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SOME APPLICATIONS OF k-REGULAR SEQUENCES AND ARITHMETIC RANK OF AN IDEAL WITH RESPECT TO MODULES

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Abstract. Let R be a commutative Noetherian ring with identity, I be an ideal of R, and M be an R-module. Let $k \geqslant -1$ be an arbitrary integer. In this paper, we introduce the notions of $\mathrm{Rad}_M(I)$ and $\mathrm{ara}_M(I)$ as the radical and the arithmetic rank of I with respect to M, respectively. We show that the existence of some sort of regular sequences can be depended on $\dim M/IM$ and so, we can get some information about local cohomology modules as well. Indeed, if $\mathrm{ara}_M(I) = n \geq 1$ and $(\mathrm{Supp}_R(M/IM))_{>k} = \emptyset$ (e.g., if $\dim M/IM = k$), then there exist n elements $x_1, ..., x_n$ in I which is a poor k-regular M-sequence and generate an ideal with the same radical as $\mathrm{Rad}_M(I)$ and so $H^i_I(M) \cong H^i_{(x_1,...,x_n)}(M)$ for all $i \in \mathbb{N}_0$. As an application, we show that $\mathrm{ara}_M(I) \leq \dim M + 1$, which is a refinement of the inequality $\mathrm{ara}_R(I) \leq \dim R + 1$ for modules, attributed to Kronecker and Forster. Then, we prove $\dim M - \dim M/IM \leq \mathrm{cd}(I, M) \leq \mathrm{ara}_M(I) \leq \dim M$, if $(R, \mathfrak{m})$ is a local ring and $IM \neq M$.

1. Introduction

Throughout this paper, R denotes a non-trivial commutative Noetherian ring with identity, I denotes an arbitrary ideal, and M denotes an R-module. The set of minimal elements of $\mathrm{Ass}_R M$ ($\mathrm{Supp}_R M$, respectively) with respect to inclusion is denoted by $\mathrm{mAss}_R M$ ($\mathrm{mSupp}_R M$,

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respectively). The symbol $\mathbb{N}_0$ denotes the set of non-negative integers and $k \geqslant -1$ is an arbitrary integer. For a subset T of $\operatorname{Spec}(R)$, we set

$$(T)_{>k} := \{\mathfrak{p} \in T \mid R/\mathfrak{p} > k\},\ (T)_{\leq k} := \{\mathfrak{p} \in T \mid \dim R/\mathfrak{p} \leq k\}.$$

In 1978, on a local ring, Schenzel, Trung, and Cuong [19] introduced the concept of filter regular M-sequence as the generalization of regular M-sequence. In 1996, on an arbitrary Noetherian ring, Ahmadi Amoli [1] introduced the notion of $I - f.\operatorname{grade}_M(\mathfrak{a})$ as the common length of all maximal I-filter regular M-sequences contained in an ideal $\mathfrak{a}$ with $\operatorname{Supp}_R(M/\mathfrak{a}M) \setminus V(I) \neq \emptyset$. This notion is a generalization of filter-depth introduced by Melkersson [16], where $(R, \mathfrak{m})$ is a local ring. It is notable that the filter-depth was defined as $f - \operatorname{depth}_{\mathfrak{a}}(M) = \min\{\operatorname{depth}_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M) \setminus V(\mathfrak{m})\}$, not by means of the common length of any sequences. In 2005, on a local ring, Nhan [17] introduced the two notions of generalized regular sequence and generalized depth which are extensions of filter regular M-sequence and filter-depth, respectively. In 2008, on a local ring, Chinh and Nhan [6] introduced the concept of k-regular M-sequence, as the extension of all kind of regular sequences mentioned above. This concept was studied basically by Ahmadi Amoli and Sanaei in 2012 [2]. In fact, for $k = -1$ any k-regular M-sequence is a regular M-sequence. Also, if $(R, \mathfrak{m})$ is a local ring, then any k-regular M-sequence is a filter regular M-sequence (generalized regular M-sequence, respectively) for $k = 0$ ($k = 1$, respectively).

The local cohomology module

$$H^i_I(M) \cong \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/I^n, M)\ (i \in \mathbb{N}_0)$$

was first introduced by Serre in 1955 [20]. This notion was generalized to algebraic geometry by Grothendieck in 1967 [8]. All kind of regular sequences give some useful information about local cohomology modules. In this paper, we deal with some applications of k-regular M-sequences in conjunction with local cohomology modules, cohomological dimension of modules, and arithmetic rank of ideals with respect to modules. Recall that the cohomological dimension of M with respect to I, $\operatorname{cd}(I, M)$, is the greatest integer $i \in \mathbb{N}_0$ such that $H^i_I(M) \neq 0$, also the arithmetic rank of I, $\operatorname{ara}_R(I)$, is the least number of elements of I required to generate an ideal with the same radical as I. We introduce the notion of $\operatorname{Rad}_M(I)$ as the radical of I with respect to M (Definition 2.1). Then, we introduce the notion of $\operatorname{ara}_M(I)$ as the arithmetic rank of I with respect to M (Definition 3.2). There is an example in which $\operatorname{ara}_M(I) < \operatorname{ara}_R(I)$ and so $\operatorname{ara}_M(I)$ may give more information about local cohomology $H_I^i(M)$ than $\operatorname{ara}_R(I)$ (Corollary 3.4 and Example 3.7).

It is difficult to determine the least number of elements $x_1, ..., x_n$ of I such that $\operatorname{Rad}_M(I) = \operatorname{Rad}_M(x_1, ..., x_n)$. In 2009, Mehrvarz et al. [15] showed that if $\operatorname{ara}_R(I) = n \geq 1$, then there is an I-filter regular M-sequence $x_1, ..., x_n$ in I such that $\operatorname{Rad}_R(I) = \operatorname{Rad}_R(x_1, ..., x_n)$. In this paper, we show the existence of some sort of regular sequences which are depended on the dimension of M/IM and then get some information about local cohomology modules as well. Indeed, we prove that if $\operatorname{ara}_M(I) = n \geq 2$ and $(\operatorname{Supp}_R(M/IM))_{>k} = \emptyset$ (e.g., if $\dim M/IM = k$), then there exists a poor k-regular M-sequence $x_1, ..., x_n \in I$ such that $\operatorname{Rad}_M(I) = \operatorname{Rad}_M(x_1, ..., x_n)$ (hence $H_I^i(M) \cong H^i_{(x_1,...,x_n)}(M)$ for all $i \in \mathbb{N}_0$) and $\operatorname{cd}((x_1, ..., x_i), M) = i$ for all $1 \leq i \leq n-1$ (Theorems 3.3 and 3.8).

By the Generalized Krull's Principal Ideal Theorem with respect to modules (Theorem 2.8) $\operatorname{ara}_M(I) \geq \operatorname{ht}_M(I)$. In the case of equality, I is called *set theoretic complete intersection with respect to* M (compare [10]). We show that under some conditions, $\operatorname{ara}_M(I) > \operatorname{ht}_M(I)$ (Corollary 3.6).

In sequel, we show that if $\dim M/IM = 1$, $(\operatorname{Supp}_R(M/IM))_{>k} = \emptyset$, and $\operatorname{ara}_M(I) = n \geqslant 2$, then there exists a poor k-regular M-sequence $x_1, ..., x_n$ such that $H_I^i(M)$ are $(x_1, ..., x_t)$-cofinite for all $1 \leqslant t \leqslant n$ and all $0 \leqslant i < t$ (Theorem 3.11). In 1882, Kronecker [11] showed that in Noetherian rings of dimension of n, any radical ideal is equal to the radical of an ideal with $n+1$ generators. That was stronger than what had been proved on polynomial rings over a field in n indeterminates. In 1964, Forster [7] showed that for any ideal I of a local ring R, $\operatorname{ara}_R(I) \leq \dim R$. In 2009, Mehrvarz et al. [15] showed that $\operatorname{ara}_R(I) \leq \dim R + 1$ for any ideal I of R (not necessarily local ring). In 2020, Azami [3, Corollary 2.4] proved that, in the local ring $(R, \mathfrak{m})$ if $H_{\mathfrak{m}}^{\dim R}(R)$ is I-cofinite, then $\operatorname{ara}_R(I) = \dim R$. One of our main results of this paper is to develop these results for modules. In fact, in Theorem 3.13, we show that $\operatorname{ara}_M(I) \leq \dim M + 1$. Also, if R is a local ring and I is an ideal for which $IM \neq M$, we show that $\operatorname{ara}_M(I) \leq \dim M$ (Corollary 3.17). By definition, on any Noetherian ring R, $\operatorname{cd}(I, M) \leq \operatorname{ara}_M(I)$. The final result of this paper is to show that on a local ring $(R, \mathfrak{m})$, if $IM \neq M$, then $\dim M - \dim M/IM \leq \operatorname{cd}(I, M) \leq \operatorname{ara}_M(I) \leq \dim M$ (Theorem 3.22).

2. Preliminaries

In this section, we introduce the notion of $\mathrm{Rad}_M(I)$ as the radical of I with respect to M which is a generalization of $\mathrm{Rad}_R(I)$. We also give some properties of this notion which are needed throughout the paper. Recall that $\mathrm{Rad}_R(I) = \{r \in R \mid \exists\, n, r^n \in I\}$ is the intersection of all prime ideals in $\mathrm{Ass}_R(R/I)$. It is natural to have the following definition.

Definition 2.1. Let I be an ideal of R and M be an R-module. The radical of I with respect to M denoted by $\mathrm{Rad}_M(I)$, is the intersection of the family of the associated prime ideals of M/IM, i.e., $\mathrm{Rad}_M(I) = \bigcap_{\mathfrak{p}\in \mathrm{Ass}_R(M/IM)} \mathfrak{p}$. It is clear that $I \subseteq \mathrm{Rad}_M(I)$.

Following, we present some properties of the radical of an ideal with respect to an R-module.

Lemma 2.2. *Let I be an ideal of R and M be a finitely generated R-module. Then*
(i) $\mathrm{Rad}_M(I) = \mathrm{Rad}_R(\mathrm{Ann}_R(M/IM)) = \mathrm{Rad}_R(\mathrm{Ann}_R(M) + I)$ *and* $\mathrm{Rad}_R(I) \subseteq \mathrm{Rad}_M(I)$.
(ii) $\mathrm{Rad}_M(I) = \{r \in R \mid \exists\, n, r^n M \subseteq IM\}$.
(iii) $IM = M$ *if and only if* $\mathrm{Rad}_M(I) = R$.
(iv) $\mathrm{Rad}_M(I) = \bigcap_{\mathfrak{p}\in \mathrm{Supp}_R(M/IM)} \mathfrak{p}$.
(v) Let J be a second ideal of R such that $I \subseteq J$. Then $\mathrm{Rad}_M(I) \subseteq \mathrm{Rad}_M(J)$.

Proof. *(i)* Let $IM = \cap_{i=1}^t N_i$ be a minimal primary decomposition of IM, where N_i is a $\mathfrak{p}_i$-primary in IM $(1 \le i \le t)$. Since $\mathrm{Ass}_R(M/IM) = \{\mathfrak{p}_1, ..., \mathfrak{p}_t\}$, then

$$\mathrm{Rad}_M(I) = \cap_{i=1}^t \mathrm{Rad}_R(\mathrm{Ann}_R(M/N_i)) = \mathrm{Rad}_R(\mathrm{Ann}_R(M/IM))$$
$$= \mathrm{Rad}_R(\mathrm{Ann}_R(M) + I).$$

(ii) It is easily followed by (i).
(iii) It is clear by (ii).
(iv) The assertion is obvious, since every ideal of $\mathrm{Supp}_R(M/IM)$ contains an ideal of $\mathrm{Ass}_R(M/IM)$.
(v) Since $\mathrm{Supp}_R(M/JM) \subseteq \mathrm{Supp}_R(M/IM)$, the assertion is followed by (iv). □

Corollary 2.3. *Let the situation be as in Lemma 2.2 and suppose that* $\mathfrak{p} \in \mathrm{Supp}_R(M)$. *Then*
(i) $\mathrm{Rad}_M(\mathfrak{p}) = \mathfrak{p}$.
(ii) $\mathrm{Rad}_M(\mathfrak{p}^n) = \mathfrak{p}$ *for all* $n \in \mathbb{N}$.

The following Lemma can be proved by definition and above results.

Lemma 2.4. *Let I, J be two ideals of R and M be a finitely generated R-module. Then we have the following.*
(i) $\mathrm{Rad}_M(I+J) = \mathrm{Rad}_M(\mathrm{Rad}_M(I) + \mathrm{Rad}_M(J))$.
(ii) $\mathrm{Rad}_M(\mathrm{Rad}_M(I)) = \mathrm{Rad}_M(I)$.
(iii) If $\mathrm{Rad}_M(I) + \mathrm{Rad}_M(J) = R$*, then* $(I+J)M = M$.

Corollary 2.5. *Let M be a finitely generated R-module. Then for any ideal I of R, there exists a positive integer υ such that* $(\mathrm{Rad}_M(I))^{\upsilon} M \subseteq IM$.

Proof. Use Lemma 2.2 (i) and [21, Lemma 8.21]. □

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a local ring, I be an ideal of R, and M be a finitely generated R-module such that $IM \neq M$. Then* $\mathrm{Rad}_M(I) = \mathfrak{m}$ *if and only if there exists a positive integer υ such that $\mathfrak{m}^{\upsilon} M \subseteq IM$.*

Proof. Use Lemma 2.2 (i) and (iii) and Corollary 2.5. □

Let M be a non-zero R-module and I be an ideal of R. Let us recall the height of I with respect to M, as $\mathrm{ht}_M(I) = \inf\{\mathrm{ht}_M(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R(M/IM)\}$, if the infimum exists, and ∞ otherwise (for a prime ideal $\mathfrak{p} \in \mathrm{Supp}_R(M)$, $\mathrm{ht}_M(\mathfrak{p}) := \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$). Also, recall that if $(R, \mathfrak{m})$ is a local ring, then $\dim M = \min\{n \in \mathbb{N}_0 \mid \exists\, x_1, ..., x_n \in \mathfrak{m}, \ell_R(M/(x_1, ..., x_n)M) < \infty\}$ ([21, Ex. 15.24]).

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Let I be an ideal of R such that* $\mathrm{Rad}_M(I) = \mathfrak{m}$*. Then $\ell_R(M/IM) < \infty$ and so $\dim M \leq m(I)$, where $m(I)$ is the minimum number of generators of I.*

Proof. By Lemma 2.2 (iv), $\mathrm{Supp}_R(M/IM) = \{\mathfrak{m}\}$. Hence $\ell_R(M/IM) < \infty$. □

To achieve the main results of this paper, we need to generalize the Krull's Principal Ideal Theorem for modules. For this purpose, Corollary 2.7 helps us.

Theorem 2.8 (Generalized Krull's Principal Ideal Theorem with respect to modules)**.** *Let I be an ideal of R generated by r elements. Assume that M is a finitely generated R-module such that $IM \neq M$. If $\mathfrak{p}$ is a minimal prime ideal of I in* $\mathrm{Supp}_R(M)$*, i.e.,* $\mathfrak{p} \in \mathrm{mSupp}_R(M/IM)$*, then* $\mathrm{ht}_M(\mathfrak{p}) \leq r$.

Proof. By assumption and Lemma 2.2 (iv), $\mathrm{Rad}_{M_{\mathfrak{p}}}(IR_{\mathfrak{p}}) = \mathfrak{p}R_{\mathfrak{p}}$. Hence by Corollary 2.5, $(\mathfrak{p}R_{\mathfrak{p}})^{\upsilon} M_{\mathfrak{p}} \subseteq (IR_{\mathfrak{p}})M_{\mathfrak{p}}$ for some $\upsilon \in \mathbb{N}$. Now, the assertion follows by Corollary 2.7. □

We end this section by showing that the local cohomology is invariant under taking radical, Rad_M. This property is used frequently in applications. Theorem 2.10 is followed by the following lemma.

Lemma 2.9. *Let I be an ideal of R and M be an R-module. Then $\Gamma_I(M) = \Gamma_{\mathrm{Rad}_M(I)}M$.*

Proof. Clearly, $\Gamma_{\mathrm{Rad}_M(I)}(M) \subseteq \Gamma_I(M)$. Conversely, let $m \in M$ be such that $I^n m = 0$ for some $n \in \mathbb{N}$. By Lemma 2.2 (i), it is enough to show that $m \in \Gamma_{\mathrm{Ann}_R(M)+I}(M)$. Assume that r and a are arbitrary elements of $\mathrm{Ann}_R(M)$ and I, respectively. Since $a^n m = 0$ and $rm = 0$, we have $(r + a)^{2n-1}m = 0$ as required. □

Theorem 2.10. *Let I and J be two ideals of R such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(J)$. Then $H^i_I(M) \cong H^i_J(M)$ for any R-module M and all $i \in \mathbb{N}_0$.*

3. Main results

In this section, we start by introducing the notion of $\mathrm{ara}_M(I)$ as the arithmetic rank of I with respect to M which generalizes $\mathrm{ara}_R(I)$. Recall that $\mathrm{ara}_R(I)$ is the least number of elements of R required to generate an ideal which has the same radical as $\mathrm{Rad}_R(I)$. First, we remind the following definition.

Definition 3.1. ([6]) A sequence $a_1, \dots, a_n$ of elements of R is called a *poor k-regular M-sequence* whenever $a_i \notin \mathfrak{p}$ for all

$$\mathfrak{p} \in \mathrm{Ass}(M/\sum_{j=1}^{i-1} a_j M), \qquad \dim R/\mathfrak{p} > k$$

for all $i = 1, \dots, n$. Moreover, if $\dim(M/\sum_{i=1}^{n} a_i M) > k$, $a_1, \dots, a_n$ is called a k-regular M-sequence.

Definition 3.2. The arithmetic rank of I with respect to M, denoted by $\mathrm{ara}_M(I)$, is defined as follows:

$$\mathrm{ara}_M(I) = \min\{i \in \mathbb{N}_0 \mid \exists\, y_1, ..., y_i \in I, \mathrm{Rad}_M(y_1, ..., y_i) = \mathrm{Rad}_M(I)\}.$$

It is clear that if $IM = M$, then $\mathrm{ara}_M(I) = 1$.

Now, we are ready to present the following theorem which plays a main role in this article. We prove the existence of a poor k-regular M-sequence (for all $k \geqslant -1$) of length $\mathrm{ara}_M(I) = n \geq 1$ such that the generated ideal has the same radical as $\mathrm{Rad}_M(I)$.

Theorem 3.3. *Let M be a finitely generated R-module and I be an ideal of R such that $(\mathrm{Supp}_R(M/IM))_{>k} = \emptyset$ (e.g., if $\dim M/IM = k$). If $\mathrm{ara}_M(I) = n \geq 1$, then there exists a poor k-regular M-sequence*

$y_1, ..., y_n \in I$ *such that* $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(y_1, ..., y_n)$. *Hence* $H^i_I(M) \cong H^i_{(y_1,...,y_n)}(M)$ *for all* $i \in \mathbb{N}_0$.

Proof. We use induction on n. By definition, there exist $x_1, ..., x_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(x_1, ..., x_n)$. One can see that $I \nsubseteq \cup_{\mathfrak{p}\in(\mathrm{Ass}_R(M))_{>k}}\mathfrak{p}$, and so $(x_1, ..., x_n) \nsubseteq \cup_{\mathfrak{p}\in(\mathrm{Ass}_R(M))_{>k}}\mathfrak{p}$. Now, by [14, Ex. 16.8], there exists $a_1 \in (x_2, ..., x_n)$ such that $x_1 + a_1 \notin \cup_{\mathfrak{p}\in(\mathrm{Ass}_R(M))_{>k}}\mathfrak{p}$. Put $y_1 := x_1 + a_1$. So that y_1 is a poor k-regular M-sequence of length 1 contained in I. By Lemma 2.2 (ii) and (v), $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(y_1, x_2, ..., x_n)$. Let $1 \le s < n$ and $y_1, ..., y_s \in I$ is a poor-k regular M-sequence with $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(y_1, ..., y_s, x_{s+1}, ..., x_n)$. We have $I \nsubseteq \cup_{\mathfrak{p}\in(\mathrm{Ass}_R M/(y_1,...,y_s)M)_{>k}}\mathfrak{p}$ and so that

$$(y_1, ..., y_s, x_{s+1}, ..., x_n) \nsubseteq \bigcup_{\mathfrak{p}\in(\mathrm{Ass}_R M/(y_1,...,y_s)M)_{>k}} \mathfrak{p}.$$

Now, choose $a_{s+1} \in (y_1, ..., y_s, x_{s+2}, ..., x_n)$ such that $x_{s+1} + a_{s+1} \notin \cup_{\mathfrak{p}\in(\mathrm{Ass}_R M/(y_1,...,y_s)M)_{>k}}\mathfrak{p}$. Put $y_{s+1} := x_{s+1} + a_{s+1}$. Then $y_{s+1} \in I$ and $y_1, ..., y_s, y_{s+1}$ is a poor k-regular M-sequence in I. We can conclude that

$$\mathrm{Rad}_M(I) = \mathrm{Rad}_M(y_1, ..., y_s, y_{s+1}, x_{s+2}, ..., x_n).$$

Therefore, the first assertion follows by induction. The last part is obvious by Theorem 2.10. □

As an application of Theorem 3.3, we prove a vanishing of local cohomology functor H^i_I which is a powerful tool for applications of local cohomology to algebraic geometry.

Corollary 3.4. *Let I be an ideal of R and M be a finitely generated R-module with* $\dim M/IM < \infty$. *Then* $H^i_I(M) = 0$ *for all* $i > \mathrm{ara}_M(I)$.

Proof. If $\mathrm{ara}_M(I) = 0$, then $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(0)$. So that $H^i_I(M) \cong H^i_{(0)}(M) = 0$ for all $i \in \mathbb{N}_0$, by Theorem 2.10. Now, assume that $\mathrm{ara}_M(I) = n \ge 1$. Let $\dim M/IM = k$. By Theorem 3.3, there exists a poor k-regular M-sequence $y_1, ..., y_n \in I$ such that $\mathrm{Rad}_M(y_1, ..., y_n) = \mathrm{Rad}_M(I)$. By Theorem 2.10, $H^i_I(M) \cong H^i_{(y_1,...,y_n)}(M)$ for all $i \in \mathbb{N}_0$. Therefore $H^i_I(M) = 0$ for all $i > n$, by [5, Theorem 3.3.1]. □

As a result of Corollary 3.4 and Theorem 2.8, $\mathrm{ht}_M(I) \le \mathrm{cd}(I, M) \le \mathrm{ara}_M(I)$.

Definition 3.5. (Compare with [10]) Let M be a finitely generated R-module. An ideal I of R with $\mathrm{ht}_M(I) = h$ is called a set theoretic complete intersection with respect to M whenever there exist

$x_1, ..., x_h \in I$ such that $\mathrm{Rad}_M(x_1, ..., x_h) = \mathrm{Rad}_M(I)$. In other words, $\mathrm{ara}_M(I) = \mathrm{ht}_M(I)$.

Corollary 3.6. *Let R be a Cohen-Macaulay ring and M be a finitely generated R-module. Suppose that I is an ideal of R such that $\mathrm{ht}_M(I) = h$ and $H_I^{h+1}(M) \neq 0$. Then I is not a set theoretic complete intersection with respect to M.*

Proof. Apply Corollary 3.4. □

The following example shows that the invariant $\mathrm{ara}_M(I)$ may give more information about local cohomology than $\mathrm{ara}_R(I)$. It shows that $H_I^i(M) = 0$ even for $i = \mathrm{ara}_R(I)$.

Example 3.7. Let k be a field and $R = k[x, y, z]$. Consider the natural homomorphism $^- : R \to R/(x^2 - yz) = \bar{R}$. Then $\bar{R}$ is a two-dimensional ring and $\bar{\mathfrak{p}} = (\bar{x}, \bar{y})$ is a prime ideal of $\bar{R}$ with $\mathrm{ht}_{\bar{R}}(\bar{\mathfrak{p}}) = 1$. Since $\mathrm{Rad}_{\bar{R}}(\bar{\mathfrak{p}}) = \mathrm{Rad}_{\bar{R}}(\bar{y})$, $\mathrm{ara}_{\bar{R}}(\bar{\mathfrak{p}}) = 1$. Now, consider the $\bar{R}$-module $M := \bar{R}/\bar{\mathfrak{p}}$. Then, it is clear that $\mathrm{Rad}_M(\bar{\mathfrak{p}}) = \mathrm{Rad}_M(\bar{0})$, and so $\mathrm{ara}_M(\bar{\mathfrak{p}}) = 0$. Therefore $\mathrm{ara}_M(\bar{\mathfrak{p}}) < \mathrm{ara}_{\bar{R}}(\bar{\mathfrak{p}})$. Consequently, by Corollary 3.4, $H_{\bar{\mathfrak{p}}}^i(M) = 0$ for all $i > 0$, especially for $i = \mathrm{ara}_{\bar{R}}(\bar{\mathfrak{p}}) = 1$.

The other application of Theorem 3.3 is as follows.

Theorem 3.8. *Let M be a finitely generated R-module and I be an ideal of R with $\dim M/IM < \infty$. Assume that $\mathrm{ara}_M(I) = n \geqslant 2$. Then there exists a poor k-regular M-sequence $y_1, ..., y_n \in I$ such that $H^i_{(y_1,...,y_i)}(M) \neq 0$ for all $1 \leqslant i \leqslant n-1$. In particular, $\mathrm{cd}((y_1, ..., y_i), M) = i$ for all $1 \leqslant i \leqslant n-1$.*

Proof. Let $\dim M/IM = k$. By Theorem 3.3, there exists a poor k-regular M-sequence $y_1, ..., y_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(y_1, ..., y_n)$. Let $1 \leqslant i \leqslant n-1$. By the Generalized Krull's Principal Ideal Theorem with respect to modules, Theorem 2.8, $\mathrm{ht}_M(\mathfrak{p}) = i$ for all $\mathfrak{p} \in (\mathrm{mAss}_R\, M/(y_1, ..., y_i)M))_{>k}$. Thus, there exists

$$\mathfrak{q} \in (\mathrm{mAss}_R\, M/(y_1, ..., y_i)M))_{>k}$$

such that $\mathfrak{q} \not\supseteq I$, and so $\dim M_{\mathfrak{q}} = i$. By the Non-Vanishing Theorem, $H^i_{(y_1,...,y_i)}(M) \neq 0$. Since for all $j > i$, $H^j_{(y_1,...,y_i)}(M) = 0$, we get $\mathrm{cd}((y_1, ..., y_i), M) = i$. □

Theorem 3.9. *Let I be an ideal of R and M be a finitely generated R-module such that $(\mathrm{Supp}_R(M/IM))_{>k} = \emptyset$. Assume that $\mathrm{ara}_M(I) = n \geq 2$ and $\mathrm{cd}(I, M) < n-1$. Then there exists a poor k-regular M-sequence $x_1, ..., x_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(x_1, ..., x_n)$ and $\mathrm{Ext}^i_R(R/I, H^{n-1}_{(x_1,...,x_{n-1})}(M)) = 0$ for all $i \in \mathbb{N}_0$.*

Proof. The first part of the assertion follows from Theorem 3.3. Hence, there is a poor k-regular M-sequence $x_1, ..., x_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(x_1, ..., x_n)$. For the last part, we use the following exact sequences ([18, Corollary 1.4]):

$$0 \to H^1_{(x_n)}(H^{n-1}_{(x_1,\dots,x_{n-1})}(M)) \to H^n_{(x_1,\dots,x_n)}(M) \to$$
$$H^0_{(x_n)}(H^n_{(x_1,\dots,x_{n-1})}(M)) \to 0$$

and

$$0 \to H^1_{(x_n)}(H^{n-2}_{(x_1,\dots,x_{n-1})}(M)) \to H^{n-1}_{(x_1,\dots,x_n)}(M) \to$$
$$H^0_{(x_n)}(H^{n-1}_{(x_1,\dots,x_{n-1})}(M)) \to 0.$$

By assumption, $H^n_{(x_1,\dots,x_n)}(M) = 0 = H^{n-1}_{(x_1,\dots,x_n)}(M)$. Thus

$$H^1_{(x_n)}(H^{n-1}_{(x_1,\dots,x_{n-1})}(M)) = 0 = H^0_{(x_n)}(H^{n-1}_{(x_1,\dots,x_{n-1})}(M)).$$

Therefore, we have the R-isomorphism

$$H^{n-1}_{(x_1,\dots,x_{n-1})}(M) \xrightarrow{x_n} H^{n-1}_{(x_1,\dots,x_{n-1})}(M),$$

which induces the following R-isomorphism,

$$\mathrm{Ext}^i_R(R/I, H^{n-1}_{(x_1,\dots,x_{n-1})}(M)) \xrightarrow{x_n} \mathrm{Ext}^i_R(R/I, H^{n-1}_{(x_1,\dots,x_{n-1})}(M)),$$

for all $i \in \mathbb{N}_0$. Since $x_n \in I$, $\mathrm{Ext}^i_R(R/I, H^{n-1}_{(x_1,\dots,x_{n-1})}(M)) = 0$ for all $i \in \mathbb{N}_0$. □

Remark 3.10. Let R be a local ring and the situation be as Theorem 3.9. By virtue of the isomorphism in the proof of Theorem 3.9, Nakayama Lemma, and Theorem 3.8, $\mathrm{Hom}_R(R/(x_1, ..., x_{n-1}), H^{n-1}_{(x_1,\dots,x_{n-1})}(M))$ is not a finitely generated R-module.

Theorem 3.11. *Let M be a finitely generated R-module and I be an ideal of R such that $\dim M/IM = 1$ and $(\mathrm{Supp}_R(M/IM))_{>k} = \emptyset$. Suppose that $\mathrm{ara}_M(I) = n \geqslant 2$. Then there exists a poor k-regular M-sequence $x_1, ..., x_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(x_1, ..., x_n)$. If $(\mathrm{Supp}_R(M/(x_1, ..., x_t)M) \setminus V(I))_{\leqslant k} = \emptyset$ for all $1 \leqslant t \leqslant n$, then the R-modules $H^i_I(M)$ are $(x_1, ..., x_t)$-cofinite for all $1 \leqslant t \leqslant n$ and all $0 \leqslant i < t$.*

Proof. By Theorem 3.3, there is a poor k-regular M-sequence $x_1, ..., x_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(x_1, ..., x_n)$. This is a k-regular M-sequence as $(\mathrm{Supp}_R(M/(x_1, ..., x_n)M))_{>k} \neq \emptyset$. Now, let $1 \leq t \leq n$. By [2, Theorem 3.2], we see that $H^i_I(M) \cong H^i_{(x_1,...,x_t)}(M)$ for all $0 \leqslant i < t$. Thus $\dim \mathrm{Supp}_R(H^i_{(x_1,...,x_t)}(M)) \leqslant \dim(M/IM) = 1$, for all $0 \leqslant i < t$, especially for all $i < \mathrm{cd}((x_1, ..., x_t), M)$. Now $H^i_{(x_1,...,x_t)}(M)$ and so $H^i_I(M)$ is $(x_1, ..., x_t)$-cofinite for all $0 \leqslant i < t$ by [4, Corollary 2.13]. □

Lemma 3.12. *Let I be an ideal of R and M be a finitely generated R-module such that $(\mathrm{Supp}_R(M/IM))_{>k} = \emptyset$. Let $\mathrm{ara}_M(I) = n \geq 2$. Then there exists a poor k-regular M-sequence $x_1, ..., x_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(x_1, ...x_n)$ and $M_\mathfrak{p}$ is a Cohen-Macaulay $R_\mathfrak{p}$-module of dimension t, for all $\mathfrak{p} \in (\mathrm{mAss}_R\, M/(x_1, ..., x_t)M)_{>k}$ and all $1 \leq t \leq n-1$.*

Proof. By Theorem 3.3, there is a poor k-regular M-sequence $x_1, ..., x_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(x_1, ...x_n)$. Let $1 \leq t \leq n-1$ and $\mathfrak{p}$ be a prime ideal in $(\mathrm{mAss}_R\, M/(x_1, ..., x_t)M)_{>k}$. Since $x_1, ..., x_t \in \mathfrak{p}$ is a poor k-regular M-sequence, $\mathrm{depth}(M_\mathfrak{p}) \geq t$, by [2, Theorem 2.3]. As $\mathfrak{p} \in (\mathrm{mSupp}_R\, M/(x_1, ..., x_t)M)_{>k}$, $\mathrm{ht}_M(\mathfrak{p}) \leq t$ by the Generalized Krull's Principal Ideal Theorem with respect to modules, Theorem 2.8. So that $t \leq \mathrm{depth}(M_\mathfrak{p}) \leq \dim M_\mathfrak{p} \leq t$. Therefore $M_\mathfrak{p}$ is Cohen-Macaulay $R_\mathfrak{p}$-module of dimension t. □

Lemma 3.12 leads to an important result of this article. As an application, we show that $\mathrm{ara}_M(I) \leq \dim M + 1$, which is a refinement of the inequality $\mathrm{ara}_R(I) \leq \dim R + 1$ for modules, due to Kronecker and Forster [11], [7] (also [12], [13]).

Theorem 3.13. *Let I be an ideal of R and M be a non-zero finitely generated R-module. Then $\mathrm{ara}_M(I) \leq \dim M + 1$.*

Proof. If $M = IM$, then there is nothing to prove because $\mathrm{ara}_M(I) = 1$. So, we may assume that $IM \neq M$ and $\dim M = d$ is finite. If $\mathrm{ara}_M(I) = 0$, the assertion is obvious. Now, let $\mathrm{ara}_M(I) = n \geq 1$. Suppose that $\dim M/IM = k$ and $\mathrm{ara}_M(I) > d + 1$. By Theorem 3.3, there exists a poor k-regular M-sequence $x_1, ..., x_n \in I$ such that $\mathrm{Rad}_M(I) = \mathrm{Rad}_M(x_1, ..., x_n)$. Also, since $\mathrm{ara}_M(I) > d + 1$, by Lemma 3.12, there exists $\mathfrak{p} \in m(\mathrm{Ass}_R\, M/(x_1, ..., x_{d+1})M)_{>k}$ such that $M_\mathfrak{p}$ is a Cohen-Macaulay $R_\mathfrak{p}$-module of dimension $d + 1$, which is a contradiction. Therefore $\mathrm{ara}_M(I) \leq d + 1$. □

Lemma 3.14. *Let I be an ideal of R and M be a finitely generated R-module. If $\mathfrak{p} \in \mathrm{Supp}_R(M/IM)$ be such that $\mathrm{ht}_M(I) = \mathrm{ht}_M(\mathfrak{p})$, then $\mathfrak{p}$ is a minimal prime ideal of I in $\mathrm{Supp}_R(M)$.*

Lemma 3.15. *Let M be a finitely generated R-module and $\mathfrak{p} \in \operatorname{Supp}_R(M)$ with $\operatorname{ht}_M(\mathfrak{p}) = n$. Then there exists an ideal $I \subseteq \mathfrak{p}$ of R which can be generated by n elements such that $\operatorname{ht}_M(I) = n$.*

Proof. Use induction on $n \geq 0$ and apply Lemma 3.14 and Theorem 2.8. □

Lemma 3.16. *Let $(R, \mathfrak{m})$ be a local ring and M be a non-zero finitely generated R-module. Then $\dim M = \operatorname{ara}_M(\mathfrak{m})$.*

Proof. Apply Theorem 2.8 and Lemmas 3.14 and 3.15. □

The following result is one of the main results of this paper.

Corollary 3.17. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module and I be an ideal of R for which $IM \neq M$. Then $\operatorname{ara}_M(I) \leq \dim M$.*

Proof. Put $d := \dim M$. If $d = 0$, then by Lemma 3.16, $\operatorname{ara}_M(\mathfrak{m}) = 0$. Thus $\operatorname{Rad}_M(0) = \mathfrak{m}$, and so $I \subseteq \operatorname{Rad}_M(0)$. Hence $\operatorname{Rad}_M(I) = \operatorname{Rad}_M(0)$ and hence $\operatorname{ara}_M(I) = 0$.
Now, let $d > 0$ and $\dim M/IM = k$. Suppose that $\operatorname{ara}_M(I) > d$. Thus, by Theorem 3.3, there exists a poor k-regular M-sequence $x_1, ..., x_t \in I$, such that $\operatorname{Rad}_M(I) = \operatorname{Rad}_M(x_1, ..., x_t)$ where $t = \operatorname{ara}_M(I)$. By Theorem 3.13, $d < \operatorname{ara}_M(I) \leq d + 1$. Thus $\operatorname{ara}_M(I) = d + 1$. This follows that $\operatorname{ara}_M(I) \geq 2$, and so by Lemma 3.12, there exists a poor k-regular M-sequence $y_1, ..., y_{d+1} \in I$ such that $\operatorname{Rad}_M(I) = \operatorname{Rad}_M(y_1, ..., y_{d+1})$ and for all $\mathfrak{p} \in \operatorname{mAss}_R(M/(y_1, ..., y_{d+1})M))_{>k}$, $M_\mathfrak{p}$ is a Cohen-Macaulay $R_\mathfrak{p}$-module of dimension $d + 1$, which is a contradiction. □

Corollary 3.18. *Let $(R, \mathfrak{m})$ be a local ring and I be an ideal of R. Let M be a finitely generated R-module of dimension d with $\dim M/IM = 1$. Suppose that $\operatorname{ht}_M(\mathfrak{p}) < d - 1$ for all minimal prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Then $\operatorname{ara}_M(I) = d$.*

Proof. First, we claim that $\operatorname{Supp}_R(H_I^{d-1}(M)) \subseteq \{\mathfrak{m}\}$. Let $\mathfrak{q}$ be a prime ideal in $\operatorname{Supp}_R(H_I^{d-1}(M))$. Then $\dim M_\mathfrak{q} \geq d - 1$. As $\operatorname{ht}_M(\mathfrak{q}) \geq d - 1$, $\mathfrak{q}$ is not a minimal prime ideal of $\operatorname{Supp}_R(M/IM)$. Hence, $\mathfrak{q} = \mathfrak{m}$ as required. Since $I \subset \mathfrak{m}$, there exists $x \in \mathfrak{m}$ such that $x \notin \cup_{\mathfrak{p} \in \operatorname{mAss}_R(M/IM)} \mathfrak{p}$. Hence $\operatorname{Rad}_M(I + Rx) = \mathfrak{m}$ and $H^i_{I+Rx}(M) \cong H^i_\mathfrak{m}(M)$ for all $i \in \mathbb{N}$, by Theorem 2.10. Using [5, Theorem 8.1.2], we have the following exact sequence

$$... \to H_I^{d-1}(M) \to H_I^{d-1}(M_x) \to H^d_{I+Rx}(M) \to H^d_I(M) \to H^d_I(M_x) \to ... .$$

Since $\dim M_x < d$, $H_I^d(M_x) = 0$. Now, considering [5, Theorem 4.2], we show that $(H_I^{d-1}(M))_x = 0$. On the contrary, let $\mathfrak{q} \in \mathrm{Supp}_R(H_I^{d-1}(M))$ be such that $\mathfrak{q} \cap \{x^i \mid i \in \mathbb{N}_0\} = \emptyset$. Thus $\mathrm{Supp}_R(H_I^{d-1}(M)) = \{\mathfrak{m}\}$, and so $\mathfrak{q} = \mathfrak{m}$. This follows that $\mathfrak{m} \cap \{x^i \mid i \in \mathbb{N}_0\} = \emptyset$, which is a contradiction. Therefore $H_I^{d-1}(M_x) = 0$. Now, by the above exact sequence, we have $H_I^d(M) \neq 0$. Hence $d = \mathrm{cd}(I, M) \leq \mathrm{ara}_M(I)$ by Corollary 3.4. Now, the assertion follows from Corollary 3.17. □

Here, we are in position to prove the last result of this paper (Theorem 3.22). In order to prove it, we need Proposition 3.21, which shows that $\dim M - \dim M/IM \leq \mathrm{cd}(I, M)$. To this end, we need the following lemmas.

Lemma 3.19. *Let $(R, \mathfrak{m})$ be a local ring, I be an ideal of R, and M be a finitely generated R-module such that $IM \neq M$. Then $\dim M - \dim M/IM \leq \mathrm{ara}_M(I)$.*

Proof. Let $\mathrm{ara}_M(I) = n$ and $x_1, ..., x_n \in I$ be such that $\mathrm{Rad}_M(x_1, ..., x_n) = \mathrm{Rad}_M(I)$. Then $\dim M - n \leq \dim M/(x_1, ..., x_n)M = \dim M/IM$, as required. □

Lemma 3.20. *Let $(R, \mathfrak{m})$ be a local ring, I be an ideal of R, and M be a finitely generated R-module. Assume that $k \geqslant -1$ is an arbitrary integer such that $\dim M > k$. Then any poor k-regular M-sequence in I is a part of a system of parameters for M.*

Proof. Let $x_1, ..., x_r \in I$ be a poor k-regular M-sequence. It is enough to prove the assertion for $r = 1$. Hence, we assume that $x \in I$ is a poor k-regular M-sequence. By the definition $(\mathrm{Ass}_R(M) \cap V(x))_{>k} = \emptyset$. Assume that $\dim M/xM = \dim M = d$. So that, there exists a chain in $\mathrm{Supp}_R(M/xM)$ as $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset ... \subset \mathfrak{p}_d$, where $\mathfrak{p}_0$ is a minimal element of $\mathrm{Supp}_R(M)$, and so $\mathfrak{p}_0 \in \mathrm{Ass}_R(M)$. Since $x \in \mathfrak{p}_0$, $\dim R/\mathfrak{p}_0 \leq k$. Hence $\dim M = d \leq \dim R/\mathfrak{p}_0 \leq k$, which is a contradiction. Therefore $\dim M/xM = \dim M - 1$, as required. □

Proposition 3.21. *Let $(R, \mathfrak{m})$ be a local ring, I be an ideal of R, and M be a finitely generated R-module such that $IM \neq M$. Then $\dim M - \dim M/IM \leq \mathrm{cd}(I, M)$.*

Proof. Put $d = \dim M$, $t = \dim M - \dim M/IM$, and $r = \mathrm{ara}_M(I)$. Let $x'_1, ..., x'_d \in \mathfrak{m}$ be a system of parameters for M. Then $\mathfrak{m}$ is the only minimal prime ideal of $(x'_1, ..., x'_d)$ in $\mathrm{Supp}_R(M)$ and so $\mathrm{Rad}_{M/IM}(x'_1 + I, ..., x'_d + I) = \mathfrak{m}/I$. As $\dim M/IM = d - t$, we may choose $x_1, ..., x_t \in I \cap \{x'_1, ..., x'_d\}$ and $y_1, ..., y_{d-t} \in \{x'_1, ..., x'_d\} \setminus I$. Thus, $\mathrm{Rad}_{M/IM}(y_1 + I, ..., y_{d-t} + I) = \mathfrak{m}/I$. As $\mathfrak{m} = \mathrm{Rad}_M((x_1, ..., x_t, y_1, ..., y_{d-t})) \subseteq \mathrm{Rad}_M(I+$

$(y_1, ..., y_{d-t}))$, we get $\mathfrak{m} = \mathrm{Rad}_M(I + (y_1, ..., y_{d-t}))$. Since $y_1, ..., y_{d-t}$ is a part of system of parameters for M, $\dim M/(y_1, ..., y_{d-t})M = \dim M - (d-t) = t$. Now, by the Non-vanishing Theorem [5, Theorem 6.1.4], the Independence Theorem [5, Theorem 4.2.1], and Theorem 2.10, we get the following

$$\begin{aligned} 0 &\neq H^t_{\mathfrak{m}}(M/(y_1, ..., y_{d-t})M) \\ &\cong H^t_{\mathfrak{m}/(y_1,...,y_{d-t})}(M/(y_1, ..., y_{d-t})M) \\ &\cong H^t_{I+(y_1,...,y_{d-t})/(y_1,...,y_{d-t})}(M/(y_1, ..., y_{d-t})M) \\ &= H^t_{I(R/(y_1,...,y_{d-t}))}(M/(y_1, ..., y_{d-t})M \\ &\cong H^t_I(M/(y_1, ..., y_{d-t})M). \end{aligned}$$

Hence $r \leq \mathrm{cd}(I, M/(y_1, ..., y_{d-t})M)$. But $\mathrm{cd}(I, M/(y_1, ..., y_{d-t})M) \leq \mathrm{cd}(I, M)$, and so $t \leq \mathrm{cd}(I, M)$. □

It is clear that on any Noetherian ring R, for an ideal I of R and R-module M, we have $\mathrm{cd}(I, M) \leq \mathrm{ara}_M(I)$ (Corollary 3.4). In the following theorem, we get useful inequalities.

Theorem 3.22. *Let $(R, \mathfrak{m})$ be a local ring, I be an ideal of R, and M be a finitely generated R-module such that $IM \neq M$. Then $\dim M - \dim M/IM \leq \mathrm{cd}(I, M) \leq \mathrm{ara}_M(I) \leq \dim M$.*

Proof. The assertion follows from Corollary 3.17 and Proposition 3.21. □

Theorem 3.23. *Let $(R, \mathfrak{m})$ be a local ring, M be finitely generated R-module of dimension $d \geq 2$, and I be an ideal of R. Assume that $\mathrm{ara}_M(\mathfrak{p}) < d$ for every $\mathfrak{p} \in \mathrm{Supp}_R(M)$ such that $\mathrm{ht}_M(\mathfrak{p}) = d-1$. Then $\mathrm{ara}_M(\mathfrak{q}) < d$ for every $\mathfrak{q} \in \mathrm{Ass}_R(M)$ such that $\dim M/qM = d$.*

Proof. Suppose contrary to our claim that there exists $\mathfrak{q} \in \mathrm{Ass}_R(M)$ such that $\dim M/\mathfrak{q}M = d$ and $\mathrm{ara}_M(\mathfrak{q}) \geq d$. By Corollary 3.17, $\mathrm{ara}_M(\mathfrak{q}) \leq d$ so that $\mathrm{ara}_M(\mathfrak{q}) = d$. By Lemma 3.12, there exists a poor d-regular M-sequence $y_1, ..., y_d \in \mathfrak{q}$ such that $\mathrm{Rad}_M(\mathfrak{q}) = \mathrm{Rad}_M(y_1, ..., y_d)$ and there exists a prime ideal $\mathfrak{p}$ in $m(\mathrm{Ass}_R(M/(y_1, ..., y_{d-1})M))_{>d}$ such that $\mathrm{ht}_M(\mathfrak{p}) = d-1$. Hence by assumption, we have $\mathrm{ara}_M(\mathfrak{p}) < d$. Since $\dim M/\mathfrak{q}M = d$, $\mathfrak{q} \nsubseteq \mathfrak{p}$. Now, we claim that $\mathrm{Rad}_M(\mathfrak{p} + \mathfrak{q}) = \mathfrak{m}$. Let $Q \in \mathrm{Supp}_R(M/(\mathfrak{p} + \mathfrak{q})M)$ be such that $Q \subsetneq \mathfrak{m}$. Since $\mathrm{ht}_M(\mathfrak{p}) = d-1$, there is a chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset ... \subset \mathfrak{p}_{d-2} \subset \mathfrak{p}_{d-1} = \mathfrak{p},$$

where $\mathfrak{p}_i \in \mathrm{Supp}_R(M)$ for all $0 \leq i \leq d-1$. Since $\mathfrak{q} \nsubseteq \mathfrak{p}$, $\mathfrak{p} \subsetneq \mathfrak{p}+\mathfrak{q} \subseteq Q \subsetneq \mathfrak{m}$. So that, we get a chain of length of $d+1$ in $\mathrm{Supp}_R(M)$ which is a contradiction. Therefore by [5, Theorem 4.2.1], Theorem 2.10, and the Non-Vanishing Theorem, we get

$$H^d_{\mathfrak{p}}(M/\mathfrak{q}M) \cong H^d_{\mathfrak{p}+\mathfrak{q}}(M/\mathfrak{q}M) \cong H^d_{\mathfrak{m}}(M/\mathfrak{q}M) \neq 0.$$

But $H^d_{\mathfrak{p}}(M/\mathfrak{q}M) \cong H^d_{\mathfrak{p}}(M) \otimes_R R/\mathfrak{q}$, by [5, Exercise 6.1.10]. Thus $H^d_{\mathfrak{p}}(M) \neq 0$. Therefore $d = \mathrm{cd}(\mathfrak{p}, M) \leq \mathrm{ara}_M(\mathfrak{p}) < d$, which is a contradiction. □

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