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Inverse eigenvalue problem of nonnegative matrices via unit lower triangular matrices (Part I)

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Abstract. This paper uses unit lower triangular matrices to solve the nonnegative inverse eigenvalue problem for various sets of real numbers. This problem has remained unsolved for many years for $n \geq 5$. The inverse of the unit lower triangular matrices can be easily calculated and the matrix similarities are also helpful to be able to solve this important problem to a considerable extent. It is assumed that in the given set of eigenvalues, the number of positive eigenvalues is less than or equal to the number of nonpositive eigenvalues to find a nonnegative matrix such that the given set is its spectrum.

Keywords: Nonnegative matrices, unit lower triangular matrices, inverse eigenvalue problem.
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1 Introduction

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a multiset $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers as the spectrum of a nonnegative matrix A. If there is such a nonnegative matrix A with spectrum σ, we say that σ is *realizable* and that A is a *realization of* σ. Some necessary conditions for the realizability of σ are

(i) $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to σ;

(ii) $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$; and

(iii) $s_k^m \leq n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$

The Perron-Frobenius theorem implies the necessity of statement (i), the necessity of statement (ii) is the observation that the trace of the k-th power of a nonnegative matrix is nonnegative and equal to the sum

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of the k-th powers of the eigenvalues; and the necessity of (iii) is known as the Johnson-Loewy-London (JLL) inequality [7, 9].

Throughout the remainder of the paper λ_1 denotes $\max\{|\lambda_i|;\lambda_i \in \sigma\}$, and σ is assumed to satisfy the necessary conditions (i), (ii) and (iii).

Many mathematicians have worked on the NIEP [3–5, 8, 11, 13–21], and there are several methods for finding realizations. In this paper we utilize a method, based on the similarity of a matrix to an upper triangular matrix, to solve several nonnegative inverse eigenvalue problems. This method was initiated by Guo in [6]. A matrix L is *unit lower triangular* provided each entry on its main diagonal equals 1, and each entry above its main diagonal is zero. The inverse of a unit lower triangular matrix also is a unit lower triangular. Recently, Nazari et al. have used unit lower triangular matrices in solving the inverse eigenvalue problem of distance matrices [12].

The paper is organized as follows.

We solve the NIEP in several cases where each element of σ is real and the number of positive elements of σ is less than or equal the number of negative elements of σ. This means that $\sigma = \{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ is a given multiset of real numbers such that $k \le n/2$ and

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k > 0 \ge \lambda_n \ge \cdots \ge \lambda_{k+1}, \tag{1}$$

we find a nonnegative matrix C such that the above set with condition (1) is its eigenvalues.

2 Real spectrum with one positive number

We first show how our method can be used for σ with just one positive element.To begin with, we present Suleimanova's Theorem [20] and provide another proof for it

Theorem 1. *Let $\sigma = \{\lambda_1,\ldots,\lambda_n\}$ be a multiset of real numbers with conditions*

$$\lambda_1 > 0 \ge \lambda_n \ge \cdots \ge \lambda_2,$$

and $\lambda_1 \ge -\sum_{i=2}^{n}\lambda_i$. Then σ is realizable.

Proof. For $n = 1$, $[\lambda_1]$ is a realization. For $n \ge 2$, let

$$A = \begin{bmatrix} \lambda_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and $t = \sum_{i=2}^{n}\alpha_i$, where α_i are nonnegative real numbers. Then

$$C = LAL^{-1} = \begin{bmatrix} \lambda_1 - t & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \lambda_1 - \lambda_2 - t & \alpha_2 + \lambda_2 & \alpha_3 & \cdots & \alpha_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1 - \lambda_{n-1} - t & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \lambda_1 - \lambda_n - t & \alpha_2 & \alpha_3 & \cdots & \alpha_n + \lambda_n \end{bmatrix}, \tag{2}$$

is similar to A. Additionally, C is nonnegative whenever

$$\lambda_1 \geq t \text{ and } \alpha_i \geq -\lambda_i, \qquad i = 2, 3, \ldots, n. \tag{3}$$

By the assumptions on σ, setting $\alpha_i = -\lambda_i$ for $i = 2, \ldots, n$ gives a solution for (3). Hence σ is realizable. □

Remark 1. *Note that if σ satisfies the hypothesis of Theorem 1 and additionally $\sum_{i=1}^{n} \lambda_i = 0$, then in order for C to be nonnegative the constraints (3) require that $\lambda_1 = t$ and $\alpha_i = -\lambda_i$ for $(i = 2, \ldots, n)$. In other words, the matrix C constructed in the proof of Theorem 1 is unique.*

Remark 2. *In Theorem 1, if $\sum_{i=1}^{n} \lambda_i > 0$, then there are infinitely many appropriate choices of α_i and hence we have many different C. In fact, for each n-tuple $d = (d_1, d_2, \ldots, d_n)$ of nonnegative real numbers with $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i$ there is a realization C of σ with main diagonal equal to d, because the equations $\alpha_i = d_i$ for $i = 2, 3, \cdots, n$, give us the value of α_i and the matrix A can be determined based on the values of α_i. For instance, consider $\sigma = \{10, -2, -2, -2, -1, -1\}$. By taking $\alpha_i = -\lambda_i$ $(i = 2, \ldots, 6)$, the resulting matrix is*

$$C = \begin{bmatrix} 2 & 2 & 2 & 2 & 1 & 1 \\ 4 & 0 & 2 & 2 & 1 & 1 \\ 4 & 2 & 0 & 2 & 1 & 1 \\ 4 & 2 & 2 & 0 & 1 & 1 \\ 3 & 2 & 2 & 2 & 0 & 1 \\ 3 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

If we want the main diagonal of (2) to be $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)$, we take $\alpha_2 = 5/2$, $\alpha_3 = 5/2$, $\alpha_4 = 5/2$, $\alpha_5 = 1$, $\alpha_6 = 1$, and the resulting matrix C is

$$C = \begin{bmatrix} 1/2 & 5/2 & 5/2 & 5/2 & 1 & 1 \\ 5/2 & 1/2 & 5/2 & 5/2 & 1 & 1 \\ 5/2 & 5/2 & 1/2 & 5/2 & 1 & 1 \\ 5/2 & 5/2 & 5/2 & 1/2 & 1 & 1 \\ 3/2 & 5/2 & 5/2 & 5/2 & 0 & 1 \\ 3/2 & 5/2 & 5/2 & 5/2 & 1 & 0 \end{bmatrix}.$$

3 Real spectrum with two positive numbers

Now, we consider σ having two positive eigenvalues. We begin with the case of $n = 4$.

Theorem 2. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a multiset of real numbers satisfying*

$$\begin{aligned} \lambda_1 \geq \lambda_2 &> 0 \geq \lambda_4 \geq \lambda_3, \\ \sum_{i=1}^{4} \lambda_i &\geq 0, \\ \lambda_1 &\geq |\lambda_3|. \end{aligned}$$

Then σ is realizable.

Proof. If $\lambda_2+\lambda_4 \geq 0$ and $\lambda_1 \geq |\lambda_3|$, then according to Theorem 1, we can find a nonnegative 2×2 matrix A_2 that realizes $\{\lambda_1,\lambda_3\}$ and a nonnegative 2×2 matrix B_2 that realizes $\{\lambda_2,\lambda_4\}$, so the matrix $C=\mathrm{diag}\{A_2,B_2\}$ has eigenvalues σ. If $\lambda_2+\lambda_4<0$ and $\lambda_1 \geq |\lambda_3|$, then we assume that $\alpha_1,\alpha_2,\alpha_3$, and α_4 be real numbers. Let $t=\sum_{i=2}^{4}\alpha_i$ and A and L be the following matrices

$$A=\begin{bmatrix} \lambda_1 & \alpha_2+\alpha_4 & \alpha_3 & 0 \\ 0 & \lambda_2 & \alpha_1 & \alpha_4 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \quad \text{and} \quad L=\begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&0&1 \end{bmatrix}.$$

Then

$$C=LAL^{-1}=\begin{bmatrix} \lambda_1-t & \alpha_2+\alpha_4 & \alpha_3 & 0 \\ \lambda_1-\lambda_2-t-\alpha_1 & \alpha_2+\lambda_2 & \alpha_3+\alpha_1 & \alpha_4 \\ \lambda_1-\lambda_3-t & \alpha_2+\alpha_4 & \alpha_3+\lambda_3 & 0 \\ \lambda_1-\lambda_2-t-\alpha_1 & \alpha_2+\lambda_2-\lambda_4 & \alpha_3+\alpha_1 & \alpha_4+\lambda_4 \end{bmatrix}, \tag{4}$$

is similar to the A. Whenever

$$\begin{aligned} -\lambda_i &\leq \alpha_i, \qquad i=2,3,4,\\ t &\leq \lambda_1,\\ -\alpha_3 &\leq \alpha_1 \leq \lambda_1-\lambda_2-t,\\ 0 &\leq \alpha_2+\alpha_4, \end{aligned} \tag{5}$$

the matrix C is nonnegative. The assumptions on σ imply that $\alpha_2=-\lambda_2, \alpha_3=-\lambda_3, \alpha_4=-\lambda_4$ and $\alpha_1=\lambda_3$ is a solution to these constraints. Hence σ is realizable. □

Example 1. Suppose that $\lambda_1+\lambda_2+\lambda_3+\lambda_4=0$. If we take $\alpha_i=-\lambda_i$, $i=2,3,4$ and $\alpha_1=\lambda_3$ then the conditions in (5) are satisfied and each diagonal entry of C will be 0. For instance let $\sigma=\{7,3,-5,-5\}$, the matrices L, L^{-1}, A and C will be obtained as follows:

$$L=\begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&0&1 \end{bmatrix}, \quad L^{-1}=\begin{bmatrix} 1&0&0&0\\-1&1&0&0\\-1&0&1&0\\0&-1&0&1 \end{bmatrix}, \quad A=\begin{bmatrix} 7&2&5&0\\0&3&-5&5\\0&0&-5&0\\0&0&0&-5 \end{bmatrix},$$

$$C=LAL^{-1}=\begin{bmatrix} 0&2&5&0\\2&0&0&5\\5&2&0&0\\2&5&0&0 \end{bmatrix}.$$

This spectrum is studied in [1] and our method gives an easily derived realization. □

Now we consider the set of σ with two positive eigenvalues and three negative eigenvalues with special conditions.

Theorem 3. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, where $\lambda_1, \dots, \lambda_5 \in \mathbb{R}$ satisfy*

$$\lambda_1 \geq \lambda_2 > 0 \geq \lambda_5 \geq \lambda_4 \geq \lambda_3,$$
$$\sum_{i=1}^{5} \lambda_i \geq 0,$$
$$\lambda_1 \geq |\lambda_3|,$$
$$\lambda_1 + \lambda_4 + \lambda_5 \geq 0.$$

Then σ is realizable.

Proof. If $\lambda_2 + \lambda_4 + \lambda_5 \geq 0$ and $\lambda_1 \geq |\lambda_3|$, then according to Theorem 1, we can find a nonnegative 3×3 matrix C_3 that realizes $\{\lambda_2, \lambda_4, \lambda_5\}$ and a nonnegative 2×2 matrix C_2 that realizes $\{\lambda_1, \lambda_3\}$, so the matrix $C = \operatorname{diag}\{C_2, C_3\}$ has eigenvalues σ. If $\lambda_2 + \lambda_4 + \lambda_5 < 0$ and $\lambda_1 \geq |\lambda_3|$, then we suppose that α_i, $i = 1, \cdots, 5$ are real numbers. Set $t = \sum_{i=2}^{5} \alpha_i$. In this case, we consider

$$A = \begin{bmatrix} \lambda_1 & \alpha_2+\alpha_4+\alpha_5 & \alpha_3 & 0 & 0 \\ 0 & \lambda_2 & \alpha_1 & \alpha_4 & \alpha_5 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then $C = LAL^{-1}$ is the matrix

$$C = \begin{bmatrix} \lambda_1 - t & \alpha_2+\alpha_4+\alpha_5 & \alpha_3 & 0 & 0 \\ \lambda_1-\lambda_2-t-\alpha_1 & \alpha_2+\lambda_2 & \alpha_3+\alpha_1 & \alpha_4 & \alpha_5 \\ \lambda_1-\lambda_3-t & \alpha_2+\alpha_4+\alpha_5 & \alpha_3+\lambda_3 & 0 & 0 \\ \lambda_1-\lambda_2-t-\alpha_1 & \alpha_2+\lambda_2-\lambda_4 & \alpha_3+\alpha_1 & \alpha_4+\lambda_4 & \alpha_5 \\ \lambda_1-\lambda_2-t-\alpha_1 & \alpha_2+\lambda_2-\lambda_5 & \alpha_3+\alpha_1 & \alpha_4 & \alpha_5+\lambda_5 \end{bmatrix}. \tag{6}$$

The matrix C is similar to the matrix A, and C is nonnegative if and only if

$$-\lambda_i \leq \alpha_i, \qquad i = 2,3,4,5$$
$$t \leq \lambda_1,$$
$$0 \leq \alpha_2 + \alpha_4 + \alpha_5,$$
$$-\alpha_3 \leq \alpha_1 \leq \lambda_1 - \lambda_2 - t.$$

With the given hypothesis on σ, one solution to this system of inequalities is

$$\alpha_2 = -\lambda_2,\ \ \alpha_3 = -\lambda_3,\ \ \alpha_4 = -\lambda_4,\ \ \alpha_5 = -\lambda_5,\ \ \alpha_1 = \lambda_3.$$

Hence σ is realizable. □

Theorem 4. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, where $\lambda_1, \dots, \lambda_5 \in \mathbb{R}$, satisfy*

$$\lambda_1 \geq \lambda_2 > 0 \geq \lambda_5 \geq \lambda_4 \geq \lambda_3,$$
$$\sum_{i=1}^{5} \lambda_i \geq 0,$$
$$\lambda_2 + \lambda_5 \leq 0.$$

Then σ is realizable.

Proof. Suppose a_3, a_4 and α_i, $i = 2, \cdots, 5$ are real numbers. Set $t = \sum_{i=2}^{5} \alpha_i$. In this case, we consider

$$A = \begin{bmatrix} \lambda_1 & \alpha_2+\alpha_5 & \alpha_3 & \alpha_4 & 0 \\ 0 & \lambda_2 & a_3 & a_4 & \alpha_5 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then $C = LAL^{-1}$ is the matrix

$$C = \begin{bmatrix} \lambda_1 - t & \alpha_2+\alpha_5 & \alpha_3 & \alpha_4 & 0 \\ \lambda_1-\lambda_2-t-a_3-a_4 & \alpha_2+\lambda_2 & \alpha_3+a_3 & \alpha_4+a_4 & \alpha_5 \\ \lambda_1-\lambda_3-t & \alpha_2+\alpha_5 & \alpha_3+\lambda_3 & \alpha_5 & 0 \\ \lambda_1-\lambda_4-t & \alpha_2+\alpha_5 & \alpha_3 & \alpha_4+\lambda_4 & 0 \\ \lambda_1-\lambda_2-t-a_3-a_4 & \alpha_2+\lambda_2-\lambda_5 & \alpha_3+a_3 & \alpha_4+a_4 & \alpha_5+\lambda_5 \end{bmatrix}. \tag{7}$$

The matrix C is similar to the matrix A, and C is nonnegative if and only if

$$\begin{aligned} -\lambda_i &\le \alpha_i, \qquad i = 2,3,4,5, \\ t &\le \lambda_1, \\ 0 &\le \alpha_2+\alpha_5, \\ -\alpha_3 &\le a_3, \\ -\alpha_4 &\le a_4, \\ a_3+a_4 &\le \lambda_1-\lambda_2-t. \end{aligned}$$

With the given hypothesis on σ, one solution to this system of inequalities is

$$\alpha_2 = -\lambda_2,\ \ \alpha_3 = -\lambda_3,\ \ \alpha_4 = -\lambda_4,\ \ \alpha_5 = -\lambda_5,\ \ a_3 = \lambda_3,\ \ a_4 = \lambda_4.$$

Hence σ is realizable. □

Example 2. Let $\sigma = \{7, 1, -3, -3, -2\}$, then consider the matrices A and L as following

$$A = \begin{bmatrix} 7 & 1 & 3 & 3 & 0 \\ 0 & 1 & -3 & -3 & 2 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$C = LAL^{-1} = \begin{bmatrix} 0 & 1 & 3 & 3 & 0 \\ 5 & 0 & 0 & 0 & 2 \\ 3 & 1 & 0 & 3 & 0 \\ 3 & 1 & 3 & 0 & 0 \\ 5 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

Now we consider the general case of two positive eigenvalues.

Theorem 5. *Let $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a multiset of $n \geq 4$ real numbers such that*

$$\lambda_1 \geq \lambda_2 > 0 \geq \lambda_n \geq \cdots \geq \lambda_3,$$

$\sum_{i=1}^{n} \lambda_i = 0$ and $\lambda_1 + \sum_{i=r}^{n} \lambda_i \geq 0$, where r is the largest positive integer such that $\lambda_2 + \sum_{i=r}^{n} \lambda_i \leq 0$. Then σ is realizable.

Proof. Let α_i for $i = 2, 3, \ldots, n$ and $a_i \geq \alpha_i$ $(i = 3, \ldots, n)$, be real numbers. Set $t = \sum_{i=2}^{n} \alpha_i$. As $\lambda_1 + \lambda_2 = -\lambda_3 - \cdots - \lambda_n$, we have $3 \leq r \leq n$. Consider the matrices

$$A = \begin{bmatrix} \lambda_1 & \alpha_2 + (\alpha_r + \cdots + \alpha_n) & \alpha_3 & \cdots & \alpha_{r-1} & 0 & \cdots & 0 \\ 0 & \lambda_2 & a_3 & \cdots & a_{r-1} & \alpha_r & \cdots & \alpha_n \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_{r-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_r & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

and

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where the second column of $L = (l_{ij})$ has $n - r + 1$ ones from entry l_{r2} to entry l_{n2}. Then the matrix $C = LAL^{-1}$ is given by

$$C = \begin{bmatrix} c_{11} & c_{12} & \alpha_3 & \alpha_4 & \cdots & \alpha_{r-1} & 0 & \cdots & 0 \\ c_{21} & \alpha_2 + \lambda_2 & \alpha_3 + a_3 & \alpha_4 + a_4 & \cdots & \alpha_{r-1} + a_{r-1} & \alpha_r & \cdots & \alpha_n \\ c_{31} & c_{32} & \lambda_3 + \alpha_3 & \alpha_4 & \cdots & \alpha_{r-1} & 0 & \cdots & 0 \\ c_{4,1} & c_{42} & \alpha_3 & \lambda_4 + \alpha_4 & \cdots & \alpha_{r-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ c_{r-1,1} & c_{r-1,2} & \alpha_3 & \alpha_4 & \cdots & \lambda_{r-1} + \alpha_{r-1} & 0 & \cdots & 0 \\ c_{r1} & c_{r2} & \alpha_3 + a_3 & \alpha_4 + a_4 & \cdots & \alpha_{r-1} + a_{r-1} & \lambda_r + \alpha_r & \cdots & \alpha_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \alpha_3 + a_3 & \alpha_4 + a_4 & \cdots & \alpha_{r-1} + a_{r-1} & 0 & \cdots & \lambda_n + \alpha_n \end{bmatrix},$$

where

$$\begin{aligned}
c_{11} =&\lambda_1 - t,\\
c_{i1} =&\lambda_1 - \lambda_i - t, i = 3,4,\cdots,r-1,\\
c_{21} =&c_{r1} = \cdots = c_{n1} = \lambda_1 - \lambda_2 - t - (a_3 + \cdots + a_{r-1}),\\
c_{12} =&c_{32} = \cdots = c_{r-1} = \alpha_2 + \alpha_r + \cdots + \alpha_n \text{ and}\\
c_{i2} =&\alpha_2 + \lambda_2 - \lambda_i, \qquad i = r,\dots,n.
\end{aligned}$$

The matrix C is nonnegative (and hence a realization of σ) if and only if

$$\begin{aligned}
-\lambda_i \le \alpha_i, &\qquad i = 2,\dots,n,\\
t \le \lambda_1, &\\
-\alpha_i \le a_i, &\qquad i = 3,\dots,r-1,\\
\lambda_1 - \lambda_i - t \ge 0, &\qquad i = 3,\dots,r-1,\\
\lambda_1 - \lambda_2 - t - (\alpha_3 + \cdots + \alpha_{r-1}) \ge 0, &\\
(\alpha_2 + \alpha_r + \cdots + \alpha_n) \ge 0, &\\
\alpha_2 + \lambda_2 - \lambda_i \ge 0, &\qquad i = r,\dots,n.
\end{aligned}$$

with

$$\alpha_i = -\lambda_i, \quad i = 2,\dots,n$$

and

$$a_i = \lambda_i, \quad i = 3,\dots,n$$

and C is a nonnegative matrix. Hence, σ is realizable. □

Example 3. Let $\sigma = \{19,1,-5,-5,-3,-3,-2,-2\}$. This spectrum is chosen from [2] and we show how to use our method to find a realization. We select

$$A = \begin{bmatrix} 19 & 1 & 5 & 5 & 3 & 3 & 2 & 0\\ 0 & 1 & -5 & -5 & -3 & -3 & -2 & 2\\ 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then the matrix $C = LAL^{-1}$ is

$$C = \begin{bmatrix} 0 & 1 & 5 & 5 & 3 & 3 & 2 & 0 \\ 17 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 5 & 1 & 0 & 5 & 3 & 3 & 2 & 0 \\ 5 & 1 & 5 & 0 & 3 & 3 & 2 & 0 \\ 3 & 1 & 5 & 5 & 0 & 3 & 2 & 0 \\ 3 & 1 & 5 & 5 & 3 & 0 & 2 & 0 \\ 2 & 1 & 5 & 5 & 3 & 3 & 0 & 0 \\ 17 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 Real spectrum with three positive numbers and its extension

Now we study σ with three positives and at least 3 non-positives.

Theorem 6. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ be a list of real numbers satisfying*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0 > \lambda_6 \geq \lambda_5 \geq \lambda_4,$$

$\sum_{i=1}^{6} \lambda_i \geq 0$*,* $\lambda_1 \geq |\lambda_4|$ *and* $\lambda_3 \leq |\lambda_6|$*, and also* $\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 \geq 0$*. Then σ is realizable.*

Proof. If $\lambda_2 + \lambda_3 + \lambda_5 + \lambda_6 \geq 0$, and $\lambda_3 \leq |\lambda_6|$, then we have $\lambda_2 + \lambda_5 \geq 0$, so $\lambda_2 \geq |\lambda_5|$ and according to Theorem 2 we can find a nonnegative 4×4 matrix C_4 that realizes $\{\lambda_2, \lambda_3, \lambda_5, \lambda_6\}$ and it is easy to find a nonnegative 2×2 matrix C_2 that realizes $\{\lambda_1, \lambda_4\}$, so the matrix $C = \mathrm{diag}\{C_2, C_4\}$ has eigenvalues σ. If $\lambda_2 + \lambda_3 + \lambda_5 + \lambda_6 \leq 0$, we choose the real numbers α_i (i=$1, \ldots, 6$), a_{24}, a_{34} and a_{35} such that

$$\begin{aligned} -\lambda_i &\leq \alpha_i, \qquad i = 2, \ldots, 6, \\ t &\leq \lambda_1, \\ -\alpha_5 &\leq a_{35} \leq \alpha_2 + \lambda_2 - \lambda_3, \\ -\alpha_4 &\leq a_{24} \leq \lambda_1 - \lambda_2 - t, \\ -\alpha_4 &\leq a_{24} + a_{34} \leq \lambda_1 - \lambda_2 - t, \\ 0 &\leq \alpha_6 + \alpha_3, \\ \alpha_4 &\leq t, \end{aligned}$$

where $t = \sum_{i=2}^{6} \alpha_i$. Let

$$A = \begin{bmatrix} \lambda_1 & \alpha_2 + \alpha_5 + \alpha_6 + \alpha_3 & 0 & \alpha_4 & 0 & 0 \\ 0 & \lambda_2 & \alpha_6 + \alpha_3 & a_{24} & \alpha_5 & 0 \\ 0 & 0 & \lambda_3 & a_{34} & a_{35} & \alpha_6 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then the matrix $C = LAL^{-1}$ is

$$\begin{bmatrix} \lambda_1 - t & t - \alpha_4 & 0 & \alpha_4 & 0 & 0 \\ \lambda_1 - \lambda_2 - t - a_{24} & \alpha_2 + \lambda_2 & \alpha_6 + \alpha_3 & \alpha_4 + a_{24} & \alpha_5 & 0 \\ \lambda_1 - \lambda_2 - t - a_{24} - a_{34} & \alpha_2 + \lambda_2 - \lambda_3 - a_{35} & \alpha_3 + \lambda_3 & \alpha_4 + a_{24} + a_{34} & \alpha_5 + a_{35} & \alpha_6 \\ \lambda_1 - \lambda_4 - t & t - \alpha_4 & 0 & \alpha_4 + \lambda_4 & 0 & 0 \\ \lambda_1 - \lambda_2 - t - a_{24} & \alpha_2 + \lambda_2 - \lambda_5 & \alpha_6 + \alpha_3 & \alpha_4 + a_{24} & \alpha_5 + \lambda_5 & 0 \\ \lambda_1 - \lambda_2 - t - a_{24} - a_{34} & \alpha_2 + \lambda_2 - \lambda_3 - a_{35} & \alpha_3 + \lambda_3 - \lambda_6 & \alpha_4 + a_{24} + a_{34} & \alpha_5 + a_{35} & \alpha_6 + \lambda_6 \end{bmatrix}. \quad (8)$$

The matrix C is nonnegative if and only if the claimed system of inequalities is consistent. To show that the above inequalities are compatible, we present the following: By selecting $\alpha_i = -\lambda_i$ for i$= 2, 3, \ldots, 6$, all entries of the last column of matrix (8) will be nonnegative. Also, by selecting $a_{35} = \lambda_5$, the entries of the fifth column of this matrix will be nonnegative. Additionally if we select $a_{24} = \lambda_4$ and $a_{34} \geq 0$, then all entries of the fourth column of matrix (8) will be nonnegative. The condition $0 \leq \alpha_6 + \alpha_3$ means that $\lambda_3 \leq -\lambda_6$ and this confirms that all the entries of the third column of matrix (8) will also be nonnegative. In second column the condition $t - \alpha_4 \geq 0$ is equivalent to $\lambda_2 + \lambda_3 + \lambda_5 + \lambda_6 \leq 0$ and the condition $\lambda_3 \leq -\lambda_6$ gives $\lambda_3 \leq -\lambda_5$ and consequently all entries of this column are nonnegative. For the first column, if we select $a_{34} = 0$, and since

$$\lambda_1 - \lambda_2 - t - a_{24} = \lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 \geq 0,$$

then we have nonnegative entries of this column. □

We now discuss the method in Theorem 6 for the general case of three positive real eigenvalues. For convenience, we illustrate this for $n = 10$ and we consider $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0 \geq \lambda_{10} \geq \cdots \geq \lambda_4$ with $\sum_{i=1}^{10} \lambda_i \geq 0$. We also assume that the three following conditions for these given eigenvalues are held:

$$\lambda_2 + \lambda_3 + \lambda_5 + \lambda_6 + \ldots + \lambda_{10} \leq 0, \quad (9)$$

$$\lambda_3 + \lambda_8 + \lambda_9 + \lambda_{10} \leq 0, \quad (10)$$

$$\lambda_1 + \lambda_3 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} \geq 0. \quad (11)$$

Let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (12)$$

and

$$A=\begin{bmatrix} \lambda_1 & \alpha_2+\alpha_3+\alpha_5+\cdots+\alpha_{10} & 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & \alpha_3+\alpha_{10}+\alpha_9+\alpha_8 & a_{24} & \alpha_5 & \alpha_6 & \alpha_7 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & a_{34} & a_{35} & a_{36} & a_{37} & \alpha_8 & \alpha_9 & \alpha_{10} \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{10} \end{bmatrix}, \tag{13}$$

where $\alpha_i \geq -\lambda_i$ for $i=2,\ldots,10$. Then the matrix C is

$$C=\begin{bmatrix} \lambda_1-t & t-\alpha_4 & 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{21} & \alpha_2+\lambda_2 & c_{23} & \alpha_4+a_{24} & \alpha_5 & \alpha_6 & \alpha_7 & 0 & 0 & 0 \\ c_{31} & c_{32} & \alpha_3+\lambda_3 & c_{34} & \alpha_5+a_{35} & \alpha_6+a_{36} & \alpha_7+a_{37} & \alpha_8 & \alpha_9 & \alpha_{10} \\ \lambda_1-\lambda_4-t & t-\alpha_4 & 0 & \alpha_4+\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{51} & \alpha_2+\lambda_2-\lambda_5 & c_{53} & \alpha_4+a_{24} & \alpha_5+\lambda_5 & \alpha_6 & \alpha_7 & 0 & 0 & 0 \\ c_{61} & \alpha_2+\lambda_2-\lambda_6 & c_{63} & \alpha_4+a_{24} & \alpha_5 & \alpha_6+\lambda_6 & \alpha_7 & 0 & 0 & 0 \\ c_{71} & \alpha_2+\lambda_2-\lambda_7 & c_{73} & \alpha_4+a_{24} & \alpha_5 & \alpha_6 & \alpha_7+\lambda_7 & 0 & 0 & 0 \\ c_{81} & c_{82} & \alpha_3+\lambda_3-\lambda_8 & c_{84} & \alpha_5+a_{35} & \alpha_6+a_{36} & \alpha_7+a_{37} & \alpha_8+\lambda_8 & \alpha_9 & \alpha_{10} \\ c_{91} & c_{92} & \alpha_3+\lambda_3-\lambda_9 & c_{94} & \alpha_5+a_{35} & \alpha_6+a_{36} & \alpha_7+a_{37} & \alpha_8 & \alpha_9+\lambda_9 & \alpha_{10} \\ c_{10,1} & c_{10,2} & \alpha_3+\lambda_3-\lambda_{10} & c_{10,4} & \alpha_5+a_{35} & \alpha_6+a_{36} & \alpha_7+a_{37} & \alpha_8 & \alpha_9 & \alpha_{10}+\lambda_{10} \end{bmatrix},$$

with

$$\begin{aligned} t &= \sum_{i=2}^{10} \alpha_i, \\ c_{32} = c_{82} = c_{92} = c_{10,2} &= \lambda_2-\lambda_3-a_{35}-a_{36}-a_{37}+\alpha_2, \\ c_{23} = c_{53} = c_{63} = c_{73} &= \alpha_3+\alpha_{10}+\alpha_9+\alpha_8, \\ c_{34} = c_{84} = c_{94} = c_{10,4} &= \alpha_4+a_{24}+a_{34}, \\ c_{21} = c_{51} = c_{61} = c_{71} &= \lambda_1-\lambda_2-t-a_{24}, \\ c_{31} = c_{81} = c_{91} = c_{10,1} &= \lambda_1-\lambda_2-t-a_{24}-a_{34}. \end{aligned}$$

Now, with conditions (9) and (10) and the following choices, it is easy to verify that the matrix C is nonnegative and then σ is realizable:

$$\begin{aligned} &\alpha_i = -\lambda_i, i=2,3,\ldots,10, \\ &a_{37}=\lambda_7,\ \ a_{34}=0,\ \ a_{36}=\lambda_6, \\ &a_{35}=\lambda_5,\ \ a_{24}=\lambda_4. \end{aligned}$$

Example 4. Let

$$\sigma = \{\lambda_1=8, \lambda_2=5, \lambda_3=1, \lambda_4=-5, \lambda_5=-3, \lambda_6=-2, \lambda_7=-1, \lambda_8=-1, \lambda_9=-1, \lambda_{10}=-1\}.$$

We see that $\lambda_3+\lambda_8+\lambda_9+\lambda_{10} = -2 \leq 0$, then by above theorem we find a nonnegative matrix C that σ

is its spectrum. At first, we choose the elements of matrix A as:

$$A = \begin{bmatrix} 8 & 3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & -5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & -2 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then with L given in (12), we have

$$C = LAL^{-1} = \begin{bmatrix} 0 & 3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

which has spectrum σ.

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