Journal of Mathematical Modeling
Vol. 12, No. 1, 2024, pp. 51–70. Research Article

JMM

Analysis of a coupled pair of Cahn-Hilliard equations with nondegenerate mobility

Ghufran A. Al-Musawi[†], **Akil J. Harfash**[†]*

[†]*Department of Mathematics, College of Sciences, University of Basrah, Basrah, Iraq*
Email(s): ghufranalmusawi@utq.edu.iq, akil.harfash@uobasrah.edu.iq

Abstract. A mathematical analysis is performed for a system consisting of two coupled Cahn-Hilliard equations. These equations incorporate a diffusional mobility that depends on concentration. This modeling approach is often used to describe the process of phase separation in a thin layer of a binary liquid mixture covering a substrate, particularly when one of the components wets the substrate. The analysis establishes the existence of a weak formulation for this problem, which is supported by the use of a Lyapunov functional. Additionally, the analysis provides insights into the regularity properties of the weak formulation.

Keywords: existence, Faedo-Galerkin, Cahn-Hilliard, weak formulation, strong solution.
AMS Subject Classification 2010: 35A01, 35D30, 35D35.

1 Introduction

Variants of Cahn-Hilliard equations have received a lot of attention recently. These equations have grown in relevance as a result of their widespread application in a variety of domains, including modeling alloys, glasses, and polymers [17, 27]. In [8], Cahn and Hilliard initially proposed Cahn-Hilliard model to describe the dynamics of binary mixture separation into two phases. This classical model has been effectively utilized to describing the so-called spinodal decomposition or phase separation phenomena, see for example [9, 25, 26] and their references for qualitative works on this subject.

Let $\mathscr{R}$ represents an open and bounded domain in $\mathbb{R}^n$, where $(n \geq 1)$. This domain has a smooth boundary denoted as $\partial\mathscr{R}$. The mathematical model under consideration involves a pair of coupled Cahn-Hilliard equations. These equations take into account a diffusional mobility that depends on the concentration of the substances involved. This model is used to describe the process of phase separation occurring on a thin film of a binary liquid mixture covering a substrate. Notably, one of the components

*Corresponding author
Received: 15 September 2023 / Revised: 22 October 2023 / Accepted: 24 October 2023
DOI: 10.22124/jmm.2023.25558.2272

is referred to as A, and the other as B. For further details and information related to this model please see [23]. The mathematical model of the problem is as follows:
(P) Find $\{u_1, u_2, w_1, w_2\}$ such that

$$\partial_t u_1 = \nabla(M(u_1)\nabla w_1), \qquad \text{in} \quad Q_T, \tag{1}$$

$$\partial_t u_2 = \nabla(M(u_2)\nabla w_1), \qquad \text{in} \quad Q_T, \tag{2}$$

where

$$w_1 = \frac{\delta F(u_1, u_2)}{\delta u_1}, \tag{3}$$

$$w_2 = \frac{\delta F(u_1, u_2)}{\delta u_2}, \tag{4}$$

$$F(u_1, u_2) = b_1 u_1^4 - a_1 u_1^2 + c_1 |\nabla u_1|^2 + b_2 u_2^4 - a_2 u_2^2 + c_2 |\nabla u_2|^2$$
$$+ D\Big(u_1 + \sqrt{\frac{a_1}{2b_1}}\Big)^2 \Big(u_2 + \sqrt{\frac{a_2}{2b_2}}\Big)^2, \tag{5}$$

where $Q_T = \mathscr{R} \times (0, T)$ and the diffusional mobility $M \in C([-1, 1])$ is assumed to satisfy

$$M(1) = M(-1) = 0, \quad \text{and} \quad m_0 \leq M(s) \leq M_0, \ \ \forall s \in (-1, 1), \ m_0, M_0 > 0. \tag{6}$$

Here $\delta F(u_1, u_2)/\delta u_i$, for $i = 1, 2$, indicates the functional derivative. The variable u_1 denotes a local concentration of A or B and u_2 indicates the presence of a liquid or a vapour phase. The constant c_i denotes the surface tension of u_i. The coefficient a_i is proportional to $T_{c_i} - T$, where T_{c_1} corresponds to the critical temperature of the $A - B$ phase separation, and T_{c_2} represents the critical temperature of the liquid-vapour phase separation.

If $a_1 > 0, a_2 > 0$, there are two equilibrium phases for each field corresponding to $u_1 = \pm\sqrt{\frac{a_1}{2b_1}}$ and $u_1 = \pm\sqrt{\frac{a_2}{2b_2}}$, denoted by u_1^+, u_1^-, u_2^+, and u_2^-, respectively. The coupling D energetically inhibits the existence of the phase denoted by the (u_1^+, u_2^+). Thus we have a three-phase system: liquid A corresponds to (u_1^-, u_2^-) regions, liquid B to (u_1^+, u_2^-) regions and the vapour phase to (u_1^-, u_2^+) regions.

To simplify the presentation, as in [23], we choose the values in (1) as follows:

$$b_1 = b_2 = \frac{1}{4}, \qquad a_1 = a_2 = \frac{1}{2}, \qquad c_1 = c_2 = \frac{\gamma}{2},$$

namely

$$F(u_1, u_2) = \psi(u_1) + \frac{\gamma}{2}|\nabla u_1|^2 + \psi(u_2) + \frac{\gamma}{2}|\nabla u_2|^2 + D\Psi(u_1, u_2), \tag{7}$$

where

$$\psi(u_1) = \frac{1}{4}(u_1^2 - 1)^2, \qquad \psi(u_2) = \frac{1}{4}(u_2^2 - 1)^2, \tag{8}$$

and

$$\Psi(u_1, u_2) = \frac{1}{2}(u_1 + 1)^2 (u_2 + 1)^2. \tag{9}$$

Despite the fact that all results can be adapted to the general situation. The constants $D > 0$ and $\gamma > 0$ are given constants. The following boundary conditions are included with this problem:

$$\frac{\partial u_1}{\partial \nu} = M(u_1)\frac{\partial w_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = M(u_2)\frac{\partial w_2}{\partial \nu} = 0, \qquad \text{on} \quad S_T, \tag{10}$$

$$u(\cdot,0) = u^0, \qquad v(\cdot,0) = v^0 \qquad \text{in} \quad \mathscr{R}, \tag{11}$$

where $S_T = \partial\mathscr{R} \times (0,T)$ and ν is the unit normal to $\partial\mathscr{R}$ point outward of $\mathscr{R}$.

Therefore, the problem takes the following form:
(P) Find $\{u_1, u_2, w_1, w_2\}$ such that

$$\partial_t u_1 = \nabla(M(u_1)\nabla w_1), \qquad \text{in} \quad Q_T, \tag{12}$$

$$\partial_t u_2 = \nabla(M(u_2)\nabla w_1), \qquad \text{in} \quad Q_T, \tag{13}$$

where

$$w_1 = \frac{\delta F(u_1,u_2)}{\delta u_1} = -\gamma\Delta u_1 + \phi(u_1) + 2D\Psi_1(u_1,u_2), \tag{14}$$

$$w_2 = \frac{\delta F(u_1,u_2)}{\delta u_2} = -\gamma\Delta u_2 + \phi(u_2) + 2D\Psi_2(u_1,u_2), \tag{15}$$

$$\frac{\partial u_1}{\partial \nu} = M(u_1)\frac{\partial w_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = M(u_2)\frac{\partial w_2}{\partial \nu} = 0, \qquad \text{on} \quad S_T, \tag{16}$$

$$u(\cdot,0) = u^0, \qquad v(\cdot,0) = v^0 \qquad \text{in} \quad \mathscr{R}, \tag{17}$$

$$\Psi_1(u_1,u_2) = \frac{\partial \Psi(u_1,u_2)}{\partial u_1}, \tag{18}$$

$$\Psi_2(u_1,u_2) = \frac{\partial \Psi(u_1,u_2)}{\partial u_2}, \tag{19}$$

$$\phi(u_1) = \psi'(u_1) \quad \text{and} \quad \phi(u_{12}) = \psi'(u_2). \tag{20}$$

If $D = 0$, the issue is reduced to two decoupled Cahn-Hilliard equations, which have been extensively explored in the mathematical literature, see for example [14, 16, 26]. We do not have liquid-vapour interfaces for this sort of situation.

The Cahn-Hilliard equation has been extensively studied in the literature, but there has been relatively little research on a diffusional mobility M that depends on the variable u, where u represents the difference in mass density between the two components of the alloy. It is worth noting that the original formulation of Cahn-Hilliard equation did include a concentration-dependent mobility, as described in [9]. As such, $M(u) = 1 - u^2$ is a thermodynamically valid choice, as discussed in [10, 19, 30]. The degeneracy of the mobility function M is a source of mathematical challenges when solving Cahn-Hilliard equation with such a mobility. However, there is hope that solutions that initially take values in the interval $[-1,1]$ will continue to do so for all positive times, which is not necessarily the case for fourth-order parabolic equations without degeneracy. It is important to emphasize that only values within the interval $[-1,1]$ have physical significance in this context. In the one-dimensional case, the existence of solutions for Cahn-Hilliard equation with degenerate mobility has been studied in [22]. Additionally, in [15], the existence of solutions is established for arbitrary spatial dimensions, and the weak form employed differs from the one presented in [22]. Moreover, in [15], singularities in the bulk energy are allowed when the mobility M degenerates. For research related to fourth-order degenerate parabolic equations in one spatial dimension, one may refer to [6]. Given the importance of this topic, recent studies have also focused on the existence and uniqueness of solutions for differential equations, as indicated in [1, 3–5, 7, 18, 21, 28, 29].

We will now give a brief overview of the paper's content. Throughout Section 2, we discuss the basic notations used in this paper for Sobolev spaces, as well as recall and demonstrate several auxiliary findings. Section 3 discusses the global existence and uniqueness of weak solutions, with the existence proof relying on Faedo-Galerkin technique and compactness arguments. In accordance with Theorem 1, we establish energy estimates for the approximate solutions, enabling us to proceed to the limit in the approximation equation. This process ultimately leads to the existence of a weak solution. The details of this demonstration are provided in Section 4, where we illustrate that the solution to problem (P) resides in higher-order Sobolev spaces.

2 Notations and auxiliary results

The $L^2(\mathscr{R})$ inner product over $\mathscr{R}$ with norm $\|\cdot\|_0$ is denoted by $(\cdot,\cdot)$. In addition, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\mathscr{R}))'$ and $H^1(\mathscr{R})$ where $(H^1(\mathscr{R}))'$ is the dual space of $H^1(\mathscr{R})$. A norm on $(H^1(\mathscr{R}))'$ is given by:

$$\|\phi\|_{(H^1(\mathscr{R}))'} := \sup_{\eta\neq 0}\frac{|\langle\phi,\eta\rangle|}{\|\eta\|_1} \equiv \sup_{\|\eta\|_1=1}|\langle\phi,\eta\rangle|. \tag{21}$$

Moreover, we denote to the function spaces which are depending on time and space as $L^\alpha(0,T;Y)$ $(1\le\alpha\le\infty)$; where Y is a Banach space. These spaces consist of all functions ϕ so that for a.e. $s\in(0,T)$ $\phi\in Y$ and the following norms are finite:

$$\|\phi(s)\|_{L^\alpha(0,T;Y)} = \left(\int_0^T \|\phi(s)\|_Y^\alpha ds\right)^{\frac{1}{\alpha}},$$

$$\|\phi(s)\|_{L^\infty(0,T;Y)} = \operatorname*{ess\,sup}_{s\in(0,T)}\|\phi(s)\|_Y.$$

$$\langle f,\eta\rangle = (f,\eta), \quad \forall f\in L^2(\mathscr{R}) \;\; and \;\; \eta\in H^1(\mathscr{R}). \tag{22}$$

Additionally, we define the spaces $L^\alpha(\mathscr{R}_T) = L^\alpha(0,T;L^\alpha(\mathscr{R}))$, $\alpha\in[1,\infty]$. Also, we define $C([0,T];Y)$, the space of continuous functions from $[0,T]$ into Y, which consists $\phi(s):[0,T]\to Y$ so that $\phi(s)\to\phi(s_0)$ in Y as $s\to s_0$. The norm which is associated to the space $C([0,T];Y)$ can be defined as [31]:

$$\|\phi(s)\|_{C([0,T];Y)} = \sup_{s\in[0,T]}\|\phi(s)\|_Y.$$

In addition, the mean integral can be defined as follows:

$$⨍\zeta = \frac{1}{|\mathscr{R}|}(\zeta,1), \quad \forall\zeta\in L^1(\mathscr{R}). \tag{23}$$

It is convenient to introduce "the inverse Laplacian Green's operator" $\mathscr{G}:\mathscr{F}_0\to V_0$ such that

$$(\nabla\mathscr{G}f,\nabla\eta) = \langle f,\eta\rangle, \quad \forall\eta\in H^1(\mathscr{R}), \tag{24}$$

where $\mathscr{F}_0 = \{f \in (H^1(\mathscr{R}))' : \langle f, 1\rangle = 0\}$ and $V_0 = \{v \in H^1(\mathscr{R}) : (v,1) = 0\}$. The well posedness of $\mathscr{G}$ can be obtained from the Lax-Milgram theorem and the following Poincaré inequality, see e.g. [32]

$$|v|_0 \le C_P(|v|_1 + |(v,1)|), \quad \forall v \in H^1(\mathscr{R}). \tag{25}$$

The norm defined in (21) on $(H^1(\mathscr{R}))'$ is also a norm on $\mathscr{F}_0$ and for convenience one can define an equivalent norm on $\mathscr{F}_0$ as (see the proof of Lemma 2.1.1 in [2]):

$$\|v\|_{-1} = |\mathscr{G}v|_1 \equiv \langle v, \mathscr{G}v\rangle^{\frac{1}{2}}, \quad \forall f \in \mathscr{F}_0. \tag{26}$$

It follows from (22) and (25), for any $v \in L^2(\mathscr{R}) \cap \mathscr{F}_0$, that

$$\|v\|_{-1}^2 = \langle v, \mathscr{G}v\rangle = (v, \mathscr{G}v) \le \|v\|_0 \|\mathscr{G}v\|_0 \le C_P \|v\|_0 |\mathscr{G}v|_1 = C_P \|v\|_0 \|v\|_{-1}, \tag{27}$$

which implies that

$$\|v\|_{-1} \le C_P \|v\|_0, \tag{28}$$

and

$$\|f\|_{(H^1(\mathscr{R}))'} = \sup_{\|v\|_1=1} |\langle f, v\rangle| = \sup_{\|v\|_1=1} |(\nabla \mathscr{G} f, \nabla v)| \le \sup_{\|v\|_1=1} \|f\|_{-1} |v|_1 \le \|f\|_{-1}. \tag{29}$$

We also recall the following well-known Sobolev results [11, 12]:

$$H^1(\mathscr{R}) \overset{c}{\hookrightarrow} L^\rho(\mathscr{R}) \hookrightarrow (H^1(\mathscr{R}))' \text{ holds for } \rho \in \begin{cases} [1,\infty] & \text{if } d = 1, \\ [1,\infty) & \text{if } d = 2, \\ [1,6] & \text{if } d = 3, \end{cases} \tag{30}$$

where $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$ are the continuous embedding and compact embedding, respectively. The following Hoders inequality is also required frequently: for $1 \le r_1, r_2 \le \infty$ such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$ if $\phi \in L^{r_1}(\mathscr{R})$ and $\psi \in L^{r_2}(\mathscr{R})$ then $\phi\psi \in L^1(\mathscr{R})$ and

$$\|\phi\psi\|_{0,1} = \int_{\mathscr{R}} |\phi\psi| d\mathbf{x} \le \left(\int_{\mathscr{R}} |\phi|^{r_1} d\mathbf{x}\right)^{\frac{1}{r_1}} \left(\int_{\mathscr{R}} |\psi|^{r_2} d\mathbf{x}\right)^{\frac{1}{r_2}} = \|\phi\|_{0,r_1} \|\psi\|_{0,r_2}. \tag{31}$$

The above inequality can be generalized by applying it twice to find that

$$\|\phi\psi\theta\|_{0,1} \le \left(\int_{\mathscr{R}} |\phi|^{r_1} d\mathbf{x}\right)^{\frac{1}{r_1}} \left(\int_{\mathscr{R}} |\psi|^{r_2} d\mathbf{x}\right)^{\frac{1}{r_2}} \left(\int_{\mathscr{R}} |\theta|^{r_3} d\mathbf{x}\right)^{\frac{1}{r_3}} = \|\phi\|_{0,r_1} \|\psi\|_{0,r_2} \|\theta\|_{0,r_3}, \tag{32}$$

for $1 \le r_1, r_2, r_3 \le \infty$ such that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$.

Let $\beta \in [1,\infty], \varpi \ge 1$ and $\upsilon \in W^{\varpi,\beta}(\mathscr{R})$, then there are constants C and $\mu = \frac{d}{\varpi}(\frac{1}{\beta} - \frac{1}{r})$ such that the inequality

$$\|\upsilon\|_{0,r} \le C \|\upsilon\|_{0,\beta}^{1-\mu} \|\upsilon\|_{\varpi,\beta}^{\mu}, \quad \text{holds for} \quad r \in \begin{cases} [\beta,\infty] & \text{if } \varpi - \frac{d}{\beta} > 0, \\ [\beta,\infty) & \text{if } \varpi - \frac{d}{\beta} = 0, \\ [\beta, -\frac{d}{\varpi - \frac{d}{\beta}}] & \text{if } \varpi - \frac{d}{\beta} < 0. \end{cases} \tag{33}$$

We also state the following result: Let $\vartheta_1, \vartheta_2, \vartheta_3 \in H^1(\mathscr{R})$, $\kappa_1 = \vartheta_1 - \vartheta_2$, $\kappa_2 = \vartheta_1^m \vartheta_2^{n_1-n_2}$, $n_1, n_2 = 0, 1, 2,$ and $n_1 - n_2 \ge 0$. We then have, for $d = 1,2,3$, that [20]

$$\left| \int_{\mathscr{R}} \kappa_1 \kappa_2 \vartheta_3 d\mathbf{x} \right| \le C \|\vartheta_1 - \vartheta_2\|_0 \|\vartheta_1\|_1^{n_2} \|\vartheta_2\|_1^{n_1-n_2} \|\vartheta_3\|_1. \tag{34}$$

3 Weak solutions

Let V be the trial space such that

$$V = \left\{ v : \int_{\mathscr{R}} (|\nabla(v)|^2 + v^2) d\mathbf{x} < \infty \right\},$$

then, by multiplying (12) and (14) by a test function $v \in V$, integrating over $\mathscr{R}$ and rearranging the terms, we find that

$$(\partial_t u_1, v) = (\nabla(M(u_1)\nabla w_1), v), \tag{35}$$

$$(w_1, v) = -\gamma(\Delta u_1, v) + (\phi(u_1), v) + 2D(\Psi_1(u_1, u_2), v). \tag{36}$$

We also have from (13) and (15) that

$$(\partial_t u_2, v) = (\nabla(M(u_2)\nabla w_2), v), \tag{37}$$

$$(w_2, v) = -\gamma(\Delta u_2, v) + (\phi(u_2), v) + 2D(\Psi_2(u_1, u_2), v). \tag{38}$$

By applying Green's formula

$$\int_{\mathscr{R}} \Delta u v d\mathbf{x} = \int_{\Gamma} v \frac{\partial u}{\partial v} d\sigma - \int_{\mathscr{R}} \nabla u \nabla v d\mathbf{x}, \tag{39}$$

we then find the weak form as follows:
(P) Find $\{u_i, w_i\}_{i=1}^2 \in H^1(\mathscr{R}) \times H^1(\mathscr{R}) \times H^1(\mathscr{R}) \times H^1(\mathscr{R}), t \in [0,T]$ such that $\forall \eta \in H^1(\mathscr{R})$,

$$(\partial_t u_1, \eta) = (M(u_1)\nabla w_1, \Delta\eta), \tag{40}$$

$$(w_1, \eta) = \gamma(\nabla u_1, \nabla\eta) + (\phi(u_1), \eta) + 2D(\Psi_1(u_1, u_2), \eta), \tag{41}$$

$$u_1(\cdot, 0) = u_1^0, \tag{42}$$

and

$$(\partial_t u_2, \eta) = (M(u_2)\nabla w_2, \nabla\eta), \tag{43}$$

$$(w_2, \eta) = \gamma(\nabla u_2, \nabla\eta) + (\phi(u_2), \eta) + 2D(\Psi_2(u_1, u_2), \eta), \tag{44}$$

$$u_2(\cdot, 0) = u_2^0. \tag{45}$$

Theorem 1. *Given $u_i^0 \in H^1(\mathscr{R}), i = 1, 2$, then there exists a solution $\{u_i, w_i\}$ to the problem (P) such that*

$$u_i(x,t) \in L^\infty(0,T;(H^1(\mathscr{R}))) \bigcap H^1(0,T;(H^1(\mathscr{R}))') \bigcap L^2(0,T;H^1(\mathscr{R})) \tag{46}$$

$$\bigcap C([0,T];L^2(\mathscr{R})) \bigcap L^2(\mathscr{R}_T), \tag{47}$$

$$w_i \in L^2(0,T;H^1(\mathscr{R})), \tag{48}$$

$$\frac{\partial u_i}{\partial t} \in L^2(0,T;(H^1(\mathscr{R}))'). \tag{49}$$

Proof. To prove the theorem, we apply Faedo-Galerkin method [24]. We separate the proof into three parts.

To prove the existence we use Faedo-Galerkin method [24]. Let $\{y_i\}_{i=1}^{\infty}u$ be an orthogonal basis for $H^1(\mathscr{R})$ and an orthonormal basis for $L^2(\mathscr{R})$, consisting of eigenfunctions for

$$-\Delta y_i + y_i = \lambda_i y_i, \text{ in } \mathscr{R}, \ \frac{\partial y_i}{\partial \nu} = 0 \text{ on } \partial\mathscr{R}, \tag{50}$$

where

$$1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \quad \text{with} \lim_{i\to\infty} \lambda_i = \infty, \tag{51}$$

is an infinite set of corresponding eigenvalues. Note $(y_i, y_j)_{H^1(\mathscr{R})} = \lambda_i \delta_{ij}$ and $(y_i, y_j)_{L^2(\mathscr{R})} = \delta_{ij}$. Now set $V^k := \text{span}\{y_i\}_{i=0}^{k} \subset H^1(\mathscr{R})$, and seek a finite-dimensional weak form corresponding to (P):
Find $\{u_1^k(x,t), u_2^k(x,t), w_1^k(x,t), w_2^k(x,t)\} \in V^k \times V^k \times V^k \times V^k$, in the form

$$u_i^k(x,t) = \sum_{j=1}^{k} c_{i,j}(t) y_i, \tag{52}$$

$$w_i^k(x,t) = \sum_{j=1}^{k} d_{i,j}(t) y_i, \tag{53}$$

such that

$$(\partial_t u_1^k, \eta^k) = (M(u_1^k)\nabla w_1^k, \Delta\eta^k), \tag{54}$$

$$(w_1^k, \eta^k) = \gamma(\nabla u_1^k, \nabla\eta^k) + (\phi(u_1^k), \eta^k) + 2D(\Psi_1(u_1^k, u_2^k), \eta^k), \tag{55}$$

$$u_1^k(\cdot, 0) = P^k u_1^0, \tag{56}$$

and

$$(\partial_t u_2^k, \eta^k) = (M(u_2^k)\nabla w_2^k, \nabla\eta^k), \tag{57}$$

$$(w_2^k, \eta^k) = \gamma(\nabla u_2^k, \nabla\eta^k) + (\phi(u_2^k), \eta^k) + 2D(\Psi_2(u_1^k, u_2^k), \eta^k), \tag{58}$$

$$u_2^k(\cdot, 0) = P^k u_2^0, \tag{59}$$

where P^k is a projection from $H^1(\mathscr{R})$ in to V^k defined by

$$P^k v = \sum_{j=1}^{k} (v, y_i) y_i, \tag{60}$$

which satisfies

$$(P^k v - v, \eta^k) = (\nabla(P^k v - v), \nabla\eta^k) = 0, \quad \forall, \ \eta^k \in V^k, \tag{61}$$

$$\|P^k\|_{\mathscr{L}(H^1, V^k)} = \|P^k\|_{\mathscr{L}(L^2, V^k)} = 1. \tag{62}$$

Simple calculations indicate that this projection operator meets, for $i = 0, 1$, the following properties:

$$\|P^k v - v\|_i \leq \|\xi^k - v\|_i, \quad \forall \xi^k \in V^k, \tag{63}$$

$$\|P^k v\|_i \leq \|v\|_i, \quad \forall v \in H^1(\mathscr{R}). \tag{64}$$

Since V^k is a dense in $H^1(\mathscr{R})$ and the injection of $H^1(\mathscr{R})$ in to $L^2(\mathscr{R})$ is compact (see [13] page 140), it follows that

$$P^k v \to v \ \text{ in } \ L^2(\mathscr{R}). \tag{65}$$

By taking $\eta^k = y_j$ in (54) and (57) we have, for $j = 1,2,...,k$, and $i = 1,2$, that

$$(\partial_t c^k_{i,j}(t) y_j, y_j) = -\left(M\left(\sum_{j=1}^{k} c^k_{i,j}(t) y_j\right) \nabla \left(\sum_{j=1}^{k} d^k_{i,j}(t) y_j\right), \nabla y_j \right).$$

Then, by using (50), (52), and (53) we can rewrite the above equations as a coupled system of first order differential equations for $j = 1,2,...,k$, and $i = 1,2$, in the following form:

$$\partial_t c^k_{i,j}(t) = -\sum_{j=1}^{k} d^k_{i,j}(t) \int_{\mathscr{R}} M\left(\sum_{j=1}^{k} c^k_{i,j}(t) y_j\right) \nabla y_j \nabla y_j d\mathbf{x}. \tag{66}$$

Furthermore, by selecting $\eta^k = y_j$ in (55) and (58), we have, for $j = 1,2,...,k$, and $i = 1,2$, that

$$(w^k_i, \eta^k) = \gamma(\nabla u^k_i, \nabla \eta^k) + (\phi(u^k_i), \eta^k) + 2D(\Psi_i(u^k_1, u^k_2), \eta^k).$$

Then, it follows from (50), (52), and (53), that

$$d^k_{i,j}(t) = \gamma \lambda_j c^k_{i,j}(t) + \int_{\mathscr{R}} (\phi(c^k_{i,j}(t))) d\mathbf{x} + 2D\Psi_i(c^k_{1,j}(t), c^k_{2,j}(t). \tag{67}$$

Since $u^k_i(\cdot,0) = P^k u^0_i, i = 1,2$, then we have, from (52) and (60), that

$$c^k_{i,j}(t)(0) = (u_0, y_j)_{L^2(\mathscr{R})}. \tag{68}$$

□

3.1 Global existence

Now, we will show the existence of a global solution, and to achieve that, all we need is some a priori estimations of $u^k_i, w^k_i, i = 1,2$ which are independent of k. Firstly, we consider the energy function in the following form:

$$E(u^k_1, u^k_2) = \int_{\mathscr{R}} \left(\frac{\gamma}{2}|\nabla u^k_1|^2 + \frac{\gamma}{2}|\nabla u^k_2|^2 + 2D\Psi(u^k_1, u^k_2) + \psi(u^k_1) + \psi(u^k_2)\right) d\mathbf{x}, \tag{69}$$

where $\Psi(\cdot,\cdot), \psi(\cdot)$ are given by (8) and (9), respectively. Since $\frac{\partial u^k_i}{\partial t} \in V^k$, for $i = 1,2$, then, by differentiating $E(u^k_1, u^k_2)$ with respect to t and rearranging the terms, we obtain that

$$\begin{aligned} \frac{\partial}{\partial t} E(u^k_1, u^k_2) =& \gamma\left(\nabla u^k_1(t), \nabla(\frac{\partial u^k_1}{\partial t})\right) + \left(\psi'(u^k_1) + 2D(\Psi_1(u^k_1, u^k_2), \frac{\partial u^k_1}{\partial t}\right) \\ &+ \gamma\left(\nabla u^k_2(t), \nabla(\frac{\partial u^k_2}{\partial t})\right) + \left(\psi'(u^k_2) + 2D(\Psi_2(u^k_1, u^k_2), \frac{\partial u^k_2}{\partial t}\right), \end{aligned} \tag{70}$$

where $\Psi_1(\cdot,\cdot)$, $\Psi_2(\cdot,\cdot)$ are given by (18) and (19) respectively. By taking $\eta = w_i, i = 1,2$, in (54) and (57), respectively, and $\eta = \frac{\partial u_i}{\partial t}, i = 1,2$, in (55) and (58), respectively, it follows, for $i = 1,2$, that

$$\int_{\mathscr{R}} \frac{\partial u_i}{\partial t} w_i d\mathbf{x} = -\int_{\mathscr{R}} M(u_i)\nabla w_i \nabla w_i d\mathbf{x}, \tag{71}$$

$$\int_{\mathscr{R}} w_i \frac{\partial u_i}{\partial t} d\mathbf{x} = \int_{\mathscr{R}} \nabla u_i \nabla \frac{\partial u_i}{\partial t} d\mathbf{x} + \left(\phi(u_i), \frac{\partial u_i}{\partial t}\right) + 2D\left(\Psi_i(u_1,u_2), \frac{\partial u_i}{\partial t}\right), \tag{72}$$

We have, from (71) and (72) and on noting (20), that

$$-\int_{\mathscr{R}} M(u_i)|\nabla w_i|^2 d\mathbf{x} = \gamma(\nabla u_i, \nabla \frac{\partial u_i}{\partial t}) + \left(\psi'(u_i), \frac{\partial u_i}{\partial t}\right) + 2D\left(\Psi_i(u_1,u_2), \frac{\partial u_i}{\partial t}\right). \tag{73}$$

By substituting (73) in (70), we find that

$$\begin{aligned}
\frac{\partial}{\partial t}E(u_1^k,u_2^k) = \; & \gamma\left(\nabla u_1^k(t), \nabla(\frac{\partial u_1^k}{\partial t})\right) + \left(\psi'(u_1^k), \frac{\partial u_1^k}{\partial t}\right) + 2D\left(\Psi_1(u_1^k,u_2^k), \frac{\partial u_1^k}{\partial t}\right) \\
& + \gamma\left(\nabla u_2^k(t), \nabla(\frac{\partial u_2^k}{\partial t})\right) + \left(\psi'(u_2^k), \frac{\partial u_2^k}{\partial t}\right) + 2D\left(\Psi_2(u_1^k,u_2^k), \frac{\partial u_2^k}{\partial t}\right) \\
= & -\int_{\mathscr{R}} M(u_1)|\nabla w_1^k|^2 d\mathbf{x} - \int_{\mathscr{R}} M(u_2)|\nabla w_2^k|^2 d\mathbf{x} \\
= & -\sum_{i=1}^{2}\int_{\mathscr{R}} M(u_i^k)|\nabla w_i^k|^2 d\mathbf{x}.
\end{aligned} \tag{74}$$

Thus, E is a Lyapunov functional. Now, integrating (74) over $(0,t)$, it follows that

$$E(u_1^k(t),u_2^k(t)) - E(u_1^k(0),u_2^k(0)) + \int_{\mathscr{R}_T}\left[M(u_1^k)|\nabla w_1^k|^2 + M(u_2^k)|\nabla w_2^k|^2\right] d\mathbf{x}dt = 0.$$

Next, by using (56) and (59), we find that

$$E(u_1^k(t),u_2^k(t)) + \int_{\mathscr{R}_T}\left[M(u_1^k)|\nabla w_1^k|^2 + M(u_2^k)|\nabla w_2^k|^2\right] d\mathbf{x}dt = E(u_1^k(0),u_2^k(0)) = E(P^k u_1^0, P^k u_2^0). \tag{75}$$

By integrating (8) over $\mathscr{R}$ and noting (33), we find that

$$\int_{\mathscr{R}} \psi(\eta_1)d\mathbf{x} \le \frac{1}{4}\int_{\mathscr{R}}(\eta_1^4+1)d\mathbf{x} \le \frac{1}{4}\|\eta_1\|_{0,4}^4 + \frac{1}{4}|\mathscr{R}| \le C\|\eta_1\|_1^4 + \frac{1}{4}|\mathscr{R}|, \tag{76}$$

Next, by integrating (9) over $\mathscr{R}$ and utilizing the Cauchy-Schwarz inequality, Young's inequality with $\varepsilon = 1$, and setting $p = q = 2$, we have that

$$\begin{aligned}
\int_{\mathscr{R}} \Psi(\eta_1,\eta_2)d\mathbf{x} = \; & \frac{1}{2}\int_{\mathscr{R}}(\eta_1+1)^2(\eta_2+1)^2 d\mathbf{x} \\
& \le 2\int_{\mathscr{R}}[\eta_1^2\eta_2^2 + \eta_1^2 + \eta_2^2 + 1]d\mathbf{x} \\
& \le 2(\|\eta_1\|_{0,4}^2\|\eta_2\|_{0,4}^2 + \|\eta_1\|_0^2 + \|\eta_2\|_0^2 + |\mathscr{R}|) \\
& \le \|\eta_1\|_{0,4}^2 + \|\eta_2\|_{0,4}^2 + 2\|\eta_1\|_0^2 + 2\|\eta_2\|_0^2 + 2|\mathscr{R}| \\
& \le C\|\eta_1\|_1^4 + C\|\eta_2\|_1^4 + 2\|\eta_1\|_0^2 + 2\|\eta_2\|_0^2 + 2|\mathscr{R}|.
\end{aligned} \tag{77}$$

By substituting (76), (77), with $\eta_1 = P^k u_1^0, \eta_2 = P^k u_2^0$ and (33) in (69), and utilizing the strong convergent of $P^k u_i^0$ to u_i^0 in $L^2(\mathscr{R}), i = 1,2$ which is given by (65) and (64), and the assumption that is $\|u_1^0\|_1 + \|u_2^0\|_1 \leq C$, we have that

$$
\begin{aligned}
E(P^k u_1^0, P^k u_2^0) &= \int_{\mathscr{R}} \left[\frac{\gamma}{2}|\nabla P^k u_1^0|^2 + \psi(P^k(u_1^0)) + \frac{\gamma}{2}|\nabla P^k u_2^0|^2 + \psi(P^k(u_2^0)) + 2D\Psi(P^k u_1^0, P^k u_2^0)\right] d\mathbf{x} \\
&\leq \frac{\gamma}{2}\|\nabla P^k u_1^0\|_0^2 + \frac{\gamma}{2}\|\nabla P^k u_2^0\|_0^2 + \frac{1}{4}\|\nabla P^k u_1^0\|_1^4 + \frac{1}{4}|\mathscr{R}| + \frac{1}{4}\|P^k u_2^0\|_1^4 + \frac{1}{4}|\mathscr{R}| \\
&\quad + 2D(C\|P^k u_1^0\|_1^4 + C\|P^k u_2^0\|_1^4 + 2\|P^k u_1^0\|_0^2 + 2\|P^k u_2^0\|_0^4) + 2|\mathscr{R}|) \\
&= C\|P^k u_1^0\|_1^4 + C\|P^k u_2^0\|_1^4 + 4D|P^k u_1^0|_0^2 + 4D\|P^k u_2^0\|_0^2 + \frac{\gamma}{2}\|P^k u_1^0\|_1^2 + \frac{\gamma}{2}|P^k u_2^0|_1^2 + C \\
&\leq C\|u_1^0\|_1^4 + C\|u_2^0\|_1^4 + 4D\|u_1^0\|_0^2 + 4D\|u_2^0\|_0^2 + \frac{\gamma}{2}|u_1^0|_1^2 + \frac{\gamma}{2}|u_2^0|_1^2 + C \\
&\leq C\big(\|u_1^0\|_1^4 + \|u_2^0\|_1^4 + \|u_1^0\|_1^2 + \|u_2^0\|_1^2 + 1\big) \quad \leq C.
\end{aligned} \tag{78}
$$

So, by combining (75) and (78), it follows that

$$
E(u_1^k(t), u_2^k(t)) + \int_{\mathscr{R}_T} \left[M(u_1^k)|\nabla w_1^k|^2 + M(u_2^k)|\nabla w_2^k|^2\right] d\mathbf{x}dt \leq C. \tag{79}
$$

Next, substituting (69) in (79), we have that

$$
\int_{\mathscr{R}} \left(\frac{\gamma}{2}|\nabla u_1^k|^2 + \frac{\gamma}{2}|\nabla u_2^k|^2 + 2D\Psi(u_1^k, u_2^k) + \psi(u_1^k) + \psi(u_2^k)\right) d\mathbf{x} \tag{80}
$$

$$
+ \int_{\mathscr{R}_T} \left[M(u_1^k)|\nabla w_1^k|^2 + M(u_2^k)|\nabla w_2^k|^2\right] - d\mathbf{x}dt \leq C. \tag{81}
$$

Since the functions $\Psi(u_1^k, u_2^k)$, $\psi(u_1^k)$, and $\psi(u_2^k))$ are positive, thus we arrive at

$$
\frac{\gamma}{2}|u_1^k|_1^2 + \frac{\gamma}{2}|u_2^k|_1^2 + \int_{\mathscr{R}_T} \left[M(u_1^k)|\nabla w_1^k|^2 + M(u_2^k)|\nabla w_2^k|^2\right] d\mathbf{x}dt \leq C, \tag{82}
$$

where C is not dependent of T and k. Next, by selecting $\eta^k = 1$ in (54) and (57), we find, for $i = 1,2$, that

$$
\left(\frac{\partial u_i^k}{\partial t}, 1\right) = -\left(M(u_i^k)\nabla w_i^k, \nabla 1\right) = 0. \tag{83}
$$

From (83), it follows, for $i = 1,2$, that

$$
\left(\frac{\partial u_i^k}{\partial t}, 1\right) = 0. \tag{84}
$$

Then, by integrating both sides of equation (84) over the interval $(0,t)$, we obtain that

$$
(u_i^k(t), 1) = (u_i^k(0), 1) = (P^k u_i^0, 1) = (u_i^0, 1) \leq C, \tag{85}
$$

which implies the following result:

$$
|(u_i^k(t), 1)| \leq C. \tag{86}
$$

By using Poincaré inequality (25), (82) and (86), we obtain that

$$
\|u_i^k(t)\|_0 \leq C_p \left(|u_i^k(t)|_1 + |(u_i^k(t), 1)|\right) \leq C, \tag{87}
$$

so, we find that

$$\sup_{t\in(0,T)} \|u_i^k(t)\|_1 \leq C,$$

i.e.

$$u_i^k(t) \in L^\infty(0,T;H^1(\mathscr{R})). \tag{88}$$

By integrating (54) and (57) over $(0,t)$ and utilizing Hoder's inequality (31), it follows, for all $\eta^k \in L^2(0,T;H^1(\mathscr{R}))$, that

$$\begin{aligned}
\left|\int_0^T \int_{\mathscr{R}} \partial_t u_i^k \eta^k d\mathbf{x}dt\right| =& \left|\int_0^T \int_{\mathscr{R}} M(u_i^k)\nabla w_i^k \nabla \eta^k d\mathbf{x}dt\right| \\
\leq& \left(\int_0^T \int_{\mathscr{R}} (M(u_i^k)|\nabla w_i^k|)^2 d\mathbf{x}dt\right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathscr{R}} |\nabla \eta^k|^2 d\mathbf{x}dt\right)^{\frac{1}{2}} \\
\leq& C\|\eta^k\|_{L^2(0,T;H^1(\mathscr{R}))}.
\end{aligned} \tag{89}$$

Then, we have that

$$\|\partial_t u_i^k\|_{L^2(0,T;(H^1(\mathscr{R}))')} = \sup_{\eta^k\neq 0} \frac{|\int_0^T \int_{\mathscr{R}} \partial_t u_i^k \eta^k d\mathbf{x}dt|}{\|\eta^k\|_{L^2(0,T;H^1(\mathscr{R}))}} \leq C,$$

i.e.

$$\partial_t u_i^k \in L^2(0,T;(H^1(\mathscr{R}))'), \quad \forall i = 1,2. \tag{90}$$

In order to show that $u_i^k(t)$ is bounded in $L^2(0,T;(H^1(\mathscr{R}))')$, we have to prove that $u_i^k(t) - \fint u_i^k(t) \in L^2(0,T;(H^1(\mathscr{R}))')$. By noting (23) and (85) we have that

$$u_i^k(t) - \fint u_i^k(t) = u_i^k(t) - \frac{1}{|\mathscr{R}|}(u_i^k(t),1) = u_i^k(t) - \frac{1}{|\mathscr{R}|}(u_i^k(0),1) = \int_0^t \frac{\partial}{\partial s}u_i^k(s)ds + u_i^k(0) - \frac{1}{|\mathscr{R}|}(u_i^k(0),1). \tag{91}$$

Hence, on utilizing Young's inequality and setting $t = T$ in the integration on the right hand side and noting (90) and (28), we obtain that

$$\begin{aligned}
\|u_i^k(t) - \fint u_i^k(t)\|_{-1} =& \left(\left\|\int_0^T \frac{\partial u_i^k}{\partial t}dt + u_i^k(0) - \frac{1}{|\mathscr{R}|}(u_i^k(0),1)\right\|_{-1}\right)^2 \\
\leq& \left(\left\|\int_0^T \frac{\partial u_i^k}{\partial t}dt\right\|_{-1} + C_p\|u_i^k(0) - \frac{1}{|\mathscr{R}|}(u_i^k(0),1)\|_{-1}\right)^2 \\
\leq& \left\|\int_0^T \frac{\partial u_i^k}{\partial t}dt\right\|_{-1}^2 + C\|u_i^k(0)\|_0^2 + C\|(u_i^k(0),1)\|_0^2 \\
\leq& \left\|\int_0^T \frac{\partial u_i^k}{\partial t}dt\right\|_{-1}^2 + C \ \leq C.
\end{aligned} \tag{92}$$

By squaring the above result, integrating over $(0,T)$, and using (29), we obtain that

$$\left\|u_i^k(t) - \fint u_i^k(t)\right\|_{L^2(0,T;(H^1(\mathscr{R}))')}^2 \leq \int_0^T \left\|u_i^k(t) - \fint u_i^k(t)\right\|_{-1}^2 dt \leq C\int_0^T dt \leq C(T). \tag{93}$$

Hence, from (90) and (93), we find that

$$\|u_i^k\|^2_{H^1(0,T;(H^1(\mathscr{R}))')} \leq C. \tag{94}$$

We have to demonstrate now that w_i^k is bounded in $L^2(0,T;H^1(\mathscr{R}))$. By using Poincaré inequality (25), we have that

$$\|w_i^k\|_1^2 = \|w_i^k\|_0^2 + |w_i^k|_1^2 \leq 2C_p(|w_i^k|_1^2 + |(w_i^k,1)|^2) + |w_i^k|_1^2 \leq C(|w_i^k|_1^2 + |(w_i^k,1)|^2). \tag{95}$$

Thus by (82) it is enough to bound $|(w_i^k,1)|$ to conclude $\|w_i^k\|_1$ is bound. Taking $\eta^k = 1$ in (55) and (58) we have, for $i = 1,2$, that

$$(w_i^k,1) = (\phi(u_i^k),1) + 2D(\Psi_i(u_1^k,u_2^k),1),$$

which implies

$$|(w_i^k,1)| \leq |(\phi(u_i^k),1)| + 2D|(\Psi_i(u_1^k,u_2^k),1)|. \tag{96}$$

Noting Young's inequality, (20), (18), (19), (8), (9) and (33), we may bound the terms on the right hand side of (96), for $i = 1,2$, as follows:

$$\begin{aligned}
|(\phi(u_i^k),1)| =& \left| \int_{\mathscr{R}} ((u_i^k)^2 - 1)u_i^k d\mathbf{x} \right| \\
\leq& \frac{1}{2}\int_{\mathscr{R}} ((u_i^k)^2-1)^2 d\mathbf{x} + \frac{1}{2}\int_{\mathscr{R}} (u_i^k)^2 d\mathbf{x} \\
=& \frac{1}{2}\int_{\mathscr{R}} [(u_i^k)^4 + 1 - (u_i^k)^2] d\mathbf{x} \\
\leq& \frac{1}{2}\int_{\mathscr{R}} (u_i^k)^4 d\mathbf{x} + \frac{1}{2}\int_{\mathscr{R}} d\mathbf{x} \\
=& \frac{1}{2}\|u_i^k\|_{0,4}^4 + \frac{1}{2}|\mathscr{R}| \\
\leq& C(\|u_i^k\|_1^4 + 1) \ \leq C,
\end{aligned} \tag{97}$$

and

$$\begin{aligned}
|(\Psi_1(u_1^k,u_2^k),1)| =& \left| \int_{\mathscr{R}} (u_1^k(t)+1)(u_2^k(t)+1)^2 d\mathbf{x} \right| \\
\leq& \int_{\mathscr{R}} (u_1^k(t)+1)^2 d\mathbf{x} + \int_{\mathscr{R}} ((u_2^k(t))^2+1)^2 d\mathbf{x} \\
\leq& \int_{\mathscr{R}} 2[(u_1^k(t))^2 + (u_2^k(t))^4 + 2] d\mathbf{x} \\
\leq& 2\|u_1^k(t)\|_{0,2}^2 + 2\|u_2^k(t)\|_{0,4}^4 + 4|\mathscr{R}| \\
\leq& C\|u_1^k(t)\|_1^2 + C\|u_2^k(t)\|_1^4 + 4|\mathscr{R}| \ \leq C.
\end{aligned} \tag{98}$$

Similarly, we can prove that

$$|(\Psi_2(u_1^k,u_2^k),1)| \leq C. \tag{99}$$

Thus, from (96), (97), (98) and (99), we conclude that

$$|(w_i^k,1)| \leq C. \tag{100}$$

By substituting (100) into (95) and integrating the result over $(0,T)$, we have that

$$\int_0^T \|w_i^k\|_1^2 ds \le C\int_0^T |w_i^k|_1^2 ds + CT, \tag{101}$$

Now, we have to prove that $\int_0^t |w_i^k|_1^2 dt \le C$. From (82), we find, for $i=1,2$, that

$$\int_{\mathscr{R}_T} M(u_i^k)|\nabla w_i^k|^2 d\mathbf{x}dt \le C.$$

By using (6), we find that

$$\int_0^t |w_i^k|_1^2 dt \le \frac{C}{m_0} = C. \tag{102}$$

By substituting (102) into (101), we have that

$$\|w_i^k\|^2_{L^2(0,T;H^1(\mathscr{R}))} = \int_0^t \|w_i^k\|_1^2 dt \le C\int_0^t |w_i^k|_1^2 dt + C\int_0^t dt \le C + CT,$$

so, it follows finally that

$$\|w_i^k\|_{L^2(0,T;H^1(\mathscr{R}))} \le C(1+T^{\frac{1}{2}}). \tag{103}$$

Now, since $L^\infty(0,T;H^1(\mathscr{R})) \subset L^2(0,T;H^1(\mathscr{R}))$, then by using (88), we arrive at

$$u_i^k \in L^2(0,T;H^1(\mathscr{R})), \tag{104}$$

Since $H^1(0,T;(H^1(\mathscr{R}))')$ and $L^2(0,T;H^1(\mathscr{R}))$ are reflexive, by compactness arguments we deduce existence of subsequence such that

$$u_i^k \rightharpoonup u_i, \quad \text{in} \quad H^1(0,T;(H^1(\mathscr{R}))')\bigcap L^2(0,T;(H^1(\mathscr{R}))'), \tag{105}$$

$$w_i^k \rightharpoonup w_i \quad \text{in} \quad L^2(0,T;H^1(\mathscr{R})), \tag{106}$$

$$\partial_t u_i^k \rightharpoonup \partial_t u_i, \quad \text{in} \quad L^2(0,T;(H^1(\mathscr{R}))'). \tag{107}$$

Since $L^\infty(0,T;H^1(\mathscr{R}))$ is dual of $L^1(0,T;(H^1(\mathscr{R}))')$ which is separable we can extract a subsequence in $L^\infty(0,T;H^1(\mathscr{R}))$ such that

$$u_i^k \rightharpoonup^* u_i \quad \text{in} \quad L^\infty(0,T;H^1(\mathscr{R})). \tag{108}$$

Note that $H^1(\mathscr{R})$ and $(H^1(\mathscr{R}))'$ are reflexive and the injection of $H^1(\mathscr{R})$ in to $L^2(\mathscr{R})$ is compact. As a result of Lions' compactness theory, see in [24, Theorem 5.1], we may extract a subsequence in $L^2(0,T;L^2(\mathscr{R}))$ such that

$$u_i^k \rightharpoonup u_i \quad \text{in} \quad L^2(\mathscr{R}_T). \tag{109}$$

Moreover, since $u_i^k \in L^2(0,T;H^1(\mathscr{R}))$ and $\frac{\partial u_i^k}{\partial t} \in L^2(0,T;(H^1(\mathscr{R}))')$, then $u_i^k \in C([0,T];L^2(\mathscr{R}))$. This result, along with (105) and the strong convergence of $P^k(u_i^0)$ to u_i^0 in $L^2(\mathscr{R})$, implies that $u_i(0) = u_i^0$. Thus, we have proven the following results:

$$u_i^k \in L^\infty(0,T;(H^1(\mathscr{R})))\bigcap H^1(0,T;(H^1(\mathscr{R}))')\bigcap L^2(0,T;H^1(\mathscr{R}))\bigcap C([0,T];L^2(\mathscr{R}))\bigcap L^2(\mathscr{R}_T),$$

$$w_i^k \in L^2(0,T;H^1(\mathscr{R})),$$

$$\frac{\partial u_i^k}{\partial t} \in L^2(0,T;(H^1(\mathscr{R}))').$$

3.2 Passage to the limit

Here, we show passage to the limit of the terms in the composite Galerkin approximation and prove that this approximation satisfies the problem (P). For any $\eta \in H^1(\mathscr{R})$, we set $\eta^k = P^k\eta$ in (54) and (55), to find that

$$(\partial_t u_1^k, P^k\eta) = -(M(u_1^k)\nabla w_1^k, \Delta P^k\eta), \tag{110}$$

$$(w_1^k, P^k\eta) = \gamma(\nabla u_1^k, \nabla P^k\eta) + (\phi(u_1^k), P^k\eta) + 2D(\Psi_1(u_1^k, u_2^k), P^k\eta). \tag{111}$$

Also by setting $\eta^k = P^k\eta$ in (57) and (58), it follows that

$$(\partial_t u_2^k, P^k\eta) = -(M(u_2^k)\nabla w_2^k, \Delta P^k\eta), \tag{112}$$

$$(w_2^k, P^k\eta) = \gamma(\nabla u_2^k, \nabla P^k\eta) + (\phi(u_2^k), P^k\eta) + 2D(\Psi_2(u_1^k, u_2^k), P^k\eta). \tag{113}$$

Passaging to the limit a.e in (110) and (112) we have (40) and (43). It is still necessary to demonstrate, in order to obtain the desired results, for $i = 1, 2$, that

$$(\phi(u_i^k), P^k\eta) \to (\phi(u_i), \eta), \quad \text{as, } k \to \infty, \tag{114}$$

$$(\Psi_i(u_1^k, u_2^k), P^k\eta) \to (\Psi_i(u_1, u_2), \eta), \quad \text{as, } k \to \infty. \tag{115}$$

From Cauchy-Schwarz and Young inequalities, and the injection of $H^1(\mathscr{R})$ in to $L^4(\mathscr{R})$, it follows that

$$\begin{aligned}
|(\phi(u_i^k), P^k\eta) - (\phi(u_i), \eta)| =& |(\phi(u_i^k), P^k\eta - \eta) + (\phi(u_i^k) - \phi(u_i), \eta)| \\
\le& |(\phi(u_i^k), P^k\eta - \eta)| + |(\phi(u_i^k) - \phi(u_i), \eta)| \\
=& |(((u_i^k)^2 - 1)u_i^k, P^k\eta - \eta)| + |((u_i^k)^3 - (u_i)^3 + u_i - u_i^k, \eta)| \\
\le& |(((u_i^k)^2 - 1)u_i^k, P^k\eta - \eta)| + |((u_i^k)^3 - (u_i)^3, \eta)| + |(u_i - u_i^k, \eta)| \\
\le& |(((u_i^k)^2 - 1)u_i^k, P^k\eta - \eta) + |((u_i^k - u_i)((u_i^k)^2 + u_i^k u_i + (u_i)^2), \eta)| + |(u_i - u_i^k, \eta)| \\
\le& C\left[\|u_i^k\|_6^3 + \|u_i^k\|_0\right]\|P^k\eta - \eta\|_0 + \|u_i^k - u_i\|_0\left[\|u_i^k\|_4^2 + \|u_i^k\|_4\|u_i\|_4 + \|u_i\|_4^2\right]\|\eta\|_\infty \\
&+ \|u_i - u_i^k\|_0\|\eta\|_0 \\
\le& C\left[\|u_i^k\|_1^3 + \|u_i^k\|_0\right]\|P^k\eta - \eta\|_0 + \|u_i^k - u_i\|_0\left[\|u_i^k\|_1^2 + \|u_i^k\|_1\|u_i\|_1 + \|u_i\|_1^2\right]\|\eta\|_\infty \\
&+ \|u_i - u_i^k\|_0\|\eta\|_0.
\end{aligned} \tag{116}$$

By using (65) and (105), we have that

$$(\phi(u_i^k), P^k\eta) \to (\phi(u_i), \eta) \quad \text{as} \quad k \to \infty.$$

To prove (115), firstly we note that

$$\begin{aligned}
(\Psi_1(u_1^k, u_2^k), P^k\eta) - (\Psi_1(u_1, u_2), \eta)| \le& |(\Psi_1(u_1^k, u_2^k) - \Psi_1(u_1^k, u_2), P^k\eta)| + |(\Psi_1(u_1^k, u_2), P^k\eta - \eta)| \\
&+ |(\Psi_1(u_1^k, u_2) - \Psi_1(u_1, u_2), \eta)| \\
=& I_1 + I_2 + I_3.
\end{aligned} \tag{117}$$

We demonstrate that each of the above terms tends to zero as $k \to \infty$. Noting Young's inequality, (18), (9), (34), and (109), it follows that

$$\begin{aligned} I_1 =& |(\Psi_1(u_1^k, u_2^k) - \Psi_1(u_1^k, u_2), P^k \eta)| \\ =& |((u_1^k+1)(u_2^k+1)^2 - (u_1^k+1)(u_2+1)^2, P^k \eta)| \\ =& |((u_1^k+1)(u_2^k - u_2)(u_2^k + u_2 + 2), P^k \eta)| \\ \leq & C\Big(\|u_2^k - u_2\|_0 \|P^k \eta\|_1 \big(\|u_1^k\|_1 \|u_2^k\|_1 + \|u_1^k\|_1 \|u_2\|_1 + \|u_1^k\|_1 \\ & + \|u_2^k\|_1 + \|u_2\|_1 \big) + \|u_2^k - u_2\|_0 \|P^k \eta\|_0 \Big) \quad \to 0 \text{ as } k \to \infty. \end{aligned} \tag{118, 119}$$

Now, we have, from the Young inequality, (18), (9), (34) and (65), that

$$\begin{aligned} I_2 =& |(\Psi_1(u_1^k, u_2), P^k \eta - \eta)| = |((u_1^k+1)(u_2+1)^2, P^k \eta - \eta)| \\ \leq & C\big(\|u_1^k\|_1 \|u_2\|_1^2 + \|u_2\|_1^2 + \|u_1^k\|_0^2 + |\mathscr{R}| \big) \|P^k \eta - \eta\|_0 \to 0 \text{ as } k \to \infty. \end{aligned} \tag{120}$$

Finally, it follows, from the Young inequality, (18), (9), (34) and (109), that

$$\begin{aligned} I_3 =& |(\Psi_1(u_1^k, u_2) - \Psi_1(u_1, u_2), \eta)| = |((u_1^k+1)(u_2+1)^2 - (u_1+1)(u_2+1)^2, \eta)| \\ =& |((u_2+1)^2(u_1^k - u_1), \eta)| \leq C\|u_1^k - u_1\|_0 \big(\|u_2\|_1^2 \|\eta\|_1 + \|\eta\|_0 \big) \to 0 \text{ as } k \to \infty. \end{aligned} \tag{121}$$

By substituting (118), (120) and (121) in (117), we arrive at

$$(\Psi_1(u_1^k, u_2^k), P^k \eta) \to (\Psi_1(u_1, u_2), \eta) \text{ as } k \to \infty.$$

Similarly, we can show that

$$(\Psi_2(u_1^k, u_2^k), P^k \eta) \to (\Psi_2(u_1, u_2), \eta) \text{ as } k \to \infty.$$

4 Regularity

Theorem 2. *For $\mathscr{R}$ sufficiently smooth, we have, for $i = 1, 2$, the following regularity results:*

$$u_i^k \in L^2(0, T; H^2(\mathscr{R})) \text{ and } \frac{\partial u_i}{\partial \nu} = 0 \text{ on } \partial\mathscr{R} \text{ for a.e } t. \tag{122}$$

Proof. From (54) and (57), it follows that

$$(\partial_t u_i^k, \eta^k) = -(M(u_i) \nabla w_i^k, \nabla \eta^k). \tag{123}$$

Also, from (55) and (58), we find that

$$(w_i^k, \eta^k) = \gamma(\nabla u_i^k, \nabla \eta^k) + (\phi(u_i^k), \eta^k) + 2D(\Psi_i(u_1^k, u_2^k), \eta^k). \tag{124}$$

If we choose $\eta^k = u_i^k$ in (124), then we get that

$$(\partial_t u_i^k, u_i^k) = -(M(u_i) \nabla w_i^k, \nabla u_i^k).$$

Since $\frac{1}{2}\frac{\partial}{\partial t}\|u_i^k\|_0^k = (\partial_t u_i^k, u_i^k)$, so we get, by using the Young's inequality, that

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_i^k\|_0^2 = -(M(u_i)\nabla w_i^k, \nabla u_i^k) \le M|\nabla w_i^k|_0|\nabla u_i^k|_0 \le \frac{M}{2}(|w_i^k|_1^2 + |u_i^k|_1^2). \tag{125}$$

In (124) , if we choose $\eta^k = -\Delta u_i^k$, then we have

$$(w_i^k, -\Delta u_i^k) = \gamma(\nabla u_i^k, \nabla(-\Delta u_i^k)) + (\phi(u_i^k), -\Delta u_i^k) + 2D(\Psi_i(u_1^k, u_2^k), -\Delta u_i^k).$$

We use integration by parts for each term in the above equation and apply Young's inequality. Consequently, we obtain that

$$\gamma\|\Delta u_i^k\|_0^2 + (\nabla\phi(u_i^k), \nabla u_i^k) + 2D(\nabla\Psi_i(u_1^k, u_2^k), \nabla u_i^k) \le \frac{1}{2}(|w_i^k|_1^2 + |u_i^k|_1^2). \tag{126}$$

From (125) and (126), it follows that

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_i^k\|_0^2 + \gamma\|\Delta u_i^k\|_0^2 + (\nabla\phi(u_i^k), \nabla u_i^k) + 2D(\nabla\Psi_i(u_1^k, u_2^k), \nabla u_i^k) \le C(|w_i^k|_1^2 + |u_i^k|_1^2). \tag{127}$$

By using (20), (8), (18), (19), and (9), we can rewrite the third and fourth terms on the left-hand side as follows, for $i, j = 1,2$ with $i \ne j$:

$$(\nabla\phi(u_i^k), \nabla u_i^k) = (\nabla((u_i^k)^3 - u_i^k), \nabla u_i^k) = (\nabla(u_i^k)^3, \nabla u_i^k) - (\nabla u_i^k, \nabla u_i^k) = 3\|u_i^k\nabla u_i^k\|_0^2 - |u_i^k|_1^2, \tag{128}$$

and

$$\begin{aligned}(\nabla\Psi_i(u_1^k, u_2^k), \nabla u_i^k) =& 2((u_i^k+1)\nabla u_j^k, (u_j^k+1)\nabla u_i^k) + ((u_j^k+1)^2, (\nabla u_i^k)^2)\\ =& 2((u_i^k+1)\nabla u_j^k, (u_j^k+1)\nabla u_i^k) + \|(u_j^k+1)\nabla u_i^k\|_0^2,\end{aligned} \tag{129}$$

We can rewrite (127) using (128) and (129) in the following form:

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\|u_i^k\|_0^2 + \gamma\|\Delta u_i^k\|_0^2 + 3\|u_i^k\nabla u_i^k\|_0^2 + 2D\|(u_j^k+1)\nabla u_i^k\|_0^2\\ &\quad\le c(|w_i^k|_1^2 + |u_i^k|_1^2) - 4D((u_i^k+1)\nabla u_j^k, (u_j^k+1)\nabla u_i^k).\end{aligned} \tag{130}$$

By summing (130), for $i, j = 1,2, i \ne j$, we have that

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}(\|u_1^k\|_0^2 + \|u_2^k\|_0^2) &+ \gamma(\|\Delta u_1^k\|_0^2 + \|\Delta u_2^k\|_0^2) + 3(\|u_1^k\nabla u_1^k\|_0^2 + \|u_2^k\nabla u_2^k\|_0^2)\\ &+ 2D(\|(u_2^k+1)\nabla u_1^k\|_0^2 + \|(u_1^k+1)\nabla u_2^k\|_0^2)\\ &\le C(|w_1^k|_1^2 + |w_2^k|_1^2 + |u_1^k|_1^2 + |u_2^k|_1^2) + 4D|((u_1^k+1)\nabla u_2^k, (u_2^k+1)\nabla u_1^k)|\\ &+ 4D|((u_1^k+1)\nabla u_1^k, (u_2^k+1)\nabla u_2^k)|.\end{aligned} \tag{131}$$

By using Cauchy-Schwarz and Young inequalities, it follows that

$$|((u_1^k+1)\nabla u_2^k, (u_2^k+1)\nabla u_1^k)| \le \frac{1}{2}(\|(u_1^k+1)\nabla u_2^k\|_0^2 + \|(u_2^k+1)\nabla u_1^k\|_0^2), \tag{132}$$

and

$$\begin{aligned}|((u_1^k+1)\nabla u_1^k,(u_2^k+1)\nabla u_1^k)| \leq & \frac{1}{2}\|(u_1^k+1)\nabla u_1^k\|_0^2+\frac{1}{2}\|(u_2^k+1)\nabla u_2^k\|_0^2 \\ \leq & \|u_2^k\nabla u_2^k\|_0^2+\|\nabla u_2^k\|_0^2+\|u_1^k\nabla u_1^k\|_0^2+\|\nabla u_1^k\|_0^2 \end{aligned} \tag{133}$$

So, we can rewrite (131) by using (132) and (133) and arrange the terms $i=1,2$, we find that

$$\frac{1}{2}\frac{d}{dt}\|u_i^k\|_0^2+\gamma\|\Delta u_i^k\|_0^2 \leq C(|w_i^k|_1^2+|u_i^k|_1^2)+4D\|u_i^k\nabla u_i^k\|_0^2 \tag{134}$$

On noting Cauchy-Schwarz inequality, Poincaré inequality (25), (85), and Young's inequality with $p=\frac{4}{d}, q=\frac{4}{4-d}$, the last term in (134) and (33) can be simplified as follows:

$$\begin{aligned}\|u_i^k\nabla u_i^k\|_0^2 \leq \|u_i^k\|_{0,4}^2|u_i^k|_{1,4}^2 \leq & C\|u_i^k\|_0^{2-\frac{d}{2}}\|u_i^k\|_1^{\frac{d}{2}}|u_i^k|_1^{2-\frac{d}{2}}|u_i^k|_2^{\frac{d}{2}} \\ \leq & C\|u_i^k\|_0^{2-\frac{d}{2}}\|u_i^k\|_1^2|u_i^k|_2^{\frac{d}{2}} \\ \leq & \frac{d\varepsilon}{4}\|u_i^k\|_2^2+C(\varepsilon)\|u_i^k\|_0^2\|u_i^k\|_1^{\frac{8}{4-d}}. \end{aligned} \tag{135}$$

Substituting (135) into (134) leads to

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_i^k\|_0^2+\gamma\|\Delta u_i^k\|_0^2 \leq C|w_i^k|_1^2+C|u_i^k|_1^2+C(\varepsilon)\|u_i^k\|_0^2\|u_i^k\|_1^{\frac{8}{4-d}}+Dd\varepsilon\|u_i^k\|_2^2. \tag{136}$$

By multiplying the equation (136) by $\frac{1}{\gamma}$ and integrating the resulting equation over $t\in[0,T]$, we have

$$\begin{aligned}\frac{1}{2\gamma}\|u_i^k(t)\|_0^2+\int_0^t\|\Delta u_i^k\|_0^2 ds \leq & C\int_0^t|w_i^k|_1^2 ds+C\int_0^t|u_i^k|_1^2 ds+C\int_0^t\|u_i^k\|_0^2\|u_i^k\|_1^{\frac{8}{4-d}}ds \\ & +\frac{Dd\varepsilon}{\gamma}\int_0^t\|u_i^k\|_2^2 ds+\frac{1}{2\gamma}\|u_i^k(0)\|_0^2. \end{aligned} \tag{137}$$

Now by using elliptic regularity property $\|u_i^k\|_2 \leq C\|\Delta u_i^k\|_0$, it follows that

$$\int_0^t\|u_i^k\|_2^2 ds \leq c\int_0^t\|\Delta u_i^k\|_0^2 ds. \tag{138}$$

Next, by inserting (137) in (138) and choosing $\varepsilon=\frac{\gamma}{4DdC}$ in (137), we have that

$$\begin{aligned}\frac{3}{4}\int_0^t\|u_i^k\|_2^2 ds \leq & C\int_0^t|w_i^k|_1^2 ds+C\int_0^t|u_i^k|_1^2 ds+C\int_0^t\|u_i^k\|_0^2\|u_i^k\|_1^{\frac{8}{4-d}}ds+\frac{1}{2\gamma}\|u_i^k(0)\|_0^2 \\ \leq & C\|w_i^k\|_{L^2(0,T;H^1(\mathscr{R}))}^2+C\|u_i^k\|_{L^2(0,T;H^1(\mathscr{R}))}^2 \\ & +C\|u_i^k\|_{L^\infty(0,T;L^2(\mathscr{R}))}^2\|u_i^k\|_{L^\infty(0,T;H^1(\mathscr{R}))}^{\frac{8}{4-d}}ds+\frac{1}{2\gamma}\|u_i^k(0)\|_0^2 \\ \leq & C\|w_i^k\|_{L^2(0,T;H^1(\mathscr{R}))}^2+C\|u_i^k\|_{L^2(0,T;H^1(\mathscr{R}))}^2+C\|u_i^k\|_{L^\infty(0,T;H^1(\mathscr{R}))}^{\frac{16-2d}{4-d}}+\frac{1}{2\gamma}\|u_i^k(0)\|_0^2. \end{aligned} \tag{139}$$

Thus, by using (88), (103) and (104) we arrive at

$$u_i^k \in L^2(0,T;H^2(\mathscr{R})).$$

Since $L^2(0,T;H^2(\mathscr{R}))$ is a reflexive Banach space, by compactness arguments, we deduce the existence of a subsequence such that

$$u_i^k \rightharpoonup u_i \text{ in } L^2(0,T;H^2(\mathscr{R})).$$

□

5 Conclusions

The paper investigates the existence of a solution for Cahn-Hilliard equation with mobility. We commence by presenting weak equivalent formulations for problem (P). The existence of solutions to problem (P) is then established using Faedo-Galerkin technique and compactness arguments, subject to certain restrictions on the initial data. We derive energy estimates for approximate solutions, enabling us to pass to the limit in the approximate equation and confirm the existence of a weak solution. Following this, we present regularity results for the weak formulation.

Acknowledgments

We would like to express our gratitude to an anonymous referee for his valuable feedback and insightful comments, which have greatly contributed to the enhancement of this manuscript.

References

[1] M.S. Abdo, S.K. Panchal, *Existence and continuous dependence for fractional neutral functional differential equations,* J. Math. Model. **5** (2017) 153–170.

[2] A. Al-Ghafli, *Mathematical and Numerical Analysis of a Pair of Coupled Cahn-Hilliard Equations with a Logarithmic Potential*, PhD thesis, Durham University, 2010.

[3] G.A. Al-Juaifri, A.J. Harfash, *Analysis of a nonlinear reaction-diffusion system of the fitzhugh-nagumo type with robin boundary conditions,* Ric. Mat. **72** (2023) 335–357.

[4] N.A. Alsarori, K.P. Ghadle, *On the mild solution for nonlocal impulsive fractional semilinear differential inclusion in banach spaces,* J. Math. Model. **6** (2018) 239–258.

[5] A. Beiranvand, A. Neisy, K. Ivaz, *Mathematical analysis and pricing of the european continuous installment call option,* J. Math. Model. **4** (2016) 171–185.

[6] F. Bernis, A. Friedman, *Higher order nonlinear degenerate parabolic equations,* J. Differ. Equ. **83** (1990) 179–206.

[7] S.P. Bhairat, *Existence and continuation of solutions of hilfer fractional differential equations,* J. Math. Model. **7** (2019) 1–20.

[8] J.W. Cahn, J.E. Hilliard, *Free energy of a nonuniform system. I. interfacial free energy,* The J. Chem. Phys. **28** (1958) 258–267.

[9] J.W. Cahn, J.E. Hilliard, Spinodal decomposition: A reprise, *Acta Metall.* **19** (1971) 151–161.

[10] J.W. Cahn, J. E. Taylor, *Overview no. 113 surface motion by surface diffusion*, Acta Metall. Mater. **42** (1994) 1045–1063.

[11] T. Cazenave, *Semilinear Schrodinger Equations*, American MathematAmerican Mathematical Society, 2003.

[12] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems*, SIAM, 2002.

[13] R. Dautray, J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology: Volume 2 Functional and Variational Methods*, Springer Science & Business Media, 1999.

[14] C. M. Elliott. The Cahn-Hilliard Model for the Kinetics of Phase Separation, *Mathematical models for phase change problems*, Springer, 1989.

[15] C.M. Elliott, H. Garcke, On the cahn–hilliard equation with degenerate mobility, *SIAM J. Math. Anal.* **27** (1996) 404–423.

[16] P.C. Fife, Models for phase separation and their mathematics, *Electron. J. Differ. Equ.* 48 (2000) 48.

[17] J.D. Gunton, R. Toral, A. Chakrabarti, Numerical studies of phase separation in models of binary alloys and polymer blends, *Phys. Scripta* **33** (1990) 12–19.

[18] A.A. Hamoud, K. P. Ghadle, Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations, *J. Math. Model.* **6** (2018) 91–104.

[19] J.E. Hilliard. Spinodal decomposition. *Phase Transit.* (1970) 497–560.

[20] M. Imran. *Numerical Analysis of a Coupled Pair of Cahn-Hilliard Equations*. PhD thesis, Durham University, 2001.

[21] A. Jawahdou, Existence of mild solutions of second order evolution integro-differential equations in the frechet spaces, *J. Math. Model.* **7** (2019) 305–318.

[22] Y. Jingxue, On the existence of nonnegative continuous solutions of the Cahn-Hilliard equation, *J. Differ. Equ.* **97** (1992) 310–327.

[23] P. Keblinski, S.K. Kumar, A. Maritan, J. Koplik, J.R. Banavar, Interfacial roughening induced by phase separation, *Phys. Rev. Lett.* **76** (1996) 1106.

[24] J.L. Lions, *Quelques Methodes de Resolution des Problemes aux Limites non Lineaires*, Dunod, Paris, 1969.

[25] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Ration. Mech. Anal.* **98** (1987) 123–142.

[26] A. Novick-Cohen, The Cahn-Hilliard equation: mathematical and modeling perspectives, *Adv. Math. Sci. Appl.* **8** (1998) 965–985.

[27] V.P. Skripov, A. Skripov, Spinodal decomposition (phase transitions via unstable states), *Sov. phys. Usp.* **22** (1979) 389.

[28] B. Sontakke, A. Shaikh, K. Nisar, Existence and uniqueness of integrable solutions of fractional order initial value equations, *J. Math. Model.* **6** (2018) 137–148.

[29] S.R. Tate, V.V. Kharat, H.T. Dinde, A nonlocal cauchy problem for nonlinear fractional integro-differential equations with positive constant coefficient, *J. Math. Model.* **7** (2019) 133–151.

[30] J.E. Taylor, J.W. Cahn, Linking anisotropic sharp and diffuse surface motion laws via gradient flows, *J. Stat. Phys.* **77** (1994) 183–197.

[31] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer Science & Business Media, 2012.

[32] A. Zenisek, J.R. Whiteman, *Nonlinear Elliptic And Evolution Problems and Their Finite Element Approximations*, Academic Press, London, 1990.