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Lions's partial derivatives with respect to probability measures for general mean-field stochastic control problem

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Abstract. In this paper, a necessary stochastic maximum principle for stochastic model governed by mean-field nonlinear controlled Itô-stochastic differential equations is proved. The coefficients of our model are nonlinear and depend explicitly on the control variable, the state process as well as of its probability distribution. The control region is assumed to be bounded and convex. Our main result is derived by applying the Lions's partial-derivatives with respect to random measures in Wasserstein space. The associated Itô-formula and convex-variation approach are applied to establish the optimal control.

Keywords: Stochastic mean-field models, stochastic control, Lions's partial-derivatives with respect to measures, maximum principle, probability measure.
AMS Subject Classification 2010: 60H10, 93E20.

1 Introduction

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,\tau]}, P)$ be a fixed filtered probability space and τ be a fixed positive real number. In this paper, we study the following mean-field-type stochastic optimal nonlinear control problem:

Problem A. Minimize a mean-field cost functional

$$J(\alpha(\cdot)) = E\int_{\mathbb{R}^d} \Phi(\mathrm{y}_\alpha(\tau), \mu^{\mathrm{y}_\alpha(\tau)})\mu(d\mathrm{y}_\alpha),$$

subject to $\mathrm{y}_\alpha(\cdot)$ solution of the (MF-SDE): $t \in [0, \tau]$

$$\begin{cases} d\mathrm{y}_\alpha(t) = \int_{\mathbb{R}^d} \varphi\left(t, \mathrm{y}_\alpha(t), \mu^{\mathrm{y}_\alpha(t)}, \alpha(t)\right)\mu(\mathrm{dy}_\alpha)dt + \int_{\mathbb{R}^d} \psi\left(t, \mathrm{y}_\alpha(t), \mu^{\mathrm{y}_\alpha(t)}, \alpha(t)\right)\mu(\mathrm{dy}_\alpha)\mathrm{d}W(t), \\ \mathrm{y}_\alpha(0) = \mathrm{y}_0, \end{cases}$$

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where, $\alpha(\cdot)$ is the control variable valued in a convex bounded subset $\mathbb{U} \subset \mathbb{R}^k$, $\mathrm{y}_\alpha(\cdot)$ is the controlled state variable, $W(\cdot)$ is a standard Brownian motion, $\mu^{\mathrm{y}_\alpha(t)}$ is the distribution of $\mathrm{y}_\alpha(t)$ and Φ, φ and ψ are a given maps.

The mean-field control theory has found important applications and has become a powerful tool in many fields, such as mathematical finance, economics, and stochastic mean-field games, see Lasry and Lions [12], Buckdahn et al. [3], and Buckdahn et al. [4]. Under partial information, necessary maximum principle of optimality for MF-SDEs has been proved in Wang et al. [18]. Stochastic optimal control of mean-field jump-diffusion systems with delay has been studied by Meng and Shen [14]. Under partial information, the necessary and sufficient conditions for optimal continuous and singular controls for mean-field SDEs with Teugels martingales have been studied in Hafayed et al. [8, 9]. Necessary conditions for mean-field FBSDEs have been studied by Hafayed et al. [10]. The general maximum principle for MF-SDEs has been established in Buckdahn et al. [2]. Mean-field game has been studied by Lions [13]. The convex maximum principle for mean-field delay SDE has been investigated in Shen et al. [17]. General maximum principle for optimal stochastic control has been established in Peng [15]. A Peng's type maximum principle for SDEs of mean-field type was proved by Buckdahn et al. [3]. A partial-derivative with respect to the measure and its application to general controlled mean-field systems have been investigated in Buckdahn et al. [4]. Forward-backward stochastic differential equations (FBSDs) and controlled McKean-Vlasov dynamics have been investigated in Carmona and Delarue [6]. Linear quadratic optimal control problem for conditional mean-field equation with random coefficients with applications has been investigated by Pham [16]. Necessary maximum principle for optimal continuous-singular control problem for general MF-SDEs, under convexity assumptions have been investigated by Hafayed et al. [11]. Second-order necessary maximum principle for MF-SDEs has been proved in Boukaf et al. [1].

In this paper, we apply the Lions's partial-derivatives with respect to probability measure to establish our maximum principle. This approach introduced by Lions [13] and later detailed in Buckdahn et al. [3, 4], Cardaliaguet [5] and Guo et al. [7]. Motivated by the recent works above, in this paper we derive the necessary maximum principle for our mean-field optimal control problem (6) -(7) Ṫhe Lions's partial-derivatives with respect to probability measure in *Wasserstein space* and the associated Itô-formula with some appropriate estimates are applied to prove our result. This approach of derivatives over Wasserstein space has turned out to be crucial in the study of our maximum principle. Our stochastic mean-field model occur naturally in the probabilistic models of financial optimization problems. Our control problem is strongly motivated by the recent study of the McKean-Vlasov games and the related McKean-Vlasov control problem.

The rest of the paper is organized as follows. The formulation of the partial derivatives with respect to probability measures, and basic notations are given in Section 2. The formulation of the control problem is given in Section 3. In Section 4 we prove our main results. Finally, to illustrate our theoretical result, we give an example in the last section.

2 Lions's partial-derivatives with respect to probability measure

We now recall briefly an important notion in mean-field control problems: The Lions's partial derivatives with respect to probability measures over *Wasserstein space* which was introduced by P. Lions [13], see also Cardaliaguet [5], and Guo et al. [7] and the recent references therein.

Throughout this paper, we let $\mathbb{K}_2(\mathbb{R}^n)$ be *Wasserstein space of probability measures* on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ with finite second-moment, i.e; $\int_{\mathbb{R}^n} |\mathrm{y}|^2 \mu(\mathrm{dy}) < \infty$, endowed with the following *Wasserstein metric*: for $\mu_1, \mu_2 \in \mathbb{K}_2(\mathbb{R}^n)$,

$$\mathbb{T}(\mu_1, \mu_2) = \inf_{\rho(\cdot,\cdot) \in \mathbb{K}_2(\mathbb{R}^{2n})} \left[\int_{\mathbb{R}^{2n}} |\mathrm{x}-\mathrm{y}|^2 \rho(\mathrm{dx}, \mathrm{dy}) \right]^{\frac{1}{2}}, \tag{1}$$

where $\rho(\cdot, \mathbb{R}^n) = \mu_1$, and $\rho(\mathbb{R}^n, \cdot) = \mu_2$. This metric is just the *Monge-Kankorovich metric* (with $p = 2$). Moreover, it has been shown that $(\mathbb{K}_2(\mathbb{R}^n), \mathbb{T}(\cdot,\cdot))$ is a complete metric space. For example, if $\mu_1 = \delta_{x_1}$ and $\mu_2 = \delta_{x_2}$ be two degenerate Dirac measures located at points x_1 and x_2 (respect.,) in $\mathbb{R}$, then we have

$$\mathbb{T}(\mu_1, \mu_2) = |x_1 - x_2|.$$

The main idea in Lions's partial-derivatives is to identify a distribution (measure of probability) $\mu \in \mathbb{K}_2(\mathbb{R}^n)$ with a random variable $\mathrm{y}(\cdot) \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$ so that $\mu = P_{\mathrm{y}}$ is the law of $\mathrm{y}(\cdot)$. We assume that probability space $(\Omega, \mathscr{F}, P)$ is *rich-enough* in the sense that for every $\mu \in \mathbb{K}_2(\mathbb{R}^n)$, there is a random variable $\mathrm{y}(\cdot) \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$ such that $\mu = P_{\mathrm{y}}$. We suppose that there is a sub-$\sigma-$field $\mathscr{G}_0 \subset \mathscr{F}$ such that $\mathscr{G}_0$ is *rich-enough i.e,*

$$\mathbb{K}_2(\mathbb{R}^n) := \left\{ \mu^{\mathrm{y}} = P_{\mathrm{y}} : \mathrm{y}(\cdot) \in \mathbb{L}^2(\mathscr{G}_0, \mathbb{R}^n) \right\}. \tag{2}$$

By $\mathscr{F} = (\mathscr{F}_t)_{t \in [0,\tau]}$, we denote the filtration generated by $W(\cdot)$, completed and augmented by $\mathscr{G}_0$. Next, for any function $f : \mathbb{K}_2(\mathbb{R}^n) \to \mathbb{R}$ we define a function $\widetilde{f} : \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n) \to \mathbb{R}$ such that

$$\widetilde{f}(\mathrm{y}) = f(\mu^{\mathrm{y}}) = f(P_{\mathrm{y}}),\ \mathrm{y}(\cdot) \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n). \tag{3}$$

Clearly, the function $\widetilde{f}$, called the *lift-function* of f, depends only on the law of $\mathrm{y} \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$ and is independent of the choice of the representative y.

Definition 1. *Let $g : \mathbb{K}_2(\mathbb{R}^n) \to \mathbb{R}$. The function g is differentiable at a distribution $\mu_0 \in \mathbb{K}_2(\mathbb{R}^n)$ if there exists $\mathrm{y}_0 \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$, with $\mu^{\mathrm{y}_0} = P_{\mathrm{y}_0}$ such that its lift $\widetilde{g}$ is* Fréchet-differentiable *at* y_0. *More precisely, there exists a continuous linear functional $\mathscr{D}\widetilde{g}(\mathrm{y}_0) : \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n) \to \mathbb{R}$ such that*

$$\widetilde{g}(\mathrm{y}_0 + \zeta) - \widetilde{g}(\mathrm{y}_0) = \langle \mathscr{D}\widetilde{g}(\mathrm{y}_0) \cdot \zeta \rangle + o(\|\zeta\|_2) = \mathscr{D}_\zeta g(\mu^{\mathrm{y}_0}) + o(\|\zeta\|_2), \tag{4}$$

where $\langle \cdot \cdot \rangle$ is the dual product on $\mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$. We call $\mathscr{D}_\zeta g(\mu^{\mathrm{y}_0})$ the Fréchet-derivative *of g at μ_0 in the direction ζ. In this case, we have*

$$\mathscr{D}_\zeta g(\mu^{\mathrm{y}_0}) = \langle \mathscr{D}\widetilde{g}(\mathrm{y}_0) \cdot \zeta \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{g}(\mathrm{y}_0 + t\zeta) \Big|_{t=0},\ \text{with } \mu^{\mathrm{y}_0} = P_{\mathrm{y}_0}. \tag{5}$$

Now, from *Riesz representation theorem*, there exists a unique random variable $\psi_0 \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$ such that $\langle \mathscr{D}\widetilde{g}(\mathrm{y}_0) \cdot \zeta \rangle = (\psi_0 \psi_0 \cdot \zeta)_2 = E\left[(\psi_0 \cdot \zeta)_2 \right]$ where $\zeta \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$. It was shown in [3] that there exists a Borel function $\Psi[\mu^{\mathrm{y}_0}](\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, depending only on the law $\mu^{\mathrm{y}_0} = P_{\mathrm{y}_0}$ but not on the choice of the representative y_0 such that $\psi_0 = \Psi[\mu^{\mathrm{y}_0}](\mathrm{y}_0)$. Thus we can write (4) as: for any $\mathrm{y} \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$, we have

$$g(\mu^{\mathrm{y}}) - g(\mu^{\mathrm{y}_0}) = (\Psi[\mu^{\mathrm{y}_0}](\mathrm{y}_0) \cdot \mathrm{y} - \mathrm{y}_0)_2 + o(\|\mathrm{y} - \mathrm{y}_0\|_2).$$

We denote $\partial_\mu g(\mu^{\mathrm{y}_0}, \mathrm{y}) = \Psi[\mu^{\mathrm{y}_0}](\mathrm{y})$, $\mathrm{y} \in \mathbb{R}^n$. Moreover, we have the following identities

$$\mathscr{D}\widetilde{g}(\mathrm{y}_0) = \psi_0 = \Psi[\mu^{\mathrm{y}_0}](\mathrm{y}_0) = \partial_\mu g(\mu^{\mathrm{y}_0}, \mathrm{y}_0),$$

and $\mathrm{D}_\zeta g(\mu^{\mathrm{y}_0}) = \left\langle \partial_\mu g(\mu^{\mathrm{y}_0}, \mathrm{y}_0) \cdot \zeta \right\rangle$, where $\zeta = (\mathrm{y} - \mathrm{y}_0)$.

Remark 1. *(1) For each* $\mu \in \mathbb{K}_2(\mathbb{R}^n)$ *, the partial derivatives* $\partial_\mu g(\mu^{\mathrm{y}}, \cdot) = \Psi[\mu^{\mathrm{y}}](\cdot)$ *are only defined in* $\mu(\mathrm{dy}) - a.e.$ *sense.*

(2) A function f *is said to be differentiable at* $\mu_0 \in \mathbb{K}_2(\mathbb{R}^n)$ *if there exists a random variable* y_0 *with law* μ_0 *such that the lift function* $\widetilde{f}$ *is Fréchet differentiable at* y_0*.*

Definition 2. *We say that the function* $g \in \mathbb{C}_b^{1,1}(\mathbb{K}_2(\mathbb{R}^n))$ *if for all* $\mathrm{y} \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$ *there exists a* $P_{\mathrm{y}}-$*modification of* $\partial_\mu g(\mu^{\mathrm{y}}, \cdot)$ *(denoted by* $\partial_\mu g$*) such that* $\partial_\mu g : \mathbb{K}_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ *is bounded and Lipschitz continuous. That is for some* $C > 0$*, it holds that*
(1) $\left|\partial_\mu g(\mu, \mathrm{y})\right| \leq C, \forall \mu \in \mathbb{K}_2(\mathbb{R}^n), \forall \mathrm{y} \in \mathbb{R}^n.$
(2) $\left|\partial_\mu g(\mu, \mathrm{y}) - \partial_\mu g(\mu', \mathrm{y}')\right| \leq C[\mathbb{T}(\mu, \mu') + |\mathrm{y} - \mathrm{y}'|], \forall \mu, \mu' \in \mathbb{K}_2(\mathbb{R}^n), \forall \mathrm{y}, \mathrm{y}' \in \mathbb{R}^n.$

We should note that if the function $g \in \mathbb{C}_b^{1,1}(\mathbb{K}_2(\mathbb{R}^n))$, the version of $\partial_\mu g(\mu^{\mathrm{y}}, \cdot)$, $\mathrm{y} \in \mathbb{L}^2(\mathscr{F}, \mathbb{R}^n)$, presented in Definition 2 is unique (see [3, Remark 2.2], and [5]). We shall denote by $\partial_\mu g(t, \mathrm{y}, \mu_0)$ the derivative with respect to μ computed at μ_0 whenever all the other variables (t, y) are held fixed, $\partial_\mu g(t, \mathrm{y}, \mu_0) = \partial_\mu g(t, \mathrm{y}, \mu)\big|_{\mu=\mu_0}$ $\mu(\mathrm{dy}) - a.e.$.

Throughout this paper, we will use the following notations, for $\psi = f, h$: $\psi_{\mathrm{y}}(t) = \frac{\partial \psi}{\partial \mathrm{y}}(t, \mathrm{y}^*(t), \mu^*, \alpha^*(t))$, $\psi_\alpha(t) = \frac{\partial \psi}{\partial \alpha}(t, \mathrm{y}^*(t), \mu^*, \alpha^*(t))$, and $\widehat{\psi}_\mu(t) = \partial_\mu \psi(t, \mathrm{y}(t), \mu, \alpha(t); \widehat{\mathrm{y}}(t))$, $\mu(\mathrm{dy}) - a.e.$.

3 Formulation of the mean-field control problem

Let $\tau > 0$ be a fixed positive real number and $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,\tau]}, P)$ be a fixed filtered probability space satisfying the usual conditions in which *one*−dimensional Brownian motion $W(t) = \{W(t) : 0 \leq t \leq \tau\}$ and $W(0) = 0$ is defined. We study optimal solutions of stochastic control problem driven by controlled mean-field model:

$$\begin{cases} d\mathrm{y}(t) = \int_{\mathbb{R}^d} \varphi\left(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t)\right) \mu(dy)dt + \int_{\mathbb{R}^d} \psi\left(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t)\right) \mu(dy)\mathrm{d}W(t),\ t \in [0, \tau] \\ \mathrm{y}(0) = \mathrm{y}_0, \end{cases} \tag{6}$$

where $\mu^{\mathrm{y}(t)} = P_{\mathrm{y}(t)}$ is the probability distribution of $\mathrm{y}(t)$. The goal of our mean-field optimal control problem is to minimize the following cost functional

$$J(\alpha(\cdot)) = E\int_{\mathbb{R}^d} \Phi(\mathrm{y}(\tau), \mu^{\mathrm{y}(\tau)})\mu(dy), \tag{7}$$

where

$$\begin{aligned} \varphi &: [0, \tau] \times \mathbb{R}^n \times \mathbb{K}_2(\mathbb{R}^n) \times \mathbb{U} \to \mathbb{R}^n, \\ \psi &: [0, \tau] \times \mathbb{R}^n \times \mathbb{K}_2(\mathbb{R}^n) \times \mathbb{U} \to \mathbb{R}^n, \\ \Phi &: \mathbb{R}^n \times \mathbb{K}_2(\mathbb{R}^n) \to \mathbb{R}, \end{aligned}$$

are given deterministic functions.

An admissible control $\alpha(\cdot)$ is an $\mathscr{F}_t-$predictable process with values in some nonempty convex subset $\mathbb{U}$ of $\mathbb{R}^k$ such that $E\int_0^\tau |\alpha(t)|^2 \mathrm{d}t < \infty$. We call $\mathbb{U}$ the control domain and denote by $\mathscr{U}([0, \tau])$

the set of all admissible controls. We assume that an optimal control exists. Any admissible control $\alpha^*(\cdot) \in \mathscr{U}([0,\tau])$ satisfying

$$J(\alpha^*(\cdot)) = \inf_{\alpha(\cdot)\in\mathscr{U}([0,\tau])} J(\alpha(\cdot)), \tag{8}$$

is called an optimal control. The maps

$$\begin{aligned} f(t,\mu,\alpha) &= \int_{\mathbb{R}^d} \varphi\left(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t)\right)\mu(dx), \\ \sigma(t,\mu,\alpha) &= \int_{\mathbb{R}^d} \psi\left(t, \mathrm{y}(t), \mu^{\mathrm{y}(t)}, \alpha(t)\right)\mu(dx), \\ h(\mu) &= \int_{\mathbb{R}^d} \Phi\left(\mathrm{y}(\tau), \mu^{\mathrm{y}(\tau)}\right)\mu(dx), \end{aligned}$$

are given deterministic functions such that

$$\begin{aligned} f &: [0,\tau]\times\mathbb{K}_2(\mathbb{R}^n)\times\mathbb{U} \to \mathbb{R}^n, \\ \sigma &: [0,\tau]\times\mathbb{K}_2(\mathbb{R}^n)\times\mathbb{U} \to \mathbb{R}^{n\times d}, \\ h &: \mathbb{K}_2(\mathbb{R}^n) \to \mathbb{R}. \end{aligned}$$

To avoid excessive complexity in the notations, we will make the simplifying assumption that all processes are 1-dimensional (i.e., $n = m = 1$) in the subsequent sections.

We define a metric $d(\cdot,\cdot)$ on the space of admissible controls $\mathscr{U}([0,\tau])$ such that $(\mathscr{U}([0,\tau]), d)$ becomes a complete metric space. For any $\alpha(\cdot)$ and $\alpha'(\cdot) \in \mathscr{U}([0,\tau])$ we set

$$d\left(\alpha(\cdot),\alpha'(\cdot)\right) = \left[E\int_0^\tau \left|\alpha(t)-\alpha'(t)\right|^2 dt\right]^{\frac{1}{2}}. \tag{9}$$

The following assumptions will be in force throughout this paper, where y denotes the state variable, and α the control variable.

- **Assumption (H1)** The control region is assumed to be bounded and convex.
- **Assumption (H2)** For fixed measure $\mu \in \mathbb{K}_2(\mathbb{R})$, for any $(\mathrm{y},\alpha) \in \mathbb{R}^d\times\mathbb{U}$, the functions φ, ψ are measurable in all variables and continuously differentiable with respect to y,α; and their partial derivatives are uniformly bounded.

  The function Φ is continuously differentiable with respect to y . Moreover $|\Phi(\mathrm{y})| \le C(1+|\mathrm{y}|^2)$, and $\left|\Phi_{\mathrm{y}}(\mathrm{y})\right| \le C(1+|\mathrm{y}|)$, where $C > 0$ is a generic positive constant, which may vary from line to line.
- **Assumption (H3)** (1) For a fixed $\mathrm{y} \in \mathbb{R}$, for all $\alpha(t) \in \mathbb{U}$: $\varphi,\psi \in \mathbb{C}_b^{1,1}(\mathbb{K}_2(\mathbb{R}^d);\mathbb{R})$ and $\Phi \in \mathbb{C}_b^{1,1}(\mathbb{K}_2(\mathbb{R});\mathbb{R})$.

  (2) All derivatives with respect to measure φ_μ, ψ_μ are bounded and Lipschitz continuous, with Lipschitz constants independent of α.

Under Assumptions (H2) and (H3), for each $\alpha(\cdot) \in \mathscr{U}([0,\tau])$, Eq. (6) has a unique strong solution $\mathrm{y}(\cdot)$ given by

$$\mathrm{y}(t) = \mathrm{y}_0 + \int_0^t \int_{\mathbb{R}^d} \varphi\left(s, \mathrm{y}(s), \mu^{\mathrm{y}(s)}, \alpha(s)\right) \mu(\mathrm{dy})\mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \psi\left(s, \mathrm{y}(s), \mu^{\mathrm{y}(s)}, \alpha(s)\right) \mu(\mathrm{dy})\mathrm{d}W(s),$$

such that $E\left[\sup_{t\in[0,\tau]} |\mathrm{y}(t)|^2\right] < \infty$, and the functional $J(\cdot)$ is well defined.

Let $\alpha^*(\cdot) \in \mathscr{U}([0,\tau])$ be an optimal control for the *problem A*, and $\mathrm{y}^*(\cdot) = \mathrm{y}^{\alpha^*}(\cdot)$ be the corresponding optimal state process.

Hamiltonian. Let us define the Hamiltonian associated to our control problem. For any $(t, \mathrm{y}, \mu, \alpha, p, q) \in [0,\tau] \times \mathbb{R} \times \mathbb{K}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$H(t, \mathrm{y}, \mu, \alpha, p(t), q(t)) = p(t) \int_{\mathbb{R}^d} \varphi(t, \mathrm{y}, \mu^{\mathrm{y}(t)}, \alpha)\mu(\mathrm{dy}) + q(t) \int_{\mathbb{R}^d} \psi(t, \mathrm{y}, \mu^{\mathrm{y}(t)}, \alpha)\mu(\mathrm{dy}), \tag{10}$$

where $(p(\cdot), q(\cdot))$ is a pair of adapted processes. The derivatives of H with respect to control variable $\alpha(\cdot)$ has the form

$$\begin{aligned} &\frac{\partial H}{\partial \alpha}(t, \mathrm{y}^*(t), \mu^{\mathrm{y}(t)}, \alpha^*(t), p(t), q(t)) \\ &\quad = \int_{\mathbb{R}^d} \varphi_\alpha\left(t, \mathrm{y}, \mu^{\mathrm{y}(t)}, \alpha\right) p(t)\mu(\mathrm{dy}) + \int_{\mathbb{R}^d} \psi_\alpha\left(t, \mathrm{y}, \mu^{\mathrm{y}(t)}, \alpha\right) q(t)\mu(\mathrm{dy}). \end{aligned} \tag{11}$$

Adjoint equation: We consider the new adjoint equation, which is the following MF-BSDE:

$$\begin{cases} \mathrm{d}p(t) = -\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}(t, \mathrm{y}, \mu, \alpha)\,\widehat{p}(t) + \int_{\mathbb{R}^d} \partial_\mu \widehat{\varphi}(t, \mathrm{y}, \mu, \alpha)\,\widehat{p}(t)\mu(\mathrm{dy})\right. \\ \qquad\qquad \left. + \,\partial_{\mathrm{y}}\widehat{\psi}(t, \mathrm{y}, \mu, \alpha)\,\widehat{q}(t) + \int_{\mathbb{R}^d} \partial_\mu \widehat{\psi}(t, \mathrm{y}, \mu, \alpha)\,\widehat{q}(t)\mu(\mathrm{dy})\right)\mathrm{d}t + q(t)\mathrm{d}W(t), \\ p(\tau) = -\widehat{E}\left[\partial_{\mathrm{y}}\widehat{\Phi}(\mathrm{y}, \mu, \alpha) + \int_{\mathbb{R}^d} \partial_\mu \widehat{\Phi}(\mathrm{y}, \mu, \alpha)\,\mu(\mathrm{dy})\right]. \end{cases} \tag{12}$$

Here, for $t \in [0,\tau]$, we have

$$\begin{aligned} \widehat{E}(\widehat{\varphi}_\mu(t)) &= \widehat{E}\left[\partial_\mu \widehat{\varphi}(t, \widehat{\mathrm{y}}^*(t), \mu^{\mathrm{y}^*(t)}, \widehat{\alpha}^*(t); z)\right]\Big|_{z=\mathrm{y}^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \varphi(t, \widehat{\mathrm{y}}^*(t,\widehat{w}), P_{\mathrm{y}^*(t,w)}, \widehat{\alpha}^*(t,\widehat{w}); \mathrm{y}^*(t,w))\mathrm{d}\widehat{P}(\widehat{w}), \end{aligned} \tag{13}$$

$$\begin{aligned} \widehat{E}(\widehat{\psi}_\mu(t)) &= E_{\widehat{P}}(\widehat{\psi}_\mu(t)) = E_{\widehat{P}}\left[\partial_\mu \widehat{\psi}(t, \widehat{\mathrm{y}}^*(t), \mu^{\mathrm{y}^*(t)}, \widehat{\alpha}^*(t); z)\right]\Big|_{z=\mathrm{y}^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \psi(t, \widehat{\mathrm{y}}^*(t,\widehat{w}), P_{\mathrm{y}^*(t,w)}, \widehat{\alpha}^*(t,\widehat{w}); \mathrm{y}^*(t,w))\mathrm{d}\widehat{P}(\widehat{w}). \end{aligned} \tag{14}$$

Similarly, we get

$$\begin{aligned} \widehat{E}(\widehat{\Phi}_\mu(\tau)) &= E_{\widehat{P}}(\widehat{\Phi}_\mu(\tau)) = E_{\widehat{P}}\left[\partial_\mu \Phi(\widehat{\mathrm{y}}^*(\tau), P_{\mathrm{y}^*(\tau)}; z)\right]\Big|_{z=\mathrm{y}^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \Phi(\widehat{\mathrm{y}}^*(\tau,\widehat{w}), P_{\mathrm{y}(\tau,w)}; \mathrm{y}^*(\tau,w))\mathrm{d}\widehat{P}(\widehat{w}). \end{aligned} \tag{15}$$

Under the Assumptions (H2) and (H3), the mean-field BSDE (12) admits a unique $\mathscr{F}_t$-adapted strong solution $(p(\cdot), q(\cdot))$ such that

$$E(\sup_{t\in[0,\tau]} |p(t)|^2 + \int_0^\tau |q(t)|^2 \mathrm{d}t) < \infty.$$

See Guo et al. [7] for some examples and different models of derivatives with respect to probability measures.

4 Main results

4.1 Maximum principle

In this paper, our purpose is to derive mean-field-type necessary maximum principle for the optimal control, where the dynamic is driven by controlled mean-field model (6). To establish our necessary optimality conditions, we apply the convex perturbation method of the optimal control. This perturbation method is described as follows: Let $\alpha^*(\cdot)$ be an optimal control and $\alpha(\cdot)$ be an arbitrary element of $\mathscr{F}_t-$measurable random variable with values in convex bounded set $\mathbb{U}$ which we consider as fixed from now on. We define a perturbed control $\alpha^\theta(\cdot)$ as follows. Let

$$\alpha^\theta(t) = \alpha^*(t) + \theta\left(\alpha(t) - \alpha^*(t)\right), \tag{16}$$

where $\theta > 0$ is sufficiently small. Since the control region $\mathbb{U}$ is convex, $\alpha^\theta(\cdot) \in \mathscr{U}([0,\tau])$. We denote by $\mathrm{y}^\theta(\cdot)$ the solution of Eq-(6) associated with $\alpha^\theta(\cdot)$.

Under Assumptions (H1), (H2) and (H3), we introduce the following new variational equation for our control problem.

Variational equation: Let $t \in [0,\tau]$, and $v(t) = \alpha(t) - \alpha^*(t)$. Then, we define the variational equation as follows:

$$\begin{cases} \mathrm{dZ}(t) = \Big[\widehat{E}\Big(\partial_{\mathrm{y}}\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t) + \int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\Big) \\ \qquad +\varphi_\alpha(t,\mathrm{y},\mu,\alpha)v(t) + \int_{\mathbb{R}^d}\varphi_\alpha(t,\mathrm{y},\mu,\alpha)v(t)\mu(\mathrm{dy})\Big]\mathrm{d}t \\ \qquad + \Big[\widehat{E}\Big(\partial_{\mathrm{y}}\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t) + \int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\Big) \\ \qquad +\psi_\alpha(t,\mathrm{y},\mu,\alpha)v(t) + \int_{\mathbb{R}^d}\psi_\alpha(t,\mathrm{y},\mu,\alpha)v(t)\mu(\mathrm{dy})\Big]\mathrm{d}W(t), \\ \mathrm{Z}(0) = 0. \end{cases} \tag{17}$$

Here the process $\mathrm{Z}(\cdot)$ is called the *first-order variational process* associated to $\alpha(\cdot)$. Since the derivatives in (17) are bounded, it follows that there exists a unique solution $\mathrm{Z}(\cdot)$ such that

$$E\left[\sup_{t\in[0,\tau]}|\mathrm{Z}(t)|^k\right] < C_k, \quad \text{for} \ \ k \geq 2. \tag{18}$$

where $C_k > 0$ is a generic positive constant depending only on k, which may vary from line to line.

We shall establish some fundamental estimates that will play the crucial roles for the proof of our stochastic maximum principle.

Our aim in this section is to establish a stochastic maximum principle for optimal stochastic control for systems driven by nonlinear controlled SDEs. Since the control domain is assumed to be convex, the proof of our result is based on convex perturbation method. Now, the main result of this paper is stated in the following theorem.

Theorem 1. *(Maximum principle in integral form via Lions's derivative). Let Assumptions (H1), (H2) and (H3) hold. Then there exists a unique pair of $\mathscr{F}_t-$adapted processes$(p(\cdot),q(\cdot))$ solution of the mean-field BSDE (12) such that for all $\alpha \in \mathbb{U}$*

$$E\int_0^\tau \frac{\partial H}{\partial \alpha}(t,\mathrm{y}^*(t),\mu^{\mathrm{y}^*(t)},\alpha^*(t),p(t),q(t))\,(\alpha(t)-\alpha^*(t))\,\mathrm{d}t \geq 0. \tag{19}$$

Corollary 1. *Under assumptions of Theorem 4.1, there exists a unique pair of $\mathscr{F}_t-$ adapted processes $(p(\cdot),q(\cdot))$ solution of mean-field BSDE-(12) such that for all $\alpha \in \mathbb{U}$*

$$\begin{aligned}\frac{\partial H}{\partial \alpha}(t,\mathrm{y}^*(t),\mu^{\mathrm{y}(t)},\alpha^*(t),p(t),q(t))\,(\alpha(t)-\alpha^*(t))\,\mathrm{d}t &\geq 0.\\ P\text{-}a.s.,\ a.e.\ t &\in [0,\tau].\end{aligned}$$

To prove Theorem 1 we need the following results.

4.2 Proof of main result

Let $(\alpha^*(\cdot),\mathrm{y}^*(\cdot))$ be the optimal solution of the control problem *(6)-(7).* We derive the variational inequality from:

$$J\left(\alpha^\theta(\cdot)\right) \geq J(\alpha^*(\cdot)), \tag{20}$$

where $\alpha^\theta(\cdot)$ is the so called convex-perturbation of $\alpha^*(\cdot)$ defined as follows: $\forall s \in [0,\tau]$

$$\alpha^\theta(s) = \alpha^*(s) + \theta(\alpha(s)-\alpha^*(s)), \tag{21}$$

where $\theta > 0$ is sufficiently small and $\alpha(s) \in \mathbb{U}$ is an element of $\mathscr{U}([0,\tau])$.

Proposition 1. *Let $\mathrm{y}^\theta(\cdot)$ and $\mathrm{y}^*(\cdot)$ be the states of (22) corresponding to $\alpha^\theta(\cdot)$ and $\alpha^*(\cdot)$, respectively. Also, let $\mathrm{Z}(\cdot)$ be the solution of (17). Then we have*

$$\lim_{\theta\to 0} E\left[\sup_{s\in[0,\tau]}\left|\mathrm{y}^\theta(s)-\mathrm{y}^*(s)\right|^{2k}\right] = 0, \tag{22}$$

$$\lim_{\theta\to 0} E\left[\sup_{s\leq\tau}\left|\theta^{-1}\left[\mathrm{y}^\theta(s)-\mathrm{y}^*(s)\right]-\mathrm{Z}(s)\right|^2\right] = 0. \tag{23}$$

Proof. By using Proposition 2 and estimate (4.8) in [3], we have

$$E\left[\sup_{s\in[0,\tau]}\left|\mathrm{y}^\theta(s)-\mathrm{y}^*(s)\right|^{2k}\right] \leq C_k\theta^k,$$

Then the proof of estimate (22) follows immediately by letting $\theta \to 0$. Let us turn to prove estimate (23). We consider

$$\gamma^{\theta}(s) = \theta^{-1}\left[\mathrm{y}^{\theta}(s) - \mathrm{y}^{*}(s)\right] - \mathrm{Z}(s),\ s \in [0,\tau]. \tag{24}$$

Since

$$D_{\xi} f(\mu^{Z_0(t)}) = \left\langle D\widetilde{f}(Z_0)\cdot\xi\right\rangle = \frac{d}{dt}\widetilde{f}(Z_0 + t\xi)\bigg|_{t=0},$$

we have the following simple form of the first order *Taylor expansion*

$$f(\mu^{Z_0(t)+\xi}) - f(\mu^{Z_0(t)}) = D_{\xi} f(\mu^{Z_0(t)}) + \mathscr{E}(\xi),$$

where $\mathscr{E}(\xi)$ is of order $O(\|\xi\|_2)$ with $O(\|\xi\|_2) \to 0$ for $\xi \in \mathbb{L}^2\left(\mathscr{F},\mathbb{R}^d\right)$. From Eq. (24), we have

$$\begin{aligned}
\gamma^{\theta}(t) &= \frac{1}{\theta}\int_0^t \int_{\mathbb{R}^d}\left[\varphi\left(s,\mathrm{y}^{\theta}(s),\mu^{\mathrm{y}^{\theta}(s)},\alpha^{\theta}(s)\right) - \varphi\left(s,\mathrm{y}^{*}(s),\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s)\right)\right]\mu(\mathrm{dy})\mathrm{d}s \\
&+\frac{1}{\theta}\int_0^t \int_{\mathbb{R}^d}\left[\psi\left(s,\mathrm{y}^{\theta}(s),\mu^{\mathrm{y}^{\theta}(s)},\alpha^{\theta}(s)\right) - \psi\left(s,\mathrm{y}^{*}(s),\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s)\right)\right]\mu(\mathrm{dy})\mathrm{d}W(s) - \mathrm{Z}(t).
\end{aligned}$$

We put

$$\begin{aligned}
f(t,\mu,\alpha) &= \int_{\mathbb{R}^d}\varphi\left(t,\mathrm{y}(t),\mu^{\mathrm{y}(t)},\alpha(t)\right)\mu(dy), \\
\sigma(t,\mu,\alpha) &= \int_{\mathbb{R}^d}\psi\left(t,\mathrm{y}(t),\mu^{\mathrm{y}(t)},\alpha(t)\right)\mu(dy), \\
h(\mu) &= \int_{\mathbb{R}^d}\Phi\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)\mu(dy).
\end{aligned} \tag{25}$$

By applying (25), we get

$$\begin{aligned}
\gamma^{\theta}(t) &= \frac{1}{\theta}\int_0^t \left[f\left(s,\mu^{\mathrm{y}^{\theta}(s)},\alpha^{\theta}(s)\right) - f\left(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s)\right)\right]\mathrm{d}s \\
&+\frac{1}{\theta}\int_0^t \left[\sigma\left(s,\mu^{\mathrm{y}^{\theta}(s)},\alpha^{\theta}(s)\right) - \sigma\left(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s)\right)\right]\mathrm{d}W(s) \\
&-\int_0^t \left\{\widehat{E}\left[f_{\mu}(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s);\widehat{\mathrm{y}}^{*}(s))\widehat{\mathrm{Z}}(s)\right] + f_{\alpha}(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s))v(s)\right\}\mathrm{d}s \\
&-\int_0^t \left\{\widehat{E}\left[\sigma_{\mu}(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s);\widehat{\mathrm{y}}^{*}(s))\widehat{\mathrm{Z}}(s)\right] + \sigma_{\alpha}(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s))v(s)\right\}\mathrm{d}W(s).
\end{aligned}$$

By simple computations, we have

$$\begin{aligned}
\int_0^t [f(s,\mu^{\mathrm{y}^{\theta}(s)},\alpha^{\theta}(s)) - f(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s))]\mathrm{d}s &= \int_0^t (f(s,\mu^{\mathrm{y}^{\theta}(s)},\alpha^{\theta}(s)) - f(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{\theta}(s)))\mathrm{d}s \\
&+\int_0^t (f(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{\theta}(s)) - f(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{*}(s)))\mathrm{d}s.
\end{aligned}$$

Applying the first-order expansion, we get

$$\begin{aligned}
&\frac{1}{\theta}\int_0^t (f(s,\mu^{\mathrm{y}^{\theta}(s)},\alpha^{\theta}(s)) - f(s,\mu^{\mathrm{y}^{*}(s)},\alpha^{\theta}(s)))\mathrm{d}s \\
&\quad= \int_0^t \int_0^1 \widehat{E}\left[\partial_{\mu} f(s,\mu^{\mathrm{y}^{*}(s)+\lambda\varepsilon(\gamma(s)+\mathrm{Z}(s))},\alpha^{\theta}(s);\widehat{\mathrm{y}}^{*}(s))(\widehat{\gamma}(s)+\widehat{\mathrm{Z}}(s))\right]\mathrm{d}\lambda\,\mathrm{d}s.
\end{aligned}$$

Using similar arguments developed above, we can easily prove that

$$\frac{1}{\theta}\int_0^t (f(s,\mu^{y^\theta(s)},\alpha^\theta(s))-f(s,\mu^{y^\theta(s)},\alpha^*(s)))\mathrm{d}s$$
$$=\int_0^t\int_0^1\left[f_\alpha\left(s,\mu^{y^\theta(s)},\alpha^*(s)+\lambda\varepsilon(\alpha(s)-\alpha^*(s))\right)v(s)\right]\mathrm{d}\lambda\,\mathrm{d}s.$$

The analogue arguments hold for σ, then we get

$$\frac{1}{\theta}\int_0^t [\sigma(s,\mu^{y^\theta(s)},\alpha^\theta(s))-\sigma(s,\mu^{y^*(s)},\alpha^*(s))]\mathrm{d}s$$
$$=\int_0^t\int_0^1\widehat{E}\left[\partial_\mu\sigma(s,\mu^{y^*(s)+\lambda\varepsilon(\gamma(s)+Z(s))},\alpha^\theta(s);\widehat{y}^*(s))(\widehat{\gamma}(s)+\widehat{Z}(s))\right]\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_0^1\left[\sigma_\alpha\left(s,\mu^{y^\theta(s)},\alpha^*(s)+\lambda\varepsilon(\alpha(s)-\alpha^*(s)\right)v(s)\right]\mathrm{d}\lambda\,\mathrm{d}s.$$

Therefore

$$E\left[\sup_{s\in[0,t]}\left|\gamma^\theta(s)\right|^2\right]\leq C_t\left[E\int_0^t\int_0^1\widehat{E}\left|f_\mu(s,\mu^{y^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{Z}(s))},\alpha^\theta(s);\widehat{y}^*(s))\widehat{\gamma}^\theta(s)\right|^2 d\lambda ds\right.$$
$$+E\int_0^t\int_0^1\widehat{E}\left|\sigma_\mu(s,\mu^{y^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{Z}(s))},\alpha^\theta(s);\widehat{y}^*(s))\widehat{\gamma}^\theta(s)\right|^2 d\lambda ds$$
$$\left.+E\left[\sup_{s\in[0,t]}\left|A^\theta(s)\right|^2\right]\right],$$

where

$$A^\theta(t)=\int_0^t\int_0^1\widehat{E}\left[f_\mu(s,\mu^{y^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{Z}(s))},\alpha^\theta(s);\widehat{y}^*(s))-f_\mu(s,\mu^{y^*(s)},\alpha^*(s);\widehat{y}^*(s))]\widehat{Z}(s)\mathrm{d}\lambda\,\mathrm{d}s\right.$$
$$+\int_0^t\int_0^1[f_\alpha\left(s,\mu^{y^*(s)},\alpha^*(s)+\lambda\varepsilon v(t)\right)-f_\alpha(s,\mu^{y^*(s)},\alpha^*(s)]v(t)\mathrm{d}\lambda\,\mathrm{d}s$$
$$+\int_0^t\int_0^1\widehat{E}\left[\sigma_\mu(s,\mu^{y^*(s)+\lambda\varepsilon(\widehat{\gamma}(s)+\widehat{Z}(s))},\alpha^\theta(s);\widehat{y}^*(s))-\sigma_\mu(s,\mu^{y^*(s)},\alpha^*(s);\widehat{y}^*(s))]\widehat{Z}(s)\mathrm{d}\lambda\,\mathrm{d}W(s)\right.$$
$$+\int_0^t\int_0^1[\sigma_\alpha\left(s,\mu^{y^*(s)},\alpha^*(s)+\lambda\varepsilon v(t)\right)-\sigma_\alpha(s,\mu^{y^*(s)},\alpha^*(s)]v(t)\mathrm{d}\lambda\,\mathrm{d}W(s).$$

Now, since the partial derivatives of f and σ with respect to μ,α are Lipschitz continuous in μ,α, then we get

$$\lim_{\theta\to 0}E\left[\sup_{s\in[0,\tau]}\left|A^\theta(s)\right|^2\right]=0.$$

Moreover, since the partial-derivatives of f and σ with respect to variables μ, and α are bounded, we obtain $\forall t\in[0,\tau]$:

$$E\left[\sup_{s\in[0,t]}\left|\gamma^\theta(s)\right|^2\right]\leq C(t)\left\{E\int_0^t\left|\gamma^\theta(s)\right|^2 ds+E\left[\sup_{s\in[0,t]}\left|A^\theta(s)\right|^2\right]\right\}.$$

By using *Gronwall's theorem*, we get

$$E\left[\sup_{s\in[0,t]}\left|\gamma^{\theta}(s)\right|^{2}\right]\leq C_{s}E\left[\sup_{s\in[0,t]}\left|A^{\theta}(s)\right|^{2}\right]\exp\left(\int_{0}^{t}C_{s}ds\right).$$

Finally, putting $t=\tau$ the proof of Proposition 1 is fulfilled when θ approaches zero. □

Proposition 2. *For any* $\alpha(\cdot)\in\mathscr{U}\left([0,\tau]\right)$, *we have*

$$0\leq E\left(\partial_{\mathrm{y}}\Phi\left(\mathrm{y}^{*}(\tau),\mu^{\mathrm{y}^{*}(\tau)}\right)+\int_{\mathbb{R}^{d}}\widehat{E}(\partial_{\mu}\Phi\left(\mathrm{y}^{*}(\tau),\mu^{\mathrm{y}^{*}(\tau)};\widehat{\mathrm{y}}^{*}(\tau)\right)\mu(\mathrm{dy})\right)\mathrm{Z}(\tau). \tag{26}$$

Proof. From (7) and (20), we have

$$0\leq J\left(\alpha^{\theta}(\cdot)\right)-J\left(\alpha^{*}(\cdot)\right)=E\left[h(\mathrm{y}^{\theta}(\tau),\mu^{\mathrm{y}^{\theta}(\tau)})-h(\mathrm{y}^{*},\mu^{\mathrm{y}^{*}(\tau)})\right].$$

By applying the first-order expansion, we get

$$\begin{aligned}
&h(\mathrm{y}^{\theta}(\tau),\mu^{\mathrm{y}^{\theta}(\tau)})-h(\mathrm{y}^{*},\mu^{\mathrm{y}^{*}(\tau)})\\
&\quad=\int_{0}^{1}\left[h_{\mathrm{y}}\left(\mathrm{y}^{*}(\tau)+\rho\Delta x^{\theta}(\tau)),\mu^{\mathrm{y}^{*}(\tau)+\rho\Delta x^{\theta}}\right)\Delta x^{\theta}(\tau)\right]\mathrm{d}\rho\\
&\qquad+\int_{0}^{1}\widehat{E}\left[h_{\mu}(\mathrm{y}^{*}(\tau)+\rho\Delta x^{\theta}(\tau),\mu^{\mathrm{y}^{*}(\tau)+\rho\Delta x^{\theta}(\tau)};\widehat{\mathrm{y}}^{*}(\tau))\Delta\widehat{\mathrm{y}}^{\theta}(\tau)\right]\mathrm{d}\rho\\
&\quad=\int_{0}^{1}\left(\partial_{\mathrm{y}}\Phi\left(\mathrm{y}^{*}(\tau)+\rho\Delta x^{\theta}(\tau),\mu^{\mathrm{y}^{*}(\tau)+\rho\Delta x^{\theta}(\tau)};\widehat{\mathrm{y}}^{*}(\tau)\right)\right)\Delta\widehat{\mathrm{y}}^{\theta}(\tau)\mathrm{d}\rho\\
&\qquad+\int_{0}^{1}\int_{\mathbb{R}^{d}}\widehat{E}(\partial_{\mu}\Phi\left(\mathrm{y}^{*}(\tau)+\rho\Delta x^{\theta}(\tau),\mu^{\mathrm{y}^{*}(\tau)+\rho\Delta x^{\theta}(\tau)};\widehat{\mathrm{y}}^{*}(\tau)\right)\mu(\mathrm{dy})\Delta\widehat{\mathrm{y}}^{\theta}(\tau)\mathrm{d}\rho,
\end{aligned}$$

where $\Delta x^{\theta}(t)=\mathrm{y}^{\theta}(t)-\mathrm{y}^{*}(t)$. Finally, by using Proposition 1, the desired result (26) is fulfilled. This completes the proof of Propositionn 2. □

Proof of Theorem 1. Itô's formula is one of the most fundamental building blocks in stochastic calculus and maximum principle, see Guo et al. [7]. By applying Itô's formula to stochastic process $p(t)\mathrm{Z}(t)$ and take expectation, where $\mathrm{Z}(0)=0$, a simple computations shows that

$$\begin{aligned}
&E(p(\tau)\mathrm{Z}(\tau))-E(p(0)\mathrm{Z}(0))\\
&\quad=E\int_{0}^{\tau}p(t)d\mathrm{Z}(t)+E\int_{0}^{\tau}\mathrm{Z}(t)dp(t)\\
&\qquad+E\int_{0}^{\tau}q(t)\left[\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t)+\int_{\mathbb{R}^{d}}\partial_{\mu}\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\right)\right.\\
&\qquad\left.+\psi_{\alpha}(t,\mathrm{y},\mu,\alpha)v(t)+\int_{\mathbb{R}^{d}}\psi_{\alpha}(t,\mathrm{y},\mu,\alpha)v(t)\mu(\mathrm{dy})\right]\mathrm{d}t\\
&\quad=I_{1}+I_{2}+I_{3},
\end{aligned} \tag{27}$$

where

$$
\begin{aligned}
I_1 &= E\int_0^\tau p(t)\mathrm{d}\mathrm{Z}(t) \\
&= E\int_0^\tau p(t)\left[\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t)+\textstyle\int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\right)\right]\mathrm{d}t \\
&\quad +E\int_0^\tau p(t)\left[\varphi_\alpha(t,\mathrm{y},\mu,\alpha)v(t)+\textstyle\int_{\mathbb{R}^d}\varphi_\alpha(t,\mathrm{y},\mu,\alpha)v(t)\mu(\mathrm{dy})\right]\mathrm{d}t.
\end{aligned}
\tag{28}
$$

Let us turn to estimate the second term I_2. From (12), we have

$$
\begin{aligned}
I_2 &= E\int_0^\tau \mathrm{Z}(t)dp(t) \\
&= -E\int_0^\tau \mathrm{Z}(t)\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{p}(t)+\textstyle\int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{p}(t)\mu(\mathrm{dy})\right)\mathrm{d}t \\
&\quad -E\int_0^\tau \mathrm{Z}(t)\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\psi}(t,\mathrm{y},\mu,\alpha)\widehat{q}(t)+\textstyle\int_{\mathbb{R}^d}\partial_\mu\widehat{\psi}(t,\mathrm{y},\mu,\alpha)\widehat{q}(t)\mu(\mathrm{dy})\right)\mathrm{d}t.
\end{aligned}
\tag{29}
$$

From (17), we have

$$
\begin{aligned}
I_3 \;=\; & E\int_0^\tau q(t)\left[\widehat{E}\left(\partial_{\mathrm{y}}\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t)+\int_{\mathbb{R}^d}\partial_\mu\widehat{\varphi}(t,\mathrm{y},\mu,\alpha)\widehat{\mathrm{Z}}(t)\mu(\mathrm{dy})\right)\right. \\
&\left. +\psi_\alpha(t,\mathrm{y},\mu,\alpha)v(t)+\int_{\mathbb{R}^d}\psi_\alpha(t,\mathrm{y},\mu,\alpha)v(t)\mu(\mathrm{dy})\right]\mathrm{d}t.
\end{aligned}
\tag{30}
$$

Substituting (28), (29) and (30) into (27), with the fact that

$$
p(\tau)=\widehat{E}\left[\partial_{\mathrm{y}}\widehat{\Phi}\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)+\int_{\mathbb{R}^d}\partial_\mu\widehat{\Phi}\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)\mu(\mathrm{dy})\right],
$$

we get

$$
\begin{aligned}
& E\left(\widehat{E}\left[\partial_{\mathrm{y}}\widehat{\Phi}\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)+\int_{\mathbb{R}^d}\partial_\mu\widehat{\Phi}\left(\mathrm{y}(\tau),\mu^{\mathrm{y}(\tau)}\right)\mu(\mathrm{dy})\right]\mathrm{Z}(\tau)\right) \\
&\quad = E\int_0^\tau p(t)\left[\varphi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))+\int_{\mathbb{R}^d}\varphi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))\mu(\mathrm{dy})\right]\mathrm{d}t \\
&\quad\quad +E\int_0^\tau q(t)\left[\psi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))+\int_{\mathbb{R}^d}\psi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))\mu(\mathrm{dy})\right]\mathrm{d}t.
\end{aligned}
$$

Applying Proposition 1, we obtain

$$
\begin{aligned}
0 \;\leq\; & E\int_0^\tau p(t)\left[\varphi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))+\int_{\mathbb{R}^d}\varphi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))\mu(\mathrm{dy})\right]\mathrm{d}t \\
& +E\int_0^\tau q(t)\left[\psi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))+\int_{\mathbb{R}^d}\psi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))\mu(\mathrm{dy})\right]\mathrm{d}t.
\end{aligned}
$$

Finally, by simple computations, with the helps of (11), we get

$$
\begin{aligned}
& E\int_0^\tau p(t)\left[\varphi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))+\int_{\mathbb{R}^d}\varphi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))\mu(\mathrm{dy})\right]\mathrm{d}t \\
& +E\int_0^\tau q(t)\left[\psi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))+\int_{\mathbb{R}^d}\psi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))\mu(\mathrm{dy})\right]\mathrm{d}t \\
& \quad = E\int_0^\tau\Big[p(t)\Big(\varphi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))+\int_{\mathbb{R}^d}\varphi_\alpha(t,\mathrm{y},\mu,\alpha)\mu(\mathrm{dy})\Big) \\
& \qquad +q(t)\Big(\psi_\alpha(t,\mathrm{y},\mu,\alpha)(\alpha(t)-\alpha^*(t))+\int_{\mathbb{R}^d}\psi_\alpha(t,\mathrm{y},\mu,\alpha)\mu(\mathrm{dy})\Big)\Big](\alpha(t)-\alpha^*(t))\mathrm{d}t \\
& \quad = E\int_0^\tau\frac{\partial H}{\partial\alpha}(t,\mathrm{y}^*(t),\mu^{\mathrm{y}^*(t)},\alpha^*(t),p(t),q(t))(\alpha(t)-\alpha^*(t))\,\mathrm{d}t.
\end{aligned}
$$

Then (19) is fulfilled which completes the proof of Theorem 1.

5 Examples: Gamma process via Lévy measure

The Gamma process is a Lévy process (of bounded variation) $(G(t))_{t\geq 0}$, with Lévy measure given by

$$
\mu(\mathrm{dy})=\frac{e^{-y}}{y}I_{\{y>0\}}\mathrm{d}y. \tag{31}
$$

It is called *Gamma process* since the probability law of $G(\cdot)$ is a Gamma distribution with mean t and scale-parameter equal to one.

5.1 Examples (Derivatives with respect to measure)

Let $(G(t))_{t\geq 0}$ be Gamma process with Lévy measure $\mu(\cdot)$ given by (31). We give some examples.
1) If $\Phi(\mu)=\int_{\mathbb{R}}\varphi(y)\mu(\mathrm{d}y)$, then the Lions's derivatives of $\Phi(\mu)$ with respect to measure at z is given by

$$
\partial_\mu\Phi(\mu)(z)=\frac{\partial\varphi}{\partial y}(z).
$$

2) If $\Phi(\mu)=\int_{\mathbb{R}}\varphi(y,\mu)\mu(\mathrm{d}y)$, then the Lions's derivatives of $\Phi(\mu)$ with respect to measure at z is given by

$$
\begin{aligned}
\partial_\mu\Phi(\mu)(z) &= \frac{\partial\varphi}{\partial y}(z,\mu)+\int_{\mathbb{R}}\frac{\partial\varphi}{\partial\mu}(y,\mu)(z)\mu(\mathrm{dy}) \\
&= h\ \frac{\partial\varphi}{\partial y}(z,\mu)+\int_{\mathbb{R}}\frac{e^{-y}}{y}\frac{\partial\varphi}{\partial\mu}(y,\mu)(z)I_{\{y>0\}}\mathrm{d}y.
\end{aligned}
$$

5.2 Maximum principle

We consider $\varphi(t,\mathrm{y}(t),\mu,\alpha(t))=\mathrm{y}(t)\alpha(t)$, $\psi(t,\mathrm{y}(t),\mu,\alpha(t))=\mathrm{y}(t)\alpha(t)$. Our purpose is to minimize $Var(\mathrm{y}(\tau))-\mu^{\mathrm{y}(\tau)}$.

From (25), a simple computations shows that

$$f(t,\mu,\alpha)=\int_{\mathbb{R}}\varphi(t,\mathrm{y}(t),\mu^{\mathrm{y}(t)},\alpha(t))\mu(dy)=\alpha(t), \tag{32}$$

$$\sigma(t,\mu,\alpha)=\int_{\mathbb{R}}\psi(t,\mathrm{y}(t),\mu^{\mathrm{y}(t)},\alpha(t))\mu(dy)=\alpha(t). \tag{33}$$

From (10) we get

$$H(t,\mathrm{y},\mu,\alpha,p(t),q(t))=\alpha(t)p(t)+\alpha(t)q(t). \tag{34}$$

Since, the Hamiltonian H is linear in the control variable $\alpha(\cdot)$, considering the first-order condition for minimizing the Hamiltonian yields

$$H_{\alpha}(t,\mathrm{y},\mu,\alpha,p(t),q(t))=p(t)+q(t)=0. \tag{35}$$

From (12) and (31), by simple computations, we have

$$\begin{cases} \mathrm{d}p(t)=q(t)\mathrm{d}W(t), \\ p(\tau)=2\left[\mathrm{y}(t)-\mu^{\mathrm{y}(\tau)}\right]-1. \end{cases} \tag{36}$$

Conjecture of the adjoint process. Looking at the terminal condition $p(\tau)$, it is reasonable to try a solution of the form:

$$p(t)=U_1(t)\left[\mathrm{y}(t)-\mu^{\mathrm{y}(\tau)}\right]+U_2(t), \tag{37}$$

where $U_1(\cdot)$, and $U_2(\cdot)$ are deterministic differentiable functions, and $U_1(\tau)=2$, and $U_2(\tau)=-1$. On the other hand, by applying Itô's formula to $U_1(t)\left(\mathrm{y}(t)-\mu^{\mathrm{y}(t)}\right)$ in (37), we get

$$\begin{aligned} \mathrm{d}p(t) &= \mathrm{d}(U_1(t)(\mathrm{y}(t)-\mu^{\mathrm{y}(t)}))+\mathrm{d}U_2(t) \\ &= U_1(t)\,\mathrm{d}(\mathrm{y}(t)-\mu)+(\mathrm{y}(t)-\mu)\,U_1'(t)\,\mathrm{d}t+U_2'(t)\,\mathrm{d}t \\ &= U_1(t)\,\alpha(t)\mathrm{d}t-U_1(t)\,\mathrm{d}\mu+(\mathrm{y}(t)-\mu)\,U_1'(t)\,\mathrm{d}t+U_2'(t)\,\mathrm{d}t+U_1(t)\,\alpha(t)\mathrm{d}W(t). \end{aligned} \tag{38}$$

From (38) and (36), we conclude

$$(\mathrm{y}(t)-\mu)\,U_1'(t)+U_1(t)\,\alpha(t)+U_1(t)\,\mu+U_2'(t)=0, \tag{39}$$

and

$$q(t)=U_1(t)\,\alpha(t). \tag{40}$$

Substituting (40) into (35), we obtain a candidate optimal control in feedback form

$$\begin{aligned} \alpha(t) &= \frac{q(t)}{U_1(t)}=\frac{-p(t)}{U_1(t)}=\frac{-U_1(t)(\mathrm{y}(t)-\mu)+U_2(t)}{U_1(t)} \\ &= -\mathrm{y}(t)+\mu-\frac{U_2(t)}{U_1(t)}. \end{aligned} \tag{41}$$

By comparing the coefficient of $y(t)$ and μ, in (39), we obtain

$$U_1(t)-U_1'(t)=0,\ U_1(\tau)=2, \tag{42}$$

and

$$U_2'(t) = 0,\ U_2(\tau) = -1. \tag{43}$$

By solving the ordinary differential equations (42)-(43), we obtain for $t \in [0, \tau]$

$$\begin{aligned} U_1(t) &= 2\exp[t-\tau], \\ U_2(t) &= -1. \end{aligned} \tag{44}$$

Finally, by substituting (41) into (44), the optimal control is given in the feedback form by

$$\alpha^*(t, \mathrm{y}^*(t), \mu^{\mathrm{y}^*(t)}) = -\mathrm{y}^*(t) + \mu^{\mathrm{y}^*(t)} + \frac{1}{2}\exp[\tau - t]. \tag{45}$$

6 Conclusion and future developments

In this paper, we have established the necessary optimality conditions in the form of Pontryagin maximum principle, for the control problem A. Our stochastic model is governed by mean-field SDE, where the coefficients depend on control variable, the state process as well as of its probability law. Lions's partial-derivatives with respect to probability measure and the associated Itô-formula are applied to proved our maximum principle. An open question is to establish the sufficient conditions for optimality for the problem A. Moreover, it would be interesting to study the general mean-field control problem for systems governed by controlled forward-backward stochastic differential equations with random jumps, where some applications in finance can be investigated.

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