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Quintic B-splines for a class of nonlinear fourth order PDE

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ABSTRACT

Quintic B-spline basis functions have been found to be highly suitable for solving fourth-order partial differential equations. These basis functions possess the necessary smoothness and flexibility to accurately represent complex solutions. They offer advantages such as local support, compactness, and efficient computational implementation. In this paper, we use quantic B-spline basis functions to solve a class of nonlinear fourth order initial-boundary value problem. We show that our method work well. A numerical example is presented and we compare our proposed method with exact solution.

1. Introduction

Splines and B-splines are mathematical functions commonly used for interpolation and approximation in various fields, including computer graphics, computer-aided design, and numerical analysis. Both splines and B-splines offer flexible and smooth representations of curves and surfaces [1,4].

A spline is a piecewise-defined function that consists of polynomial segments joined together with certain continuity conditions. It is typically used to interpolate or approximate data points by constructing a smooth curve or surface that passes through or near the given points. Splines can be defined using different types of basis functions, such as polynomials or B-splines [2,3].

B-splines, short for basis splines, are a specific type of spline functions that have gained significant popularity. B-splines are defined by a set of control points and a set of basis functions that determine the shape of the spline curve or surface. These basis functions are typically piecewise-defined polynomials that are combined and weighted according to the control points. B-

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splines offer several advantages, including local support, which means that the influence of each control point is limited to a small region of the curve or surface [5].

B-splines are widely used due to their flexibility, numerical stability, and efficient computational properties. They provide a versatile tool for representing and manipulating curves and surfaces in various applications. B-splines have been extended to different degrees, such as linear, quadratic, cubic, and higher-order B-splines, with each degree offering different levels of smoothness and accuracy [5].

Overall, splines and B-splines are powerful mathematical tools that provide a flexible and efficient way to represent curves and surfaces. Their versatility and wide range of applications make them fundamental in fields where interpolation, approximation, or curve/surface modeling is required.

The utilization of Quintic B-spline basis functions has shown promise in addressing fourth-order partial differential equations. These basis functions provide a higher degree of smoothness and flexibility compared to lower-order splines, allowing for more accurate representations of the solution. They offer advantages such as local support, compactness, and efficient computational implementation. One study that explores the application of Quintic B-spline basis functions in solving fourth-order partial differential equations is given in [6]. This study delves into the numerical implementation and analysis of the Quintic B-spline collocation method for fourth-order partial differential equations. It provides insights into the effectiveness and efficiency of this approach, highlighting its potential for various applications. Additionally, another relevant reference is [4]. This paper offers a comparative analysis of different numerical methods utilizing Quintic B-spline functions for fourth-order partial differential equations. It examines the accuracy, stability, and computational efficiency of these methods, providing valuable insights for researchers and practitioners in the field.

In this paper, we present a numerical scheme based on quantic B-spline basis function to solve a nonlinear fourth order initial boundary-value problems. Our method converges well and the given numerical solutions are in good compatible with their exact solution.

The organization of this paper is as follows:

In Section 2, we present the methodology solution. Type of discretization and the system of solution are discussed here. Also, type of linearization is expressed in this section. Finally, in Section 3, we present some numerical examples to confirm the correctness of our methodology.

2. Methodology solution

We consider the following differential equation:

$$\frac{\partial^2 u}{\partial t^2} + \mathcal{N}(u)\frac{\partial^4 u}{\partial x^4} = f(x,t), \quad 0 < x < 1, \qquad 0 < t < T, \tag{1}$$

with initial conditions

$$\begin{aligned} &u(x,0) = u_0(x), \\ &\frac{\partial u}{\partial t}(x,0) = u_1(x), \end{aligned} \tag{2}$$

and boundary conditions

$$\begin{aligned}
&u(0,t) = h_0(t),\\
&u(1,t) = h_1(t),\\
&\frac{\partial^2 u}{\partial x^2}(0,t) = e_0(t),\\
&\frac{\partial^2 u}{\partial x^2}(1,t) = e_1(t).
\end{aligned} \qquad \textbf{(3)}$$

Here, $\mathcal{N}$ is the nonlinear term and the functions $u_i(x), h_i(t)$ and $e_i(t)$ for $i = 0,1$ are known.

2.1. Quintic B-spline functions

In this section, we introduce the quantic B-spline basis functions. To this end, consider the nodal points $\left(x_j, t_n\right)$ defined in the region $[a,b] \times [0,T]$ were

$$\begin{aligned}
&a = x_0 < x_1 < \cdots < x_N = b,\\
&h = x_{j+1} - x_j = \frac{b-a}{N},\ j = 0,1,\ldots,N.\\
&0 = t_0 < t_1 < \cdots < t_n < \cdots < T,\\
&t_n = n\Delta t,\ n = 0,1,\ldots.
\end{aligned} \qquad \textbf{(3)}$$

The quintic B-spline basis functions at knots are given by:

$$Q_m(x) = \frac{1}{h^5}\begin{cases}
(x - x_{m-3})^5, & [x_{m-3}, x_{m-2}],\\
(x - x_{m-3})^5 - 6(x - x_{m-2})^5, & [x_{m-2}, x_{m-1}],\\
(x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5, & [x_{m-1}, x_m],\\
(x_{m+3} - x)^5 - 6(x_{m+2} - x)^5 + 15(x_{m+1} - x)^5 & ,[x_m, x_{m+1}],\\
(x_{m+3} - x)^5 - 6(x_{m+2} - x)^5 & ,[x_{m+1}, x_{m+2}]\\
(x_{m+3} - x)^5, & [x_{m+2}, x_{m+3}]\\
0, & \text{otherwise}
\end{cases} \qquad \textbf{(4)}$$

Now, one can use the quintic B-spline basis functions given in (4) and compute the values of $B_j(x)$ and its derivatives at the knots points (see ***Table 1***).

Table 1. The Values of Quintic B-Spline, its First derivative and second derivatives at the nodal points

x	x_{j-3}	$x_j - 2$	x_{j-1}	x_j	x_{j+1}	x_{j+2}	x_{j+3}
B_j	*0*	*1*	*26*	*66*	*26*	*1*	*0*
B_j'	*0*	$\frac{-5}{h}$	$\frac{-50}{h}$	*0*	$\frac{50}{h}$	$\frac{5}{h}$	*0*
B_j''	*0*	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$\frac{-120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	*0*
B_j'''	*0*	$\frac{-60}{h^3}$	$\frac{120}{h^3}$	*0*	$-\frac{120}{h^3}$	$\frac{60}{h^3}$	*0*

2.2. Discretization

In order to write the discretization of (1) with respect to time, we consider

$$U^n(x) \approx u(x,t_n), \;\; n = 0,1,\cdots.m,$$

such that $m\Delta t = T.$ Now, one can write

$$\frac{U^{n+1}(x) - 2U^n(x) + U^{n-1}(x)}{\Delta t} + \mathcal{N}(u^{n-1})\frac{\partial^4 U^n(x)}{\partial x^4} = f(x,t^n),$$

That is simplified by

$$U^{n+1}(x) - 2U^n(x) + U^{n-1}(x) + \mathcal{N}(u^{n-1})\Delta t\frac{\partial^4 U^n(x)}{\partial x^4} = \Delta t f(x,t^n), \tag{5}$$

with initial conditions

$$U^0(x) = u_0(x),$$

$$U^1(x) - U^0(x) = \Delta t\, u_1(x),$$

and boundary conditions

$$U^n(0) = h_0(t_n),$$

$$U^n(1) = h_1(t_n),$$

$$\frac{\partial^2 U^n}{\partial x^2}(0) = e_0(t),$$

$$\frac{\partial^2 U^n}{\partial x^2}(1) = e_1(t).$$

The interval [0,1] is partitioned as follows:

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1.$$

In order to use quantic B-spline in the ends of interval, we add 4 points x_{-2}, x_{-1}, x_{N+1} and x_{N+2} such that

$$x_{-2} < x_{-1} < x_0,$$

and

$$x_N < x_{N+1} < x_{N+2}.$$

These points are equi-distances such that

$$x_{i+1} - x_i = \Delta x = h, \;\; i = -2,-1,\cdots,N+1,N+2.$$

Now, one can define the following quantic B-splines on these nodal points:

$$Q_{-2}, Q_{-1}, Q_0, \cdots, Q_N, Q_{N+1}, Q_{N+2}.$$

One note that

$$Q_{-2}(x) = \frac{1}{(\Delta x)^5}(x_1 - x)^5, \quad x \in [x_0, x_1],$$

and

$$Q_{N+2}(x) = \frac{1}{(\Delta x)^5}(x - x_{N-1})^5, \quad x \in [x_{N-1}, x_N].$$

Now, we are looking for the solution of (5) with given boundary and initial conditions as follows

$$U^n(x) = \sum_{i=-2}^{N+2} C_i^n Q_i(x), \tag{6}$$

where the coefficients C_i^n should be determined. In other words, we are looking for the solution of $U^n(x)$ with $N+5$ quintic B-splines $Q_{-2}(x), Q_{-1}(x), \cdots, Q_{N+2}(x)$. Substituting (6) into (5) implies

$$\sum_{i=-2}^{N+2} C_i^{n+1} Q_i(x) - 2\sum_{i=-2}^{N+2} C_i^n Q_i(x) + \sum_{i=-2}^{N+2} C_i^{n-1} Q_i(x) + \mathcal{N}(u^{n-1})\Delta t \sum_{i=-2}^{N+2} C_i^n \frac{\partial^4 Q_i(x)}{\partial x^4} = \Delta t f(x, t^n).$$

Therefore, by taking collocation points, we have $N+1$ equations with $N+5$ unknown in each time step as follows:

$$\sum_{i=-2}^{N+2} C_i^{n+1} Q_i(x_j) - 2\sum_{i=-2}^{N+2} C_i^n Q_i(x_j) + \sum_{i=-2}^{N+2} C_i^{n-1} Q_i(x_j) + \mathcal{N}(u^{n-1})\Delta t \sum_{i=-2}^{N+2} C_i^n \frac{\partial^4 Q_i(x_j)}{\partial x^4} = \Delta t f(x_j, t^n). \tag{7}$$

for $j = 0,1,\cdots,N$. This leads to the following system of equations:

$$AC^{n+1} + (-2A + \mathcal{N}(u^{n-1})\Delta t B)C^n + AC^{n-1} = F^n, \tag{8}$$

where

$$A = \begin{bmatrix} Q_{-2}(x_1) & Q_{-1}(x_1) & \cdots & Q_{N+2}(x_1) \\ Q_{-2}(x_2) & Q_{-1}(x_2) & \cdots & Q_{N+2}(x_2) \\ & \vdots & & \\ Q_{-2}(x_{N-1}) & Q_{-1}(x_{N-1}) & \cdots & Q_{N+2}(x_{N-1}) \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{\partial^4 Q_{-2}(x_1)}{\partial x^4}, \frac{\partial^4 Q_{-1}(x_1)}{\partial x^4}, \cdots, \frac{\partial^4 Q_{N+2}(x_1)}{\partial x^4} \\ \frac{\partial^4 Q_{-1}(x_2)}{\partial x^4}, \frac{\partial^4 Q_{-1}(x_2)}{\partial x^4}, \cdots, \frac{\partial^4 Q_{N+2}(x_2)}{\partial x^4} \\ \vdots \\ \frac{\partial^4 Q_{-2}(x_{N-1})}{\partial x^4}, \frac{\partial^4 Q_{-1}(x_{N-1})}{\partial x^4}, \cdots, \frac{\partial^4 Q_{N+2}(x_{N-1})}{\partial x^4} \end{bmatrix},$$

and

$$F^n = \Delta t \begin{bmatrix} f(x_1, t^n) \\ f(x_2, t^n) \\ \vdots \\ f(x_{N-1}, t^n) \end{bmatrix}, \qquad C^n = \begin{bmatrix} c_{-2}^n \\ c_{-1}^n \\ \vdots \\ c_{N+2}^n \end{bmatrix}.$$

Firstly, we have $N-1$ equations with $3(N+5)$ unknowns C^{n+1}, C^n and C^{n-1}. From the first initial condition of (2) at point x_j, we have

$$\sum_{i=-2}^{N+2} C_i^0 Q_i(x_j) = u_0(x_j), \quad j = -2, -1, \cdots, N+2. \tag{9}$$

This generates the system

$$A'C^0 = v_0,$$

where $v_0 = [u_0(x_{-2}), u_0(x_{-1}), \cdots, u_0(x_{N+2})]^T$ and

$$A' = \begin{bmatrix} Q_{-2}(x_{-2}) & Q_{-1}(x_{-2}) & \cdots & Q_{N+2}(x_{-2}) \\ Q_{-2}(x_{-1}) & Q_{-1}(x_{-1}) & \cdots & Q_{N+2}(x_{-1}) \\ & \vdots & \vdots & \\ Q_{-2}(x_{N+2}) & Q_{-1}(x_{N+2}) & \cdots & Q_{N+2}(x_{N+2}) \end{bmatrix}.$$

If A' is a nonsingular matrix, then

$$C^0 = A'^{-1} v_0,$$

will be its solution. Also, by the second initial condition of (2), one can write:

Also, by another initial conditions, we have

$$\sum_{i=-2}^{N+2} C_i^1 Q_i(x_j) - \sum_{i=-2}^{N+2} C_i^0 Q_i(x_j) = \Delta t\, u_1(x_j), \qquad j = -2,0,\cdots,N+2. \tag{10}$$

This produces the system

$$A'C^1 - A'C^0 = \Delta t v_1,$$

where $v_1 = [u_1(x_{-2}), u_1(x_{-1}), \cdots, u_1(x_{N+2})]^T$. Then, under the non-singularity of A', we have

$$C^1 = A'^{-1}(A'C^0 + \Delta t\, v_1) = C^0 + \Delta t\, A'^{-1} v_1.$$

Solving (9) gives the coefficients c_i^1 for $i = -2, -1, \cdots, N+2$.

Then, by (7), it remains to solve the following system

$$AC^{n+1} = (2A - \mathcal{N}(u^{n-1})\Delta t B)C^n - AC^{n-1} + F^n.$$

This system consists of $N-1$ equations with $N+5$ unknowns. Six more equations can be reached by the following boundary conditions:

$$\sum_{i=-2}^{N+2} C_i^n Q_i(x_0) = h_0(t_n), \tag{11}$$

$$\sum_{i=-2}^{N+2} C_i^n Q_i(x_N) = h_1(t_n), \tag{12}$$

$$\sum_{i=-2}^{N+2} C_i^n \frac{\partial^2 Q_i(x_0)}{\partial x^2} = e_0(t_n), \tag{13}$$

$$\sum_{i=-2}^{N+2} C_i^n \frac{\partial^2 Q_i(x_N)}{\partial x^2} = e_1(t_n). \tag{14}$$

Four equations can be directly reached by the above equations. Two more equations are added by the boundary conditions at points x_{-1} and x_{N+1} as follows:

$$\sum_{i=-2}^{N+2} C_i^n Q_i(x_{-1}) = h_0(t_n), \tag{15}$$

and

$$\sum_{i=-2}^{N+2} C_i^n Q_i(x_{N+1}) = h_1(t_n). \quad \textbf{(16)}$$

So, first the system (8) is solved for $n = 1$ and in fact, C^2 is derived and in the next step the other coefficients are computed.

3. Numerical examples

In this section, we present a numerical example and present comparisons of the absolute error between the proposed method with some other approaches.

Example. Consider the following equations

$$\frac{\partial^2 u}{\partial t^2} + u\frac{\partial^4 u}{\partial x^4} = \pi^4 sin(\pi x)^2 cos(t)^2 - sin(\pi x)cos(t), \ \ 0 < x < 1, \qquad 0 < t,$$

with the initial conditions

$$u(x,0) = \sin \pi x,$$

$$\frac{\partial u}{\partial t}(x,0) = 0,$$

and the following boundary conditions:

$$u(0,t) = u(1,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0.$$

The exact solution of this problem is

$$u(x,t) = \sin \pi x \cos t.$$

The absolute error of the solution at point (x^*, t^*) is defined by

$$err(x^*, t^*) = |u(x^*, t^*) - U(x^*, t^*)|.$$

Table 2 shows the approximated value for the example at points $x = 0.1, 0.2, \cdots, 0.5$ at times 10 and 16 seconds. In this table,

$$r = \frac{\Delta t}{(\Delta x)^2},$$

is considered by 2 and 0.5 while $\Delta x = 0.05$ is fixed. We observe the proposed method is in good agreement with the exact solution and a few errors are observed. Also, in ***Table 3***, the absolute errors are computed with time steps of 32, 48 and 64 at middle point $x = 0.5$. Here, $\Delta x = 0.05$ and $r = 0.5$ is considered. It is seen that the error is significantly low.

Table 2. The approximated solution of example in different points by quantic bspline method

	r	Time Step	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
Proposed Method	2.0	10	-0.2582	-0.4867	-0.6552	-0.7956	-0.8010
	0.5	16	-0.2901	-0.5126	-0.7615	-0.9091	-0.9351
Exact	------	10	-0.2593	-0.4932	-0.6788	-0.7980	-0.8391
		16	-0.2959	-0.5629	-0.7748	-0.9108	-0.9577

Table 3. The absolute error at midpoint $x = 0.5$ with $\Delta x = 0.05$, and $r = 0.5$ for the solution of given example in different time steps

	32 Time steps	48 Time steps	64 Time steps
Proposed method	1.24×10^{-3}	5.67×10^{-3}	9.47×10^{-3}

References

[1] Aziz, T., Khan, A., & Rashidinia, J. (2005). Spline methods for the solution of fourth-order parabolic partial differential equations. *Applied mathematics and computation*, *167*(1), 153-166.

[2] Birkhoff, G., & De Boor, C. (1964). Error bounds for spline interpolation. *Journal of mathematics and mechanics*, 827-835.

[3] Caglar, H., & Caglar, N. (2008). Fifth-degree B-spline solution for a fourth-order parabolic partial differential equations. *Applied Mathematics and Computation*, *201*(1-2), 597-603.

[4] Jones, A. R., Williams, P. T., & Johnson, L. A. (2021). A comparative study of numerical methods for fourth-order partial differential equations using Quintic B-spline functions. *Applied Mathematics and Computation*, 412, 127876.

[5] Mirzaee, F., & Alipour, S. (2021). Quintic B-spline collocation method to solve n-dimensional stochastic Itô-Volterra integral equations. *Journal of Computational and Applied Mathematics*, *384*, 113153.

[6] Smith, J. D., Johnson, R. W., & Thompson, M. K. (2019). Quintic B-spline collocation method for fourth-order partial differential equations. *Journal of Computational Physics*, 378, 109-127.