and $k\leq t\leq k+1$,

$$\frac{\ln V(t)}{\ln t}\leq\frac{\ln\left(\sup_{k\leq t\leq k+1}V(t)\right)}{\ln t}\leq\frac{\ln k^{1+\varepsilon}}{\ln t}\leq 1+\varepsilon.\tag{11}$$

So the desired assertion (1) follows from letting $\varepsilon\to 0^+$ in (11). □

6 An example and its numerical simulations

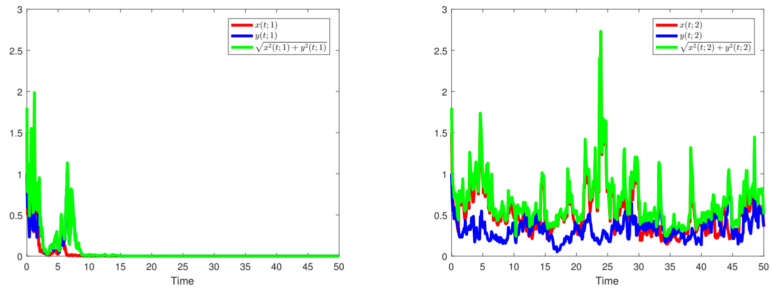
By the method in [6], an example and its numerical simulations are given to support the theoretical results. System (2) may be regarded as the result of regime switching between the following two subsystems:

$$\begin{cases}\mathrm{d}x(t)=x(t^-)\left[\left(1.5-x(t^-)-\frac{2y(t^-)}{1+x(t^-)}\right)\mathrm{d}t+2\mathrm{d}B(t)+\int_{\mathbb{Z}}0.5\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\right],\\ \mathrm{d}y(t)=y(t^-)\left[\left(0.5+\frac{x(t^-)}{1+x(t^-)}-2y(t^-)\right)\mathrm{d}t+1.5\mathrm{d}B(t)+\int_{\mathbb{Z}}0.5\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\right],\end{cases}\tag{1}$$

$$\begin{cases}\mathrm{d}x(t)=x(t^-)\left[\left(1.75-2x(t^-)-\frac{3y(t^-)}{1+x(t^-)}\right)\mathrm{d}t+0.5\mathrm{d}B(t)+\int_{\mathbb{Z}}0.5\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\right],\\ \mathrm{d}y(t)=y(t^-)\left[\left(0.75+\frac{x(t^-)}{1+x(t^-)}-3y(t^-)\right)\mathrm{d}t+0.5\mathrm{d}B(t)+\int_{\mathbb{Z}}0.5\widetilde{N}(\mathrm{d}t,\mathrm{d}\mu)\right].\end{cases}\tag{2}$$

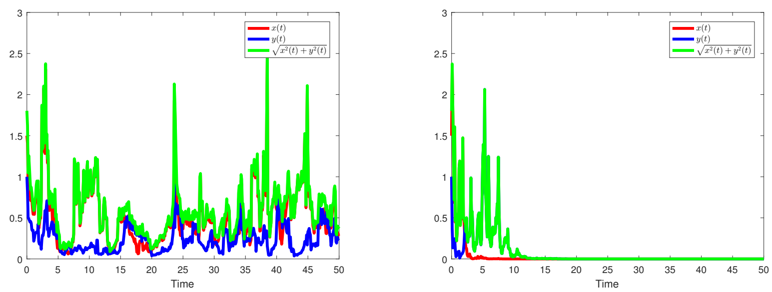
Here, $\lambda(\mathbb{Z})=1$ and

$$\begin{cases}r_1(1)=1.5,\ r_2(1)=0.5,\ a_{11}(1)=1,\ a_{12}(1)=2,\ a_{21}(1)=1,\ a_{22}(1)=2,\\ r_1(2)=1.75,\ r_2(2)=0.75,\ a_{11}(2)=2,\ a_{12}(2)=3,\ a_{21}(2)=1,\ a_{22}(2)=3,\\ \sigma_1(1)=2,\ \sigma_2(1)=1.5,\ \sigma_1(2)=0.5,\ \sigma_2(2)=0.5,\\ \gamma_1(\mu,1)=0.5,\ \gamma_2(\mu,1)=0.5,\ \gamma_1(\mu,2)=0.5,\ \gamma_2(\mu,2)=0.5.\end{cases}\tag{3}$$



(a) (b)

Figure 1: The solutions to subsystems (1) and (2).



(a) (b)

Figure 2: The left and right subfigures are, respectively, the numerical simulations of the solutions to system (2) for Case 1 and Case 2.

Based on (3), we compute

$$\begin{cases} B_1(1) = -1 + \ln\frac{3}{2}, & B_2(1) = -\frac{9}{8} + \ln\frac{3}{2}, \quad B(1) = -\frac{9}{8} + \ln\frac{3}{2}, \\ B_1(2) = \frac{9}{8} + \ln\frac{3}{2}, & B_2(2) = \frac{1}{8} + \ln\frac{3}{2}, \quad B(2) = \frac{1}{8} + \ln\frac{3}{2}. \end{cases} \tag{4}$$

From Corollary 1, system (1) is extinctive. By Corollary 2, system (2) is stochastically permanent.

Case 1. Let $\Gamma = \begin{pmatrix} -5 & 5 \\ 1 & -1 \end{pmatrix}$. Then $\pi = (\pi_1, \pi_2) = \left(\frac{1}{6}, \frac{5}{6}\right)$. Thus,

$$\mathscr{B} = -\frac{1}{12} + \ln\frac{3}{2} > 0. \tag{5}$$

According to Theorem 3, system (2) is stochastically permanent.

Case 2. Let $\Gamma = \begin{pmatrix} -1 & 1 \\ 9 & -9 \end{pmatrix}$. Then, $\pi = (\pi_1, \pi_2) = \left(\frac{9}{10}, \frac{1}{10}\right)$. Hence,

$$\mathscr{B}_1 = -\frac{63}{80} + \ln\frac{3}{2} < 0, \ \mathscr{B}_2 = -1 + \ln\frac{3}{2} < 0. \tag{6}$$

Based on Theorem 2, system (2) is extinctive.

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