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A new numerical method for discretization of the nonlinear Klein-Gordon model arising in light waves

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Abstract. Due to the importance of the generalized nonlinear Klein-Gordon equation (NL-KGE) in describing the behavior of light waves and nonlinear optical materials, including phenomena such as optical switching by manipulating the dispersion and nonlinearity of optical fibers and stable solitons, we explain the approximation of this model by evaluating different classical and fractional terms in this paper. To estimate the fundamental function, we use a first-order finite difference approach in the temporal direction and a collocation method based on Gegenbauer polynomials (GP) in the spatial direction to solve the NL-KGE model. Moreover, the stability and convergence analysis is proved by examining the order of the new method in the time direction as $\mathscr{O}(\delta t)$. To demonstrate the efficiency of this design, we presented numerical examples and made comparisons with other methods in the literature.

Keywords: Nonlinear Klein-Gordon equation, nonlinear optical materials, fractional calculus, collocation method, Gegenbauer polynomial, stability.
AMS Subject Classification 2010: 65L60, 65N12, 35R11.

1 Introduction

The nonlinear Klein-Gordon equation (NL-KGE) is a partial differential equation that mathematically represents and explains the behavior of scalar fields in space and time [25]. This equation considers the nonlinearity of the field, implying that the field's behavior is affected by its magnitude. [5, 7]. NL-KGE has diverse applications in various fields for simulating a wide range of phenomena [14, 24]. The linear Klein-Gordon equation (L-KGE) can only describe a free scalar field, which implies that the field is unaffected by external forces or interactions. [12, 16]. This equation is a simplified version of the NL-KGE and is commonly like a starting point for analyzing scalar fields in physics and other sciences. NL-

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KGE is a more advanced version of L-KGE that accounts for nonlinearity in the scalar field. [23]. It is an influential tool for modeling various phenomena in physics, engineering, and mathematics and is widely used in scientific research and engineering applications [2, 23]. The NL-KGE model is a valuable tool for analyzing the behavior of light waves in nonlinear optical materials, including phenomena in optical switching, self-focusing, and solitons. Solitons, in particular, can be studied using the equation to develop new types of optical communication systems [1, 6]. By manipulating the dispersion and nonlinearity of optical fibers, stable solitons that transmit data at high speeds can be created over long distances. An example of the NL-KGE model in optical communication involves the behavior of solitons in optical fibers. [26]. Optical fibers are commonly used in modern communication systems to transmit information over long distances. The NL-KGE model can be used to describe the behavior of a soliton in an optical fiber [3, 4, 26]. This equation enables researchers to analyze and design new communication systems based on the properties of soliton propagation. The NL-KGE model is crucial in studying the behavior of light waves in nonlinear optical materials and has numerous applications in optic communication and laser technology. This study aims to obtain a numerical solution for a one-dimensional time-dependent NL-KGE that can display a nonlinear phenomenon [19]

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + au(x,t) + bu^3(x,t) = f(x,t), \ \ x \in (0,1), \ \ t \in (0,T], \tag{1}$$

where $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ is the classical derivative for $\alpha = 2$ and Caputo derivative for $1 < \alpha < 2$, which is defined as follows

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = {}^C D_x^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^\xi \frac{\partial^2 u(\xi,t)}{\partial^2 \xi} (x-\xi)^{(1-\alpha)} d\xi.$$

The following assumptions are considered as initial and boundary conditions when dealing with the given Eq (1)

$$\begin{cases} u(x,0) = q_1(x), \ \ \frac{\partial u(x,t)}{\partial t}\big|_{t=0} = q_2(x), & x \in (0,1), \\ u(0,t) = p_1(t), \ \ u(1,t) = p_2(t), & t \in (0,T], \end{cases}$$

in which the function $f(x,t)$ is the source term.

In Eq. (1), the function $u(x,t)$ represents the wave displacement at a specific spatial position x and time t with the constants a and b having predetermined values. Additionally, the self-interaction in the system is mathematically described by the nonlinear function $f(x,t)$. A variety of analytical and numerical methods have been used to solve this equation. In this section, we will briefly describe some of these studies. In [18], amplitude instability analysis is investigated to solve NL-KGE by the finite difference method. Earlier than this, spectral methods based on the [15] present the Legendre orthogonal polynomials. Dehghan in [8] proposed a method to tackle NL-KGE in optoelectronic devices. Sirendaoreji later introduced a new scheme for NL-KGE using auxiliary equations [6]. A cubic B-spline collocation method to gain the numerical result of NL-KGE is presented by Rashidinia et. al [20]. Using the Tension spline scheme, they tackled the solution of NL-KGE in paper [21]. Dehghan et al. utilized radial basis functions and fourth-order compact schemes and solved the NL-KGE. in papers [11] and [10]. In 2010, a combination of collocation and finite difference scheme was adopted to get a new numerical method for the NL-KGE model in [17]. Later, the procedure of the dual reciprocity boundary integral model was applied to solve the NL-KGE model in [9]. All the mentioned methods such as the implicit finite difference method used only on the mesh points. Moreover, the accuracy of these techniques decreases in non-smooth and irregular regions. Therefore, we present a new, unconditionally stable approach to solving this equation that improves the convergence of the numerical method.

The rest of this paper deals with these sections. Numerical procedures and structure for the time discretization are present in Section 2. Section 3 provides a detailed explanation of the GP approximation operators. In Section 4, we discuss the semi-design stability studies of numerical methods. Section 5 compares numerical example results with other studies.

2 Semi-discretization scheme

Let us select grid points $t_j = j\delta t,\ j = 0,1,\ldots,N_t$ for the time interval $[0,T]$, in which $\delta t = T/N_t$ is a uniform time step length. In this case, the domain $[0,T]$ is covered by $\Omega_{\delta t}$, where $\Omega_{\delta t} = \{t_j | 0 \le j \le N_t\}$. Let $\mathfrak{u}^j = \{u^j(x) | 0 \le j \le N_t\}$ be the discrete function in time on the mesh $\Omega_{\delta t}$. To clarify the notation for any grid function $u^j(x) \in \mathfrak{u}^j$, describe it as follows

$$\frac{\partial^2 u(x,t_j)}{\partial t^2} = \frac{u^{j-1}(x) - 2u^j(x) + u^{j+1}(x)}{\delta t^2} + \mathscr{O}(\delta t^2), \quad \frac{\partial u(x,t_j)}{\partial t} = \frac{u^{j+1}(x) - u^{j-1}(x)}{2\delta t} + \mathscr{O}(\delta t^2), \tag{2}$$

where $u^j(x) = u(x,t_j)$ and $\mathscr{O}(\delta t^2)$ indicates the local truncation error of the approximation. Considering (1) at the point t_{j+1}, we have

$$\frac{u^{j-1}(x) - 2u^j(x) + u^{j+1}(x)}{\delta t^2} - \frac{\partial^\alpha u^{j+1}(x)}{\partial x^\alpha} + au^{j+1}(x) + b(u^{j+1}(x))^3 = f^{j+1}(x) + \mathscr{O}(\delta t^2), \tag{3}$$

where $f^{j+1}(x) = f(x,t_{j+1})$. Simplifying (3) leads to

$$(1 + a\delta t^2)u^{j+1}(x) - \delta t^2 \frac{\partial^\alpha u^{j+1}(x)}{\partial x^\alpha} + b\delta t^2 (u^{j+1}(x))^3 = \delta t^2 f^{j+1}(x) - u^{j-1}(x) + 2u^j(x) + \mathscr{O}(\delta t^4). \tag{4}$$

As U^3 has a continuous first-order derivative, we can represent it in the following way

$$u^3(x,t_{j+1}) = u^3(x,t_j) + \mathscr{O}(\delta t).$$

Then, we can easily rewrite Eq. (4) as below

$$(1 + a\delta t^2)u^{j+1}(x) - \delta t^2 \frac{\partial^\alpha u^{j+1}(x)}{\partial x^\alpha} = \delta t^2 f^{j+1}(x) - b\delta t^2 (u^j(x))^3 - u^{j-1}(x) + 2u^j(x) + \mathscr{O}(\delta t). \tag{5}$$

Directly, by inserting $j = 0$ in (5), one gets

$$(1 + a\delta t^2)u^1(x) - \delta t^2 \frac{\partial^\alpha u^1(x)}{\partial x^\alpha} + a\delta t^2 u^1(x) = \delta t^2 f^1(x) - b\delta t^2 (u^1(x))^3 - u^{-1}(x) + 2u^0(x) + \mathscr{O}(\delta t), \tag{6}$$

and applying the following relation using notation (2)

$$\frac{u^1(x) - u^{-1}(x)}{2\delta t} = \frac{\partial u(x,0)}{\partial t} = q_2(x),$$

one derives that

$$u^{-1}(x) = u^1(x) - 2\delta t \times q_2(x).$$

By replacing the above relation in Eq. (6), one reaches the following relation for $j=0$

$$(2+a\delta t^2)u^1(x)-\delta t^2\frac{\partial^\alpha u^1(x)}{\partial x^\alpha}=\delta t^2 f^1(x)-b\delta t^2(u^1(x))^3+2\delta t\times q_2(x)+2u^0(x)+\mathscr{O}(\delta t^2). \quad (7)$$

Let $U^j(x)$ be the approximate solution $u^j(x)$ in the relations (5) and (7). Then the semi-discrete procedure is obtained as

$$\begin{cases}(2+a\delta t^2)U^1(x)-\delta t^2\frac{\partial^\alpha U^1(x)}{\partial x^\alpha}=\delta t^2 f^1(x)-b\delta t^2(U^0(x))^3+2\delta t\times q_2(x)+2U^0(x), & j=0,\\ (1+a\delta t^2)U^{j+1}(x)-\delta t^2\frac{\partial^\alpha U^{j+1}(x)}{\partial x^\alpha}=\delta t^2 f^{j+1}(x)-b\delta t^2(U^j(x))^3-U^{j-1}(x)+2U^j(x), & 1\le j\le N_t-1.\end{cases} \quad (8)$$

3 Full-discretization scheme

To find the full discretization, we first explain the basis function to approximate the space in $\Omega_x=\{x_i\}_{i=0}^{N_x}$ in the domain $[0,1]$ where N_x is a positive integer. For convenience, we denote $u^j(x_i)=u_i^j$ and $f^j(x_i)=f_i^j$. This approximation is defined by the summation of a coefficient ρ_i^j in each temporal step j which is multiplied by the orthogonal polynomials $\varphi_i(x)$ as

$$u^j(x)=\sum_{i=0}^{N_x}\rho_{i+1}^j\varphi_{i+1}(x), \quad (9)$$

where ρ_{i+1}^j is the coefficient and $\rho_{i+1}^j=\langle u^j(x),\varphi_{i+1}(x)\rangle$. We can use many polynomials as $\{\varphi_{i+1}(x)\}_{i=0}^{N_x}$ to do so. In this paper, we use shifted GPs that is the well-known Jacobi polynomials $P_n^{(\alpha,\beta)}(x)|_{x\to 2x-1}$ with $\alpha=\beta=\gamma-\frac{1}{2},(\gamma>-\frac{1}{2})$ and is defined as below in the interval $[0,1]$

$$\varphi_{i+1}(x)=\sum_{k=0}^{i}\sum_{r=0}^{k}\mathscr{G}_{k,r}^{i,\gamma}x^r,\ \ i=0,1,\dots,N_x, \quad (10)$$

where

$$\mathscr{G}_{k,r}^{i,\gamma}=\frac{(-1)^{k-r}(i+2\gamma-1)!(2\gamma+i+k-1)!(\gamma-0.5)!}{r!(i+2\gamma-i-1)!(i-k)!(2\gamma+i-1)!(\gamma+k-0.5)!(k-r)!}.$$

To obtain the fully discrete scheme, we have to approximate each term of Eq. (8) by the relation (9). In Eq. (8) there is a Caputo fractional derivative of $u^j(x)$, which can be estimated at node x_i. Here, we approximate this term as follows

$${}_0D_x^\alpha\varphi_{i+1}(x)=\sum_{k=0}^{i}\sum_{r=\lceil\alpha\rceil}^{k}G_{k,r}^{i,\gamma}x^{r-\alpha},\quad i=\lceil\alpha\rceil,\lceil\alpha\rceil+1,\dots,N_x, \quad (11)$$

in which $\lceil\alpha\rceil$ is the ceiling of the fractional term α $G_{k,r}^{i,\gamma}=\frac{(r)!}{(r-\alpha)!}\rho_i^j$, and for $0\le i<\lceil\alpha\rceil$ we get ${}_0D_x^\alpha\varphi_{i+1}(x)=0$. As a result, employing the last series in (9), we have

$$\frac{\partial^\alpha u^j(x)}{\partial x^\alpha}=\sum_{i=0}^{N_x}\rho_{i+1}^j\frac{\partial^\alpha\varphi_{i+1}(x)}{\partial x^\alpha}=\sum_{i=0}^{N_x}\sum_{k=0}^{i}\sum_{r=\lceil\alpha\rceil}^{k}\rho_{i+1}^j G_{k,r}^{i,\gamma}x^{r-\alpha}=\Lambda^j\Phi^\alpha, \quad (12)$$

where $\Lambda^j = [\rho_1^j, \rho_2^j, \ldots, \rho_{i+1}^j]$, and Φ^α is a vector that each of its entries is obtained from

$$\sum_{k=0}^{i} \sum_{r=\lceil\alpha\rceil}^{k} \rho_{i+1}^j G_{k,r}^{i,\gamma} x^{r-\alpha}.$$

Moreover, we can write Eq. (9) in the matrix form as below

$$u^j(x) = \sum_{i=0}^{N_x} \sum_{k=0}^{i} \sum_{r=0}^{k} \rho_{i+1}^j \mathscr{G}_{k,r}^{i,\gamma} x^r = \Lambda^j \Phi, \tag{13}$$

where Φ is a vector that each of its entries is obtained from

$$\sum_{k=0}^{i} \sum_{r=0}^{k} \rho_{i+1}^j \mathscr{G}_{k,r}^{i,\gamma} x^r.$$

Now, according to Eqs. (12) and (13), and substituting in Eq. (8), we can immediately obtain the following nonlinear equations

$$\begin{cases} \Lambda^1\Phi^1 = \Lambda^0\Phi^0 + F^1, & j=0, \\ \Lambda^{j+1}\Phi^{j+1} = \Lambda^{j-1}\Phi^{j-1} + \Lambda^j\Phi^j + F^{j+1}, & 1 \le j \le N_t - 1, \end{cases} \tag{14}$$

where

$$\Phi^1 = (2 + a\delta t^2)\Phi - \delta t^2 \Phi^\alpha, \quad \Phi^0 = 2\Phi - b\delta t^2 (\Phi)^3, \quad F^1 = \delta t^2 f^1(x) + 2\delta t \times q_2(x),$$
$$\Phi^{j+1} = (1 + a\delta t^2)\Phi - \delta t^2 \Phi^\alpha, \quad \Phi^{j-1} = -\Phi, \quad \Phi^j = 2\Phi - b\delta t^2 (\Phi)^3, \quad F^{j+1} = \delta t^2 f^{j+1}(x).$$

We use the collocation manner to solve the nonlinear system of equations (14). For this objective, we get the roots of the shifted GP, $\varphi_{N_x-1}(x)$, as the collocation points and substitute them in the system to obtain nonlinear equations at each time step j. The function $\varphi_{N_x-1}(x)$ has $N-1$ roots. So we need two more conditions to obtain a system with $N_x + 1$ equations, which can be obtained using the following boundary conditions for $i = 0, 1, \ldots N_x$ and $k = 1, 2, \ldots, N_t$

$$u(0,t_j) = \sum_{i=0}^{N_x} \rho_{i+1}^j \varphi_{i+1}(0) = p_1(t_j), \quad u(1,t_j) = \sum_{i=0}^{N_x} \rho_{i+1}^j \varphi_{i+1}(1) = p_2(t_j).$$

To start the iteration method, we need the initial condition as

$$u(x,0) = \sum_{i=0}^{N_x} \rho_{i+1}^0 \varphi_{i+1}(x) = q_1(x),$$

in which

$$\rho_{i+1}^0 = \langle q_1(x), \varphi_{i+1}(x) \rangle = \int_0^1 \frac{2(k+\gamma)k!\Gamma(\gamma)^2}{\pi 2^{1-2\gamma}\Gamma(k+2\gamma)} \times (4x - 4x^2)^{\gamma-0.5} q_1(x) \varphi_{i+1}(x) dx,$$

where the coefficient multiplied by the above integral is the weight function of the shifted GPs.

4 Study of the stability

For analyzing the stability of the numerical method (8), let us discuss the homogeneous part of the second relation, which may be expressed as

$$(1+a\delta t^2)U^{j+1}(x)-\delta t^2\frac{\partial^\alpha U^{j+1}(x)}{\partial x^\alpha}=-b\delta t^2(U^j(x))^3-U^{j-1}(x)+2U^j(x),\quad 1\le j\le N_t-1. \tag{15}$$

To exhibit the unconditional stability of the new method, it is necessary to check that the error in each time step is less than the prior step, i.e.,

$$\|\varepsilon^{j+1}(x)\|\le\|\varepsilon^j(x)\|,\quad 1\le j\le N_t-1,$$

where $\varepsilon^j(x)=U^j(x)-u^j(x)$ that $U^j(x)$ and $u^j(x)$ are the approximate and exact solution, respectively. Suppose the functional space with the Hilbert space $L^2(\Omega)$ in Ω and the standard norm $\|u^j(x)\|_2^2=\langle u^j(x),u^j(x)\rangle$ is as following

$$H_\Omega^n(u^j(x))=\{u^j(x)\in L^2(\Omega),D^\alpha u^j(x)\in L^2(\Omega),\forall|\alpha|\le n\},$$

where D^α is the fractional derivative. Using (15) and multiplying $\varepsilon^{j+1}(x)$ and integrating on Ω, we obtain the following relation

$$\begin{aligned}&(1+a\delta t^2)\langle\varepsilon^{j+1}(x),\varepsilon^{j+1}(x)\rangle-\delta t^2\langle\frac{\partial^\alpha\varepsilon^{j+1}(x)}{\partial x^\alpha},\varepsilon^{j+1}(x)\rangle\\&\quad=-b\delta t^2(\langle\varepsilon^j(x),\varepsilon^{j+1}(x)\rangle)^3-\langle\varepsilon^{j-1}(x),\varepsilon^{j+1}(x)\rangle+2\langle\varepsilon^j(x),\varepsilon^{j+1}(x)\rangle.\end{aligned} \tag{16}$$

Now we state the following lemmas which are the properties of inner multiplication for fractional functions.

Lemma 1 ([13, 22]). *The following relation holds for $\alpha\in(1,2)$ and $u,v\in H_\Omega^2$ in the domain (a,b)*

$$\langle {}_aD_x^\alpha u(x),v(x)\rangle=\langle {}_aD_x^{\frac{\alpha}{2}}u(x),{}_xD_b^{\frac{\alpha}{2}}v(x)\rangle,$$

where ${}_aD_x^{\frac{\alpha}{2}}u(x)$ and ${}_xD_b^{\frac{\alpha}{2}}v(x)$ are the left and right Caputo or Riemann-Liouville fractional derivatives, respectively.

Lemma 2 ([13, 22]). *If $u(x)\in H_\Omega^n$, then the following conditions hold for $\alpha\ge 0$*

$$\langle {}_aD_x^\alpha u(x),{}_xD_b^\alpha u(x)\rangle=\cos(\pi\alpha)\|{}_xD_b^\alpha u(x)\|^2=\cos(\pi\alpha)\|{}_xD_b^\alpha u(x)\|^2.$$

It is clear from Lemmas 1 and 2 that the second term of Eq. (16) is nonnegative. Then, we have

$$(1+a\delta t^2)\langle\varepsilon^{j+1}(x),\varepsilon^{j+1}(x)\rangle\le-b\delta t^2(\langle\varepsilon^j(x),\varepsilon^{j+1}(x)\rangle)^3-\langle\varepsilon^{j-1}(x),\varepsilon^{j+1}(x)\rangle+2\langle\varepsilon^j(x),\varepsilon^{j+1}(x)\rangle.$$

According to the nonlinear term, it summarizes to

$$(1+a\delta t^2)\langle\varepsilon^{j+1}(x),\varepsilon^{j+1}(x)\rangle\le-\langle\varepsilon^{j-1}(x),\varepsilon^{j+1}(x)\rangle+(2-b\delta t^2)\langle\varepsilon^j(x),\varepsilon^{j+1}(x)\rangle.$$

Since $1+a\delta t^2 \geq 1$ and $\langle \varepsilon^{j+1}(x), \varepsilon^{j+1}(x)\rangle = \|\varepsilon^{j+1}(x)\|^2$, we get

$$\|\varepsilon^{j+1}(x)\|^2 \leq -\langle \varepsilon^{j-1}(x), \varepsilon^{j+1}(x)\rangle + 2\langle \varepsilon^{j}(x), \varepsilon^{j+1}(x)\rangle.$$

So

$$\|\varepsilon^{j+1}(x)\|^2 \leq \langle \varepsilon^{j-1}(x), \varepsilon^{j+1}(x)\rangle + 2\langle \varepsilon^{j}(x), \varepsilon^{j+1}(x)\rangle.$$

On the other hand, employing the Cauchy-Schwarz inequality, we get

$$\|\varepsilon^{j+1}(x)\| \leq \|\varepsilon^{j-1}(x)\| + 2\|\varepsilon^{j}(x)\|,$$

which is equivalent to

$$\|\varepsilon^{j+1}(x)\| - \|\varepsilon^{j}(x)\| \leq \|\varepsilon^{j-1}(x)\| + \|\varepsilon^{j}(x)\|. \tag{17}$$

Hence, summing up the relation (17) from $j=1$ to $j=N_t$, we acquire the following relation

$$\sum_{j=1}^{N_t}(\|\varepsilon^{j+1}(x)\| - \|\varepsilon^{j}(x)\|) \leq \sum_{j=1}^{N_t} \|\varepsilon^{j-1}(x)\| + \sum_{j=1}^{N_t} \|\varepsilon^{j}(x)\|.$$

Then, we get

$$\|\varepsilon^{N_t+1}(x)\| - \|\varepsilon^{1}(x)\| \leq \sum_{j=1}^{N_t} \|\varepsilon^{j-1}(x)\| + \sum_{j=1}^{N_t} \|\varepsilon^{j}(x)\|,$$

which gives

$$\|\varepsilon^{N_t+1}(x)\| \leq \|\varepsilon^{1}(x)\| + 2\sum_{j=1}^{N_t-1} \|\varepsilon^{j-1}(x)\| + \|\varepsilon^{N_t}(x)\|.$$

For different values of N_t, the above relation can be written as follows

$$\begin{aligned} &\text{for } N_t = 1 : \|\varepsilon^{2}(x)\| \leq 2\|\varepsilon^{1}(x)\|,\\ &\text{for } N_t = 2 : \|\varepsilon^{3}(x)\| \leq 5\|\varepsilon^{1}(x)\|,\\ &\text{for } N_t = 3 : \|\varepsilon^{4}(x)\| \leq 12\|\varepsilon^{1}(x)\|. \end{aligned}$$

With the continuation of the previous relationship, we see that

$$\|\varepsilon^{N_t+1}(x)\| \leq C\|\varepsilon^{1}(x)\|,$$

where C is a nonnegative constant. This inequality shows that the numerical method is unconditional stable in the time direction.

5 Numerical results and implementation

In this section, we apply the new scheme to solve a numerical example (Eq. (1)) to demonstrate its potential. All examples are programmed on a Dell Inspiron Intel (R) Core i72630QM 2.00GHz using Wolfram Mathematica 11. In some examples, the exact solution is utilized to estimate the order of convergence so that the arrangement of the connecting stability of the strategy might be well demonstrated. The fourth example is taken from [11] so that we can compare our results with the numerical results of this paper. To show the error, we use L_∞-norm and L_2-norm errors as below

$$E_\infty = \max_{0\leq i\leq N_x} |u_i^{N_t} - \widehat{u}_i^{N_t}|, \quad E_2^2 = \sum_{i=0}^{N_x} |u_i^{N_t} - \widehat{u}_i^{N_t}|^2, \quad E_{RMS}^2 = \frac{1}{N_x+1}\sum_{i=0}^{N_x} |u_i^{N_t} - \widehat{u}_i^{N_t}|^2,$$

where $u_i^{N_t}$ and $\widehat{u}_i^{N_t}$ are the exact and approximated solution in the final step of the temporal variable, respectively. Moreover, the observed order of the numerical method about the temporal variable is denoted as follows

$$\mathscr{O} = \log_2\left(\frac{e_{\delta t}}{e_{\frac{\delta t}{2}}}\right),$$

where $e_{\delta t}$ is defined as the error of the numerical solution with the step size δt at the final time T.

Example 1. Consider the following nonlinear Klein-Gordon model in $0 < x < 1,\ 0 \leq t \leq T$,

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) + bu^3(x,t) = f(x,t),$$

with the initial and boundary conditions as

$$\begin{cases} u(x,0) = x(1-x), \quad \frac{\partial u(x,t)}{\partial t}|_{t=0} = -x(1-x), & x \in (0,1), \\ u(0,t) = u(1,t) = 0, & t \in (0,T], \end{cases}$$

and the source term $f(x,t)$ is

$$f(x,t) = \exp(-t)\big((1+a)x(x-1) + b\exp(-2t)x^3(x-1)^3 - 2\big).$$

The exact solution is given by $u(x,t) = \exp(-t)x(x-1)$. Our principal claim is to make the numerical convergence rate of the suggested scheme over time. Table 1 illustrates the discrete error and convergence order of t for the new method. In this table, we choose $N_x = 5$, and $N_x = 7$. It can be seen that the numerical results are in the perfect match with the exact solution and the convergence order is approaching the predicted order 1. Figure 1 illustrates the numerical results that converge to the exact solution and confirms that the approach is numerically convergent.

Table 1: The L_2 and L_∞ errors of Example 1 with $a = b = 1$, and $N_x = 5$, and $N_x = 7$, at $T = 1$.

	Error and the convergence order by the new method with $N_x = 5$				Error and the convergence order by the new method with $N_x = 7$			
N_t	E_∞	$\mathscr{O}$	E_2	$\mathscr{O}$	E_∞	$\mathscr{O}$	E_2	$\mathscr{O}$
20	1.22533$E-3$	–	2.74532$E-3$	–	1.22448$E-3$	–	2.74524$E-3$	–
40	6.54549$E-4$	0.904591	1.46758$E-3$	0.903541	6.54431$E-4$	0.903858	1.46756$E-3$	0.903520
80	3.39070$E-4$	0.948919	7.60810$E-4$	0.947827	3.39186$E-4$	0.948164	7.60808$E-4$	0.947811
160	1.72658$E-4$	0.973666	3.87630$E-4$	0.972856	1.72781$E-4$	0.973132	3.87632$E-4$	0.972845
320	8.71313$E-5$	0.986655	1.95684$E-4$	0.986156	8.72121$E-5$	0.986343	1.95685$E-4$	0.986150

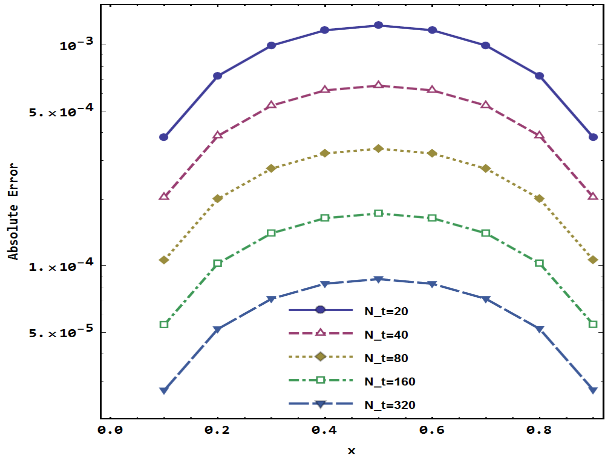

Figure 1: $E_\infty-$error the Example 1 with $N_x = 5$ and the various temporal size at $T = 1$.

Example 2. Consider the following nonhomogeneous problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) + bu^3(x,t) = f(x,t), \quad (x,t) \in (0,1) \times (0,T],$$

with the initial and boundary conditions as

$$\begin{cases} u(x,0) = \cos(x), & \frac{\partial u(x,t)}{\partial t}|_{t=0} = -\sin(x), \quad x \in (0,1), \\ u(0,t) = \cos(t), & u(1,t) = \cos(1+x), \qquad t \in (0,T], \end{cases}$$

and the source term is $f(x,t) = \cos(x+t)\big(a + b\cos^2(x+t)\big)$. The exact solution is given by $u(x,t) = \cos(x+t)$. Numerical outcomes are demonstrated in Table 2, Figures 2, and 3. From Table 2 it is obvious that the new strategy corresponds to the first order convergence. It also presents the results that converge in the spatial direction. This means that each time we decrease the step length in space and/or time, numerical results converge to the exact solution, confirming that the method is numerically convergent. Figure 2 shows the approximated solution and its error with $N_T = 200$, $N_x = 7$ at $T = 1$. It is clear from this figure that the suggested numerical procedure has high convergence order and good accuracy. Moreover, in Figure 3, the absolute error is shown with the various values of the temporal direction and $N_x = 7$ at $T = 1$, where decreasing the step length in the time direction reduces the error in the numerical results.

Example 3. Consider the NL-KGE model with the fractional term in the spatial derivative direction

$$\frac{\partial^2 u(x,t)}{\partial t^2} - {}^C D_x^\alpha u(x,t) + au(x,t) + bu^3(x,t) = f(x,t), \tag{18}$$

with the initial and boundary conditions as

$$\begin{cases} u(x,0) = x^2 + 1, & \frac{\partial u(x,t)}{\partial t}|_{t=0} = -x^2 - 1, \quad x \in (0,1), \\ u(0,t) = \exp(-t), & u(1,t) = 2\exp(-t), \quad t \in (0,T], \end{cases}$$

Table 2: The L_2 and L_∞ errors of Example 2 with $a=b=1$, and $N_x=5$, and $N_x=7$, at $T=1$.

	Error and the convergence order by the new method with $N_x=5$				Error and the convergence order by the new method with $N_x=7$			
N_t	E_∞	$\mathscr{O}$	E_2	$\mathscr{O}$	E_∞	$\mathscr{O}$	E_2	$\mathscr{O}$
20	$8.04389E-3$	–	$1.84052E-2$	–	$8.05236E-3$	–	$1.83980E-2$	–
40	$4.21166E-3$	0.933505	$9.64901E-3$	0.931659	$4.21290E-3$	0.934599	$9.64369E-3$	0.931890
80	$2.15573E-3$	0.966215	$4.94726E-3$	0.963750	$2.15420E-3$	0.967663	$4.94390E-3$	0.963935
160	$1.09018E-3$	0.983604	$2.50598E-3$	0.981256	$1.08872E-3$	0.984519	$2.50401E-3$	0.981409
320	$5.47875E-4$	0.992651	$1.26135E-3$	0.990405	$5.47146E-4$	0.992633	$1.26023E-3$	0.990557

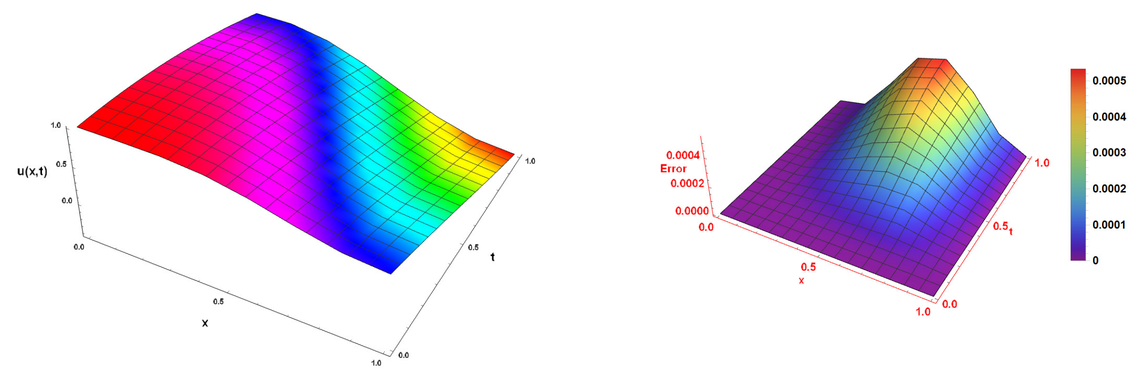

Figure 2: The approximated solution (left panel) and the L_∞−error (right panel) with $N_t=200$, $N_x=7$ for Example 2.

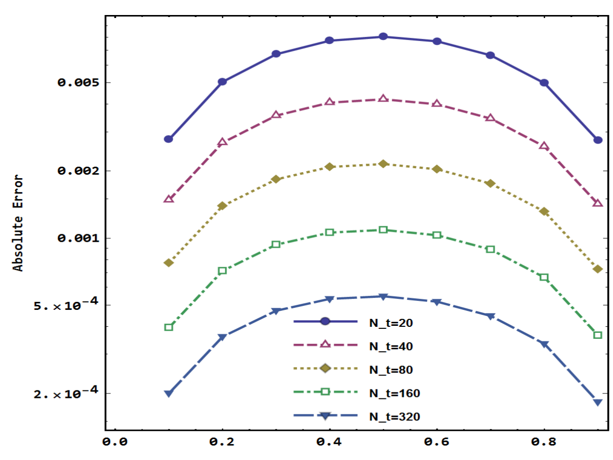

Figure 3: The absolute error for Example 2 with $N_x=7$ at $T=1$.

and the source term is

$$f(x,t)=\exp(-t)\big((1+a)(x^2+1)-\frac{2}{\Gamma(3-\alpha)}x^{2-\alpha}+b\exp(-2t)(x^2+1)^3\big).$$

Table 3: The L_2 and L_∞ errors of Example 3 with $a = b = 1$, and $N_x = 5$, at $T = 1$.

	Error and the convergence order by the new method with $\alpha = 1.8$				Error and the convergence order by the new method with $\alpha = 1.5$			
N_t	E_∞	$\mathscr{O}$	E_2	$\mathscr{O}$	E_∞	$\mathscr{O}$	E_2	$\mathscr{O}$
20	5.78084$E-3$	–	4.50402$E-2$	–	2.49446$E-2$	–	4.55574$E-2$	–
40	2.88622$E-3$	1.0021	2.25322$E-2$	0.999225	1.24683$E-2$	1.00046	2.30457$E-2$	0.98319
80	1.44001$E-3$	1.0031	1.12714$E-2$	0.999325	6.22787$E-3$	1.00146	1.16571$E-2$	0.98329
160	7.17962$E-4$	1.0041	5.63793$E-3$	0.999425	3.10863$E-3$	1.00246	5.89603$E-3$	0.98339
320	3.57715$E-4$	1.0051	2.81989$E-3$	0.999525	1.55059$E-3$	1.00346	2.98195$E-3$	0.98349

The exact solution of Eq. (18) under the above conditions is given by $u(x,t) = \exp(-t)(x^2+1)$. We solved Example 3 with the new method presented in Section 3 with the various values of α for different N_x and different step sizes δt. Numerical results are shown in Table 2 and Figure 4. It is clear from Table 2 that our new method is of the first order of convergence and shows that the numerical strategy has convergence in the spatial direction. Figure 4 demonstrates that each time we decrease the step length in the time direction, the numerical results converge to the exact solution that accredits the numerical convergence of the new procedure.

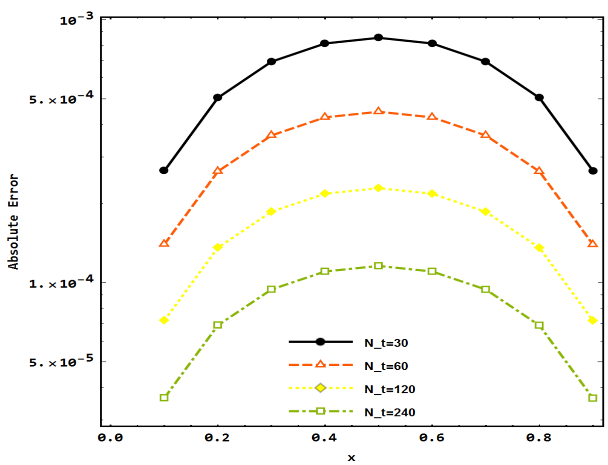

Figure 4: The absolute error for the Example 3 with $N_x = 6$, and $\alpha = 1.9$ at $T = 1$.

The paper [11] studied a nonhomogeneous problem investigating NL-KGE using thin-plate splines radial basis functions. In this paper, we compare the results with the new scheme.

Example 4. Consider NL-KGE studied in paper [11] defined by Eq. (1) as

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u^3(x,t) = f(x,t),$$

with the initial and boundary conditions as

$$\begin{cases} u(x,0) = 0, & \frac{\partial u(x,t)}{\partial t}|_{t=0} = 0, \quad x \in (0,1), \\ u(0,t) = 0, & u(1,t) = t^3, \qquad t \in (0,T], \end{cases}$$

Table 4: Comparing the error of method [11] with the new method for Example 4 with $\delta t = 0.0001$, and $N_x = 5$, at $T = 1,2,3,4,5$.

	Errors and the CPU time of [11] with $N_x = 20$				Errors and the CPU time of the new method with $N_x = 5$			
T	E_∞	E_2	E_{RMS}	$Time(s)$	E_∞	E_2	E_{RMS}	$Time(s)$
1	$1.1012E-5$	$5.4998E-5$	$5.4725E-6$	6	$7.52894E-6$	$8.48962E-6$	$7.89025E-6$	5.492
2	$1.6496E-4$	$1.1522E-3$	$1.1465E-4$	14	$7.59203E-5$	$9.58702E-5$	$9.70026E-5$	10.725
3	$5.9728E-4$	$3.2588E-3$	$3.2426E-4$	25	$4.89350E-5$	$7.56843E-5$	$6.79205E-5$	14.839
4	$1.8264E-3$	$9.8191E-3$	$9.7704E-4$	37	$5.86302E-4$	$6.87904E-4$	$1.00720E-4$	19.982
5	$3.6915E-3$	$1.9139E-2$	$1.9044E-3$	52	$3.92380E-4$	$6.19430E-3$	$1.36401E-4$	28.079

and the source term is $f(x,t) = 6xt(x^2 - t^2) + x^3t^3$. The exact solution under the above conditions is given by $u(x,t) = x^3t^3$. We solved Example 4 with the new scheme with the fixed value N_x and different values of T. The numerical results are shown in Table 4. It is clear from Table 4 that our new scheme is better than the numerical results of [11] . Moreover, it shows that whenever we reduce step size N_x in the spatial direction, the numerical results are better than those in [11] . Also, the computational time in the new method is much less compared to the one in [11] .

6 Conclusions

In this paper, we have presented a first-order method to describe solving the Klein-Gordon model using fractional and classical terms in spatial differentiation, such that we define the fractional terms in the Caputo sense. We also demonstrated the stability and convergence analysis of the presented scheme by examining the order of the numerical method in the time direction as $\mathcal{O}(\delta t)$. To validate the implementation of our numerical strategy, we have considered four examples. Numerical results presented in tabular and graphical form demonstrate the high-order and stable performance of the proposed scheme. Moreover, we theoretically proved the stability of numerical techniques using the energy method. We observed good performance for nonlinear and fractional terms.

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