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Stability analysis of fractional-order predator-prey model with anti-predator behaviour and prey refuge

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Abstract. This article investigates a fractional-order predator-prey model incorporating prey refuge and anti-predator behaviour on predator species. For our proposed model, we prove the existence, uniqueness, non-negativity and boundedness of solutions. Further, all biologically possible equilibrium points and their stability analysis for the proposed system are carried out with the linearization process. Moreover, by using an appropriate Lyapunov function, the global stability of the co-existence equilibrium point is studied. Finally, we provide numerical simulations to demonstrate how the theoretical approach is consistent.

Keywords: Caputo fractional derivative, prey refuge, anti-predator, stability analysis, Hopf bifurcation.
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1 Introduction

Among the range of interactions between various types of living species and non-living components of the environment, the predator-prey relationship plays a crucial role, which was first developed by Lotka and Volterra [16, 30]. The Lotka and Volterra model was expanded by adding a logistic growth for the prey species and a variety of functional responses. They are ratio-dependent functional response, Hassell-Varley, Holling I-IV functional response, Beddington-DeAngelis functional response, Crowley-Martin functional response and others. Over the past few decades, many authors have investigated these predator-prey models [1,3,6,12,14,18,27,28,33]. In the ecological system, increasing the prey's survival rate is required to maintain the environmental balance. So, an appropriate method for preserving the prey population is the prey refuge, which was proposed by Gause [7,8]. After introducing Gause work, various

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investigation done by several researchers. For example, Chen [2] discussed interaction of predator-prey model with refuge. Ghosh [9] explained the predator-prey relationship with additional food for predators and incorporating a prey refuge. In [11] Chakraborty proposed the bifurcation analysis of delayed predator-prey model with refuge on prey species [10, 15, 17, 22, 31]. In reality, prey adopts various strategies to reduce predation pressure during their interactions. To protect against predators, several animals use their spikes, claws and fangs, which is a common example of anti-predator behaviour. In [26] Tang and Xiao represented predator-prey model with anti-predator behaviour which takes the form

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \frac{\beta xy}{a+x^2}, \\ \frac{dy}{dt} &= -dy + \frac{\mu\beta xy}{a+x^2} - \eta xy, \end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ denotes densities of prey and predator respectively, r is the intrinsic growth rate, the environmental carrying capacity is denoted by k, β is the attack rate, μ is the conversion efficiency, d denotes predator's mortality rate, and η is the rate of anti-predator behaviour. Note that in this case, using an anti-predator has no impact on the number of prey, but it reduces the actual size of the predator population and also the author describes that the presence of anti-predator inhibits the predator density and raise the density of the prey population. Similar findings on the predatorprey model with anti-predator behaviour were obtained by [4, 23–25].

Based on the above discussions, we modify model (1) and assume that the predator-prey model with incorporating prey refuge which takes the following form:

$$\begin{aligned} \frac{du}{dt} &= \alpha u - \beta u^2 - \frac{(1-\theta)^2 u^2 v}{\gamma + (1-\theta)^2 u^2}, \\ \frac{dv}{dt} &= -\delta v + \frac{\eta(1-\theta)^2 u^2 v}{\gamma + (1-\theta)^2 u^2} - \mu uv. \end{aligned} \tag{2}$$

Here, $\theta \in [0,1)$, θu denotes the prey density at time t that is protected due to the refuge. Therefore, $(1-\theta)u$ of prey is available to the predator.

Fractional differential equations give additional benefits than usual integer-order since they may capture the complete time state of a biological process whereas integer-order can relate only a slight variation to a certain time of that process. Due to their widespread existence and applicability, the fractional theory has been proposed and extensively studied in a number of fields, including mechanics, finance, signal and image processing, electrical engineering, mechatronics, biology, physics, biophysics and control theory [5, 13, 19–21, 29, 32]. When comparing to the integer-order, fractional-order differential equations are more accurate in describing population dynamics and showing the relationships between prey and predator species. For the mathematical formulation of our system, we used the Caputo fractional derivative.

This paper consists of the following sections. In Section 2, we provide the construction of fractional-order model with anti-predator and incorporating prey refuge and also preliminary results. In Sections 3 and 4, we mainly examine the non-negativity and boundedness of solutions and existence and uniqueness of the solutions of the system, respectively. Section 5 provides the existence of equilibria of the system and local stability analysis of the equilibrium points. We examine Hopf bifurcation in Section 6. In

Section 7, we provide the sufficient condition for the global stability. Finally, Section 8 we analyze the importance of anti-predator and prey refuge and give a few numerical results using MATLAB software to verify our theoritical findings.

2 Fractional-order model

From (2), we consider the fractional-order predator-prey model with anti-predator behaviour and incorporating prey refuge as the following form:

$$\begin{aligned} {}^{c}D^{q}u(t) &= \alpha u - \beta u^2 - \frac{(1-\theta)^2u^2v}{\gamma+(1-\theta)^2u^2}, \\ {}^{c}D^{q}v(t) &= -\delta v + \frac{\eta(1-\theta)^2u^2v}{\gamma+(1-\theta)^2u^2} - \mu uv, \end{aligned} \tag{3}$$

with the non-negative initial values $u(0) = u_0$ and $v(0) = v_0$, where $q \in (0,1)$ and $\alpha, \beta, \theta, \gamma, \delta, \eta$ and μ are non-negative.

Table 1: Description of the parameters used in model (3).

Parameter	Description
$u(t)$	prey population
$v(t)$	predator population
α	intrinsic growth rate
$\frac{\alpha}{\beta}$	carrying capacity of the prey species
θ	prey refuge term
δ	death rate of the predator speies
η	conversion efficiency from predator
$\frac{(1-\theta)^2u^2v}{\gamma+(1-\theta)^2u^2}$	Holling type III response function
μ	rate of anti-predator behaviour

2.1 Preliminary results

Definition 1. *([19]). The Caputo differential operator for $q > 0$ is given by*

$$ {}^{c}D^{q}f(t) = \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^m(s)}{(t-s)^{1+q-m}} ds, $$

where $\Gamma(.)$ is the Gamma function and $m \in \mathbb{Z}^+$ such that $q \in (m-1,m)$. For $q \in (0,1)$

$$ {}^{c}D^{q}f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} ds. $$

Definition 2. *A point X^* is an equilibrium point of the following system*

$$ {}^{c}D^{q}X(t) = f(t, X(t)), \quad with \quad X(0) = X_0, \tag{4}$$

if and only if $f(t, X^*) = 0$.

Lemma 1. *([32]). Suppose that* $f(t) \in C[a,b]$ *and* ${}^cD^q f(t) \in C(a,b)$, $0 < q \leq 1$. *If* ${}^cD^q\, f(t) \geq 0$, $\forall$ $t \in (a,b)$, *then* $f(t)$ *is a nondecreasing function* $\forall\, t \in [a,b]$ *and if* ${}^cD^q f(t) \leq 0$, $\forall\, t \in (a,b)$, *then* $f(t)$ *is a nonincreasing function* $\forall\, t \in [a,b]$.

Lemma 2. *([13]). Let* $u(t)$ *be a continuous function on* $(0,T]$ *that satisfies* ${}^cD^q u(t) \leq -au(t) + b, \quad u(0) = u_0 > 0, \quad 0 < q < 1$, *where* a, $b \in \mathbb{R}^2$, $a \neq 0$. *Then*

$$u(t) \leq \left(u_0 - \frac{b}{a}\right) E_q[-at^q] + \frac{b}{a}.$$

Lemma 3. *([20]). Consider system (4), where* $f : \aleph \times (0,T] \to \mathbb{R}^n$, $\aleph \subset \mathbb{R}^n$. *If* $f(t,X)$ *follows the local Lipschitz condition, then there exists a unique solution of (4) on* $\aleph \times (0,T]$.

Theorem 1. *([19]). The equilibrium points of the system (4) are locally asymptotically stable if the eigenvalue* λ *of the Jacobian matrix* $j = \frac{\partial f}{\partial x}$ *which satisfy* $|arg(\lambda)| > \frac{q\pi}{2}$ *and it is unstable if the eigenvalue* λ *satisfy* $|arg(\lambda)| < \frac{q\pi}{2}$.

3 Non-negativity and boundedness of the solution

Here, we examine the uniform boundedness and non-negativity of the system's solution. Assume that $\Omega^+ = \{(u,v) \in \Omega : u, v \in \mathbb{R}^+\}$.

Theorem 2. *Every solutions of (3) initiates in* $\mathbb{R}^+$ *are non-negative and uniformly bounded.*

Proof. For any solution $u(t) \in \mathbb{R}^+$ we need to show that $u(t)$ is non-negative. Assume that $u_0 > 0$ and $v_0 > 0$ for $t = 0$. If $u(t) > 0$ is not true, then there exists $t_1 > 0$, such that $u(t) > 0$ for $0 \leq t < t_1$, $u(t) = 0$ for $t = t_1$ and $u(t) < 0$ for $t > t_1$. According to the first equation of (3), ${}^cD^q u(t)\Big|_{u(t_1)} = 0$.

By Lemma 1, $u(t_1^+) = 0$, which contradicts to our assumption $u(t_1^+) < 0$ that is $u(t) < 0$, $t > t_1$. Therefore, we obtain $u(t) \geq 0$, $\forall t \geq 0$. Similarly, we can prove $v(t) \geq 0$, $\forall\, t \geq 0$.

It is enough to show that the function $P(t) = u(t) + \frac{v(t)}{\eta}$ is bounded with non-negative initial conditions. Taking Caputo fractional derivative on both sides of $P(t)$, we have

$${}^cD^q P(t) = {}^cD^q u(t) + \frac{1}{\eta}{}^cD^q v(t) = \alpha u - \beta u^2 - \frac{(1-\theta)^2 u^2 v}{\gamma + (1-\theta)^2 u^2} - \frac{\delta}{\eta} v + \frac{(1-\theta)^2 u^2 v}{\gamma + (1-\theta)^2 u^2} - \frac{\mu}{\eta} uv,$$

which gives

$${}^cD^q P(t) = \alpha u - \beta u^2 - \frac{\delta}{\eta} v - \frac{\mu}{\eta} uv.$$

Now, for any positive number δ,

$$\begin{aligned} {}^{c}D^{q}P(t)+\delta P(t) &= \alpha u-\beta u^{2}-\frac{\delta}{\eta}v+\delta u+\frac{\delta}{\eta}v-\frac{\mu}{\eta}uv \\ &= -\beta\left[u^{2}-\frac{\alpha+\delta}{\beta}u\right] \\ &= -\beta\left[u-\frac{\alpha+\delta}{2\beta}\right]^{2}+\frac{(\alpha+\delta)^{2}}{4\beta} \\ &\leq \frac{(\alpha+\delta)^{2}}{4\beta}. \end{aligned}$$

By using Lemma 2, we have

$$P(t)\leq\left(P(0)-\frac{(\alpha+\delta)^{2}}{4\beta}\right)E_{\alpha}\left[-\delta t^{\alpha}\right]+\frac{(\alpha+\delta)^{2}}{4\beta}\rightarrow\frac{(\alpha+\delta)^{2}}{4\beta}\ as\ t\ \rightarrow\ \infty.$$

That is to say, every solutions of system (3) initiating in Ω^{+} remains in $\sigma=\{(u,v)\in\Omega^{+}:u+\frac{v}{\eta}\leq\frac{(\alpha+\delta)^{2}}{4\beta}+\varepsilon,\ \varepsilon>0\}$. □

4 Existence and uniqueness of the solution

In this section, we investigate the existence of the solution model (3) which is unique in $\aleph\times(0,T]$, where $\aleph$ = $\{(u,v)\in \mathrm{R}^{2}$: max $\{\,|u|,|v|\,\}\leq$M$\}$.

Theorem 3. *For each $\vartheta_0=(u_0,v_0)\in\aleph$, there exists a unique solution $\vartheta(t)\in\aleph$ of model (3) with initial condition ϑ_0, $\forall\, t>0$.*

Proof. The mapping

$$\chi(\vartheta)=(\chi_{1}(\vartheta),\chi_{2}(\vartheta)),$$

is considered, where

$$\begin{aligned} \chi_{1}(\vartheta) &= \alpha u-\beta u^{2}-\frac{(1-\theta)^{2}u^{2}v}{\gamma+(1-\theta)^{2}u^{2}}, \\ \chi_{2}(\vartheta) &= -\delta v+\frac{\eta(1-\theta)^{2}u^{2}v}{\gamma+(1-\theta)^{2}u^{2}}-\mu uv. \end{aligned}$$

For arbitrary $\vartheta,\overline{\vartheta}\in\aleph$, it follows that

$$
\begin{aligned}
&\|\chi(\vartheta)-\chi(\overline{\vartheta})\| \\
&=|\chi_1(\vartheta)-\chi_1(\overline{\vartheta})|+|\chi_2(\vartheta)-\chi_2(\overline{\vartheta})| \\
&=\left|\alpha u-\beta u^2-\frac{(1-\theta)^2u^2v}{\gamma+(1-\theta)^2u^2}-\alpha\overline{u}+\beta\overline{u}^2+\frac{(1-\theta)^2\overline{u}^2\overline{v}}{\gamma+(1-\theta)^2\overline{u}^2}\right| \\
&\quad+\left|-\delta v+\frac{\eta(1-\theta)^2u^2v}{\gamma+(1-\theta)^2u^2}-\mu uv+\delta\overline{v}-\frac{\eta(1-\theta)^2\overline{u}^2\overline{v}}{\gamma+(1-\theta)^2\overline{u}^2}+\mu\overline{uv}\right| \\
&\leq\left|\alpha(u-\overline{u})-\beta(u^2-\overline{u}^2)-\left[\frac{(1-\theta)^2u^2v(\gamma+(1-\theta)^2\overline{u}^2)-(1-\theta)^2\overline{u}^2\overline{v}\gamma-(1-\theta)^4u^2\overline{u}^2\overline{v}}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2\overline{u}^2)}\right]\right| \\
&\quad+\Bigg|\left[\frac{\eta(1-\theta)^2u^2v(\gamma+(1-\theta)^2\overline{u}^2)-\eta(1-\theta)^2\overline{u}^2\overline{v}(\gamma+(1-\theta)^2u^2)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2\overline{u}^2)}\right] \\
&\quad-\delta(v-v_1)-\mu(uv-\overline{uv})\Bigg| \\
&\leq\Bigg|\Bigg[\alpha(u-\overline{u})-2\beta M(u-\overline{u})-\frac{2\gamma M^2(1-\theta)^2(u-\overline{u})}{(\gamma+(1-\theta)^2M^2)^2}+\frac{2\eta\gamma M^2(1-\theta)^2(u-\overline{u})}{(\gamma+(1-\theta)^2M^2)^2} \\
&\quad-\mu M(u-\overline{u})\Bigg]\Bigg|+\Bigg|\Bigg[-\frac{\gamma M^2(1-\theta)^2(v-\overline{v})}{(\gamma+(1-\theta)^2M^2)^2}+\frac{\eta\gamma M^2(1-\theta)^2(v-\overline{v})}{(\gamma+(1-\theta)^2M^2)^2} \\
&\quad-\frac{M^4(1-\theta)^4(v-\overline{v})}{(\gamma+(1-\theta)^2M^2)^2}+\frac{\eta\gamma M^4(1-\theta)^4(v-\overline{v})}{(\gamma+(1-\theta)^2M^2)^2}-\delta(v-\overline{v})-\mu M(v-\overline{v})\Bigg]\Bigg| \\
&\quad=\left[\alpha+2\beta M+(1+\eta)2\gamma M^2(1-\theta)^2+\mu M\right]|u-\overline{u}| \\
&\quad+\left[(1+\eta)\gamma M^2(1-\theta)^2+(1+\eta)M^4(1-\theta)^4+\delta+\mu M\right]|v-\overline{v}| \\
&\leq\mathbb{M}\|\vartheta-\overline{\vartheta}\|,
\end{aligned}
$$

where M= max $\{m_1,m_2\}$, $m_1=\alpha+2\beta M+2\gamma M^2(1-\theta)^2(1+\eta)+\mu M$ and $m_2=\gamma M^2(1-\theta)^2(1+\eta)+M^4(1-\theta)^4(1+\eta)+\delta+\mu M$. Thus, $\chi(\vartheta)$ satisfies Lipschitz condition, it follows from Lemma 3 with initial condition $\vartheta_0=(u_0,v_0)$ has a unique solution $\vartheta(t)$. □

5 Local stability of equilibria

By solving the following equations, we can find the equilibrium points of model (3)

$$
\begin{cases}
{}^cD^qu(t)=0,\\
{}^cD^qv(t)=0,
\end{cases}
$$

that is,

$$\alpha u - \beta u^2 - \frac{(1-\theta)^2 u^2 v}{\gamma + (1-\theta)^2 u^2} = 0,$$
$$-\delta v + \frac{\eta (1-\theta)^2 u^2 v}{\gamma + (1-\theta)^2 u^2} - \mu u v = 0.$$

The equilibrium points are as follows

(i) $E_0(0,0)$,

(ii) $E_1\left(\frac{\alpha}{\beta}, 0\right)$,

(iii) $E^*\left(u^*, \frac{(\alpha-\beta u^*)(\gamma+(1-\theta)^2 u^*)}{(1-\theta u^*)}\right)$. Here, u^* is given by the following equation

$$\omega_1 u^{*^3} + \omega_2 u^{*^2} + \omega_3 u^* + \omega_4 = 0, \tag{5}$$

where $\omega_1 = \mu(1-\theta)^2$, $\omega_2 = (1-\theta)^2(\delta-\eta)$ with $\eta > \delta$, $\omega_3 = \mu\gamma$, $\omega_4 = \delta\gamma$. By using Descartes rule of signs, we have $\omega_1 > 0$, $\omega_2 < 0$, $\omega_3 > 0$ and $\omega_4 > 0$. So the maximum number of positive real roots is two. Additionally, v^* will be positive if $u^* < \frac{\alpha}{\beta}$ holds.

Next, we check the stability behaviour of all feasible equilibrium points by using the standard linearization method. First, we have derived the Jacobian matrix of the model (3)

$$J(u,v) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = \alpha - 2\beta u + \frac{2u^3 v (1-\theta)^4}{(\gamma + (1-\theta)^2 u^2)^2} - \frac{2uv(1-\theta)^2}{(\gamma + (1-\theta)^2 u^2)},$$
$$a_{12} = -\frac{u^2(1-\theta)^2}{(\gamma + (1-\theta)^2 u^2)},$$
$$a_{21} = -\mu v - \frac{2u^3 v \eta (1-\theta)^4}{(\gamma + (1-\theta)^2 u^2)^2} + \frac{2uv\eta(1-\theta)^2}{(\gamma + (1-\theta)^2 u^2)},$$
$$a_{22} = -\delta - \mu u + \frac{\eta u^2 (1-\theta)^2}{(\gamma + (1-\theta)^2 u^2)}.$$

At E_0 the Jacobian matrix is given by

$$J(E_0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\delta \end{pmatrix}.$$

Their corresponding eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = -\delta$, satisfying $|\arg(\lambda_1)| = 0 < \frac{q\pi}{2}$ and $|\arg(\lambda_2)| = \pi > \frac{q\pi}{2}$, where $0 < q < 1$. Therefore, E_0 is unstable by Theorem 1.
The Jacobian matrix at $E_1\big(K, 0\big)$, where $K = \frac{\alpha}{\beta}$ is as follows

$$J(E_0) = \begin{pmatrix} -\alpha & \frac{(1-\theta)^2K^2}{\gamma+(1-\theta)^2K^2} \\ & \\ 0 & -\delta - \mu K + \frac{\eta(1-\theta)^2K^2}{\gamma+(1-\theta)^2K^2} \end{pmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = -\alpha$ and $\lambda_2 = \frac{\eta(1-\theta)^2K^2}{\gamma+(1-\theta)^2K^2} - (\delta + \mu K)$. Now, $|\arg(\lambda_1)| = \pi > \frac{q\pi}{2}$ and $|\arg(\lambda_2)| = \pi > \frac{q\pi}{2}$ for $0 < q < 1$, otherwise $|\arg(\lambda_2)| = 0 < \frac{q\pi}{2}$ for $0 < q < 1$, which proves that by Theorem 1 $E_1\big(K, 0\big)$ is locally asymptotically stable, when $\frac{\eta(1-\theta)^2K^2}{\gamma+(1-\theta)^2K^2} < (\delta + \mu K)$ and otherwise unstable.

The Jacobian matrix for E^* is as follows:

$$J(E^*) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

where

$$\begin{aligned} b_{11} &= \alpha - 2\beta u^* + \frac{2(1-\theta)^4 u^{*^3} v^*}{(\gamma+(1-\theta)^2 u^{*2})^2} - \frac{2(1-\theta)^2 u^* v^*}{(\gamma+(1-\theta)^2 u^{*2})}, \\ b_{12} &= -\frac{(1-\theta)^2 u^{*^2}}{(\gamma+(1-\theta)^2 u^{*2})}, \\ b_{21} &= -\mu v^* - \frac{2\eta(1-\theta)^4 u^{*^3} v^*}{(\gamma+(1-\theta)^2 u^{*2})^2} + \frac{2\eta(1-\theta)^2 u^* v^*}{(\gamma+(1-\theta)^2 u^{*2})}, \\ b_{22} &= -\delta - \mu u^* + \frac{\eta(1-\theta)^2 u^{*^2}}{(\gamma+(1-\theta)^2 u^{*2})}. \end{aligned}$$

The characteristic equation of $J(E^*)$ is $\lambda^2 - \text{trace}(J(E^*))\lambda + \det(J(E^*)) = 0$, where $\text{trace}(J(E^*)) = b_{11} + b_{22} = T$ and $\det(J(E^*)) = b_{11}b_{22} - b_{12}b_{21} = D$. Therefore, the eigenvalues for the characteristic equation are

$$\lambda_1 = \frac{1}{2}\Big(T + \sqrt{T^2 - 4D}\Big), \qquad \lambda_2 = \frac{1}{2}\Big(T - \sqrt{T^2 - 4D}\Big).$$

Case (i) Assume $T \leq 0$ and $D > 0$. We have three sub-cases here.

Sub − case (i) If $T = 0$, then we obtain two complex conjugate eigenvalues with real part zero, verifies the condition $|\arg(\lambda_{1,2})| = \frac{\pi}{2}$. Therefore, the equilibrium point E^* is locally asymptotically stable by Theorem 1.

Sub − case (ii) If $T^2 \geq 4D$ and $T < 0$, then the eigenvalues are negative real values $\lambda_{1,2} < 0$. Since $|\arg(\lambda_{1,2})| = \pi > \frac{q\pi}{2}$ then equilibrium point E^* is locally asymptotically stable by Theorem 1.

Sub − case (iii) For $T^2 < 4D$ and $T < 0$, we have a complex conjugate eigenvalues $(\lambda_1 = \overline{\lambda_2})$. By the hypothesis $T < 0$, we have $Re(\lambda_1) = Re(\lambda_2) = T < 0$ and $|\arg(\lambda_{1,2})| > \frac{q\pi}{2}$. Thus, E^* is locally asymptotically stable if $T < 0$.

Case (ii) For $T > 0$, the eigenvalues λ_1 and λ_2 verify that $|\arg(\lambda_1)| = |\arg(\lambda_2)| = \tan^{-1}\frac{\sqrt{4D-T^2}}{T} > \frac{q\pi}{2}$, when $\sqrt{4D-T^2} > T\,\tan\frac{q\pi}{2}$. In this condition, we have a pair of complex conjugate eigenvalues λ_1, λ_2. The real and imaginary parts of eigenvalues have the following properties

$$Im(\lambda_1) = -Im(\lambda_2) = \frac{1}{2}\sqrt{4D-T^2} > 0 \quad \text{and} \quad Re(\lambda_1) = Re(\lambda_2) = \frac{T}{2}.$$

From the assumptions, we obtain $\left|Im(\lambda_{1,2})\right| > Re(\lambda_{1,2})\,\tan\frac{q\pi}{2}$, which satisfy the condition $|\arg(\lambda_{1,2})| > \frac{q\pi}{2}$. Therefore, the equilibrium point E^* is locally asymptotically stable by Theorem 1. The following theorems are derived from these findings.

Theorem 4. *Model (3) is locally asymptotically stable around E^*, if one of the following conditions holds:*

1. $T \leq 0,\ D > 0.$
2. $T > 0,\ \sqrt{4D-T^2} > T\tan\frac{q\pi}{2}.$

Theorem 5. *Model (3) exhibits unstable behaviour near E^*, if one of the following is holds*

1. $T > 0,\ T^2 - 4D \geq 0.$
2. $T > 0,\ T^2 - 4D < 0$ *and* $\sqrt{4D-T^2} < T\ tan\frac{q\pi}{2}.$

Proof. **Case (i)** For $T^2 - 4D \geq 0$ and $T > 0$, we get $\lambda_1 > 0,\ \lambda_2 > 0$, which lead to $|\arg(\lambda_{1,2})| < \frac{q\pi}{2}$ and by Theorem 1, we conclude that E^* is unstable.

Case (ii) If $T > 0,\ T^2 - 4D < 0$ and $\sqrt{4D-T^2} < T\,\tan\frac{q\pi}{2}$, then $\lambda_1 = \overline{\lambda_2},\ \lambda_2 = \overline{\lambda_1}$. This implies that $Im(\lambda_1) = -Im(\lambda_2) = \frac{1}{2}\sqrt{4D-T^2} > 0$. Hence, $|\arg(\lambda_{1,2})| = \tan^{-1}\frac{\sqrt{4D-T^2}}{T} < \frac{q\pi}{2}$ and by Theorem 1, we conclude that E^* is unstable . □

6 Hopf-bifurcation analysis

The system undergoes a Hopf-bifurcation when the Jacobian matix of the linearized system at any equilibrium point has a pair of conjugate eigenvalues. We consider the fractional-order system as follows,

$$^{c}D^{q}u = g(\zeta, u),\ \text{where } q \in (0,1), u \in \mathbb{R}^2. \tag{6}$$

Assume, system (6) has a equilibrium point E^*, then around E^* the Hopf-bifurcation occurs w.r.t the parameter ζ at $\zeta = \zeta^*$ if

(i) the Jacobian matrix of model (6) at E^* has two complex-conjugate eigenvalues $\lambda_{1,2} = a_j \pm ib_j$ become purely imaginary at $\zeta = \zeta^*$,

(ii) $\phi_{1,2}(q, \zeta^*) = 0,$

(iii) $\frac{\partial \phi_{1,2}}{\partial \zeta}|_{\zeta=\zeta^*} \neq 0,$

such that $\phi_i(q,\zeta) = \frac{q\pi}{2} - \min_{i=1,2} |\arg(\lambda_i(\zeta))|$.
As a result of the observations made in the previous sections that the derivative order significantly affects the stability of the system dynamics, we examine q as a Hopf-bifurcation parameter as follows:

(i) the Jacobian matrix of system (6) at E^* has two complex conjugate eigenvalues $\lambda_{1,2} = a_j \pm ib_j$ become purely imaginary at $q = q^*$,

(ii) $\varphi_{1,2}(q^*) = 0$,

(iii) $\frac{\partial \varphi_{1,2}}{\partial \zeta}|_{q=q^*} \neq 0$,

such that $\varphi_i(q) = \frac{q\pi}{2} - \min_{i=1,2} |\arg(\lambda_i(q))|$.

Now, we examine the Hopf-bifurcation criteria of our constructed system (3) in the following theorem.

Theorem 6. *Model (3) undergoes Hopf-bifurcation around $E^*(u^*,v^*)$ at $q = q^* = \tan^{-1}\left|\frac{\sqrt{4D-T^2}}{T}\right|$ where $4D > T^2,\ T \neq 0$.*

Proof. Since, $\lambda_{1,2} = \rho_1 \pm i\rho_2$, where $\rho_1 = \frac{T}{2} > 0,\ \rho_2 = \frac{\sqrt{4D-T^2}}{2}$, we have

$$\varphi_{1,2}(q^*) = \frac{q^*\pi}{2} - \tan^{-1}\left|\frac{\rho_2}{\rho_1}\right| = \tan^{-1}\left|\frac{\rho_2}{\rho_1}\right| - \tan^{-1}\left|\frac{\rho_2}{\rho_1}\right| = 0,$$

and $\frac{\partial \varphi_{1,2}}{\partial \zeta}|_{q=q^*} = \frac{\pi}{2} \neq 0$. Hence, all the conditions are satisfied. Therefore, Hopf-bifurcation occurs around the equilibrium point E^*. □

Remark 1. *The occurrence of Hopf-bifurcation for the parameters θ (prey refuge) and μ (anti-predator) is quite difficult analytically. So, we study the Hopf-bifurcation for these parameters numerically.*

7 Global stability of equilibria

Here, we establish the global stability behaviour of system (3) at $E^*(u^*,v^*)$.

Lemma 4. *([29]) Let $u(t)$ be a continuously differentiable function. Then, for any time $t > 0$*

$$^cD^q\left(u(t) - \kappa - \kappa\log\frac{u(t)}{\kappa}\right) \leq \left(1 - \frac{\kappa}{u(t)}\right)\, ^cD^q u(t),\ u(t),\ \kappa \in \mathbb{R}^+, \tag{7}$$

where $0 < q < 1$.

Theorem 7. *If $\frac{v(1-\theta)^2(\gamma-(1-\theta)^2uu^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*2})} < \beta$ and $\frac{u(1-\theta)^2(\gamma-(1-\theta)^2uu^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*2})} < \frac{\gamma}{\eta}$ then E^* is globally asymptotically stable.*

Proof. We define Lyapunov function as follows:

$$\mathbb{W}(u,v) = u - u^*\left(1+\log\frac{u}{u^*}\right) + \frac{1}{\eta}\left(v - v^*\left(1+\log\frac{v}{v^*}\right)\right). \tag{8}$$

Using Lemma 4, we have

$$\begin{aligned}
{}^cD^q\mathbb{W}(u,v) &\le \left(\frac{u-u^*}{u}\right) {}^cD^q u(t) + \frac{1}{\eta}\left(\frac{v-v^*}{v}\right) {}^cD^q v(t) \\
&= \left(\frac{u-u^*}{u}\right)\left[\alpha u - \beta u^2 - \frac{(1-\theta)^2u^2v}{\gamma+(1-\theta)^2u^2}\right] + \frac{1}{\eta}\left(\frac{v-v^*}{v}\right)\left[-\delta v + \frac{\eta(1-\theta)^2u^2v}{\gamma+(1-\theta)^2u^2} - \mu uv\right] \\
&\le (u-u^*)\left[-\beta(u-u^*) - \left[\frac{(1-\theta)^2uv\gamma + (1-\theta)^2uu^{*^2} - (1-\theta)^2\gamma u^*v^* - (1-\theta)^4u^*u^2y^*}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})}\right]\right] \\
&\quad + (v-v^*)\left[\frac{(1-\theta)^2\gamma u^2 - (1-\theta)^2\gamma u^{*^2}}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} - \frac{\mu}{\eta}(u-u^*)\right] \\
&\le -\beta(u-u^*)^2 - \frac{\gamma(1-\theta)^2u^*(u-u^*)(v-v^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} - \frac{\gamma(1-\theta)^2v(u-u^*)^2}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} \\
&\quad + \frac{(1-\theta)^4uu^*v(u-u^*)^2}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} - \frac{(1-\theta)^4u^2u^*(u-u^*)(v-v^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} \\
&\quad + \frac{\gamma(1-\theta)^2(u+u^*)(u-u^*)(v-v^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} - \frac{\mu(u-u^*)(v-v^*)}{\eta} \\
&\le \left[\frac{v(1-\theta)^2(\gamma-(1-\theta)^2uu^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} - \beta\right](u-u^*)^2 \\
&\quad + \left[\frac{u(1-\theta)^2(\gamma-(1-\theta)^2uu^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} - \frac{\gamma}{\eta}\right](u-u^*)(v-v^*).
\end{aligned}$$

Therefore, ${}^cD^q\mathbb{W}(u,v) \le 0$ if $\frac{v(1-\theta)^2(\gamma-(1-\theta)^2uu^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} < \beta$ and $\frac{u(1-\theta)^2(\gamma-(1-\theta)^2uu^*)}{(\gamma+(1-\theta)^2u^2)(\gamma+(1-\theta)^2u^{*^2})} < \frac{\gamma}{\eta}$. Hence, E^* is globally asymptotically stable. □

8 Numerical simulations

Numerical simulations for system (3) are performed by generalized Adams-Bashforth-Moulton Predictor Corrector method [5]. Numerical simulations are carried out by changing the values of anti-predator term (μ), prey refuge (θ) and fractional derivative order (q). Here, we classify three section to verify the analytical results of our formulated model through MATLAB.

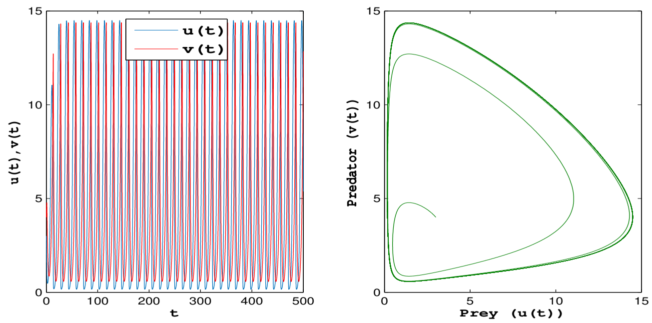

Figure 1: Numerical values of $u(t)$, $v(t)$ of system (3) for $q = 1$ and $\theta = 0.3$.

8.1 Absence of anti-predator

In the absence of anti-predator ($\mu = 0$), system (3) has the following form:

$$^cD^q u(t) = \alpha u - \beta u^2 - \frac{(1-\theta)^2 u^2 v}{\gamma + (1-\theta)^2 u^2},$$
$$^cD^q v(t) = -\delta v + \frac{\eta(1-\theta)^2 u^2 v}{\gamma + (1-\theta)^2 u^2}.$$

We have discussed the significance of prey refuge in both the integer-order and the fractional-order by varying the parameters θ and q, which are described in the next two cases:

Case 1 : We take $\alpha = 1,\ \beta = 0.05,\ \gamma = 0.5,\ \eta = 1.2,\ \delta = 0.8,\ \mu = 0$ and the initial time $u_0 = 3,\ v_0 = 4$ with step size 2^{-6} and we fix the value of q as $q = 1$ and slowly raise the amount of prey refuge θ in the interval $[0.3, 0.9]$. It is noted that the dynamical behaviour of model (3) near E^* switches from unstable limit cycle to stable through Hopf-bifurcation, which is shown in Figure 1 and Figure 2 respectively. Figure 1 shows the phase portrait of system (3) near E^*, for $\theta = 0.3$, demonstrates the unstable limit cycle. Figure 2 shows the phase portrait of system (3) for $\theta = 0.9$ and demonstrates the stable state of E^*. It follows that when the value of the prey refuge θ passes a threshold value $\theta^* = 0.81$, the dynamics convert from an unstable to a stable state through Hopf-bifurcation.

Case 2 : We fix the prey refuge level at $\theta = 0.3$ and slowly reduce the value of q. By decreasing the value of q, system (3) switches to stable steady state around E^*, which is shown in Figures.3, 4 and 5. Figures.3, 4 and 5 show the phase portraits of system (3) for q= 0.99, q= 0.90 and q= 0.89 respectively. From Figure 5, we clearly observe that when the value of q reduces to 0.89, the system switches to stable steady state.

8.2 Absence of prey refuge

In the absence of prey refuge ($\theta = 0$), system (3) has the following form:

$$^cD^q u(t) = \alpha u - \beta u^2 - \frac{u^2 v}{\gamma + u^2},$$
$$^cD^q v(t) = -\delta v + \frac{\eta u^2 v}{\gamma + u^2} - \mu u v.$$

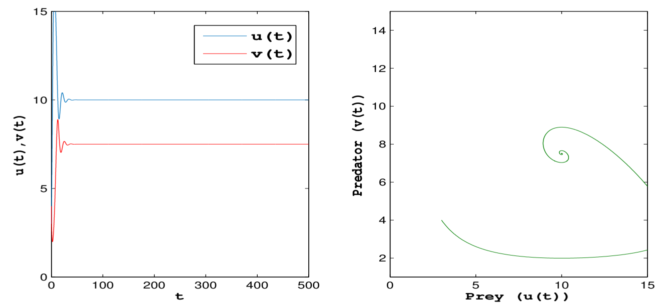

Figure 2: Numerical values of $u(t)$, $v(t)$ of system (3) for $q = 1$ and $\theta = 0.9$.

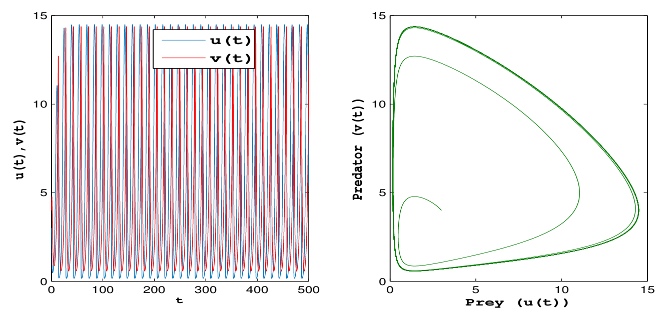

Figure 3: Numerical values of $u(t)$, $v(t)$ of system (3) for $\theta = 0.3$ and $q = 0.99$.

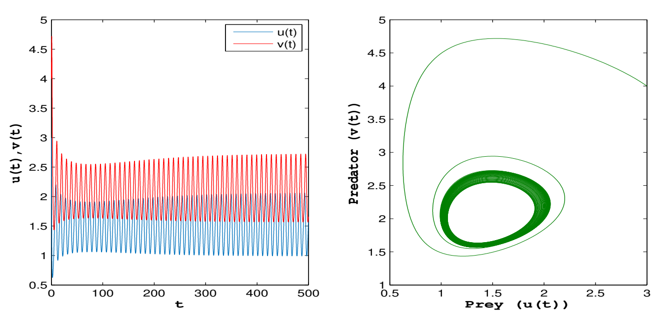

Figure 4: Numerical values of $u(t)$, $v(t)$ of system (3) for $\theta = 0.3$ and $q = 0.90$.

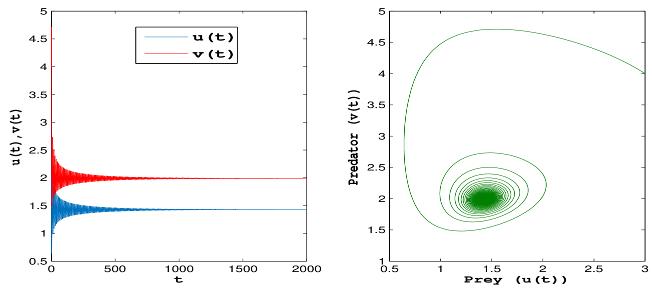

Figure 5: Numerical values of $u(t)$, $v(t)$ of system (3) for $\theta = 0.3$ and $q = 0.89$

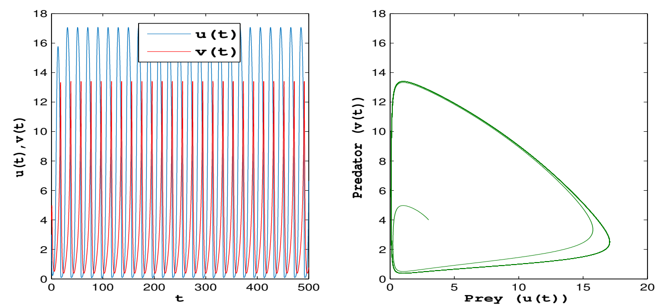

Figure 6: Numerical values of $u(t)$, $v(t)$ of system (3) for $q = 1$ and $\mu = 0.01$.

We have discussed the significance of anti-predator in both the integer-order and the fractional-order by varying the parameters μ and q, which are described in the next two cases:

Case 1 : We take $\alpha = 1,\ \beta = 0.05,\ \gamma = 0.5,\ \eta = 1.2,\ \delta = 0.8,\ \theta = 0$ and the initial time $u_0 = 3,\ v_0 = 4$ with step size 2^{-6} and we fix the value of q as $q = 1$ and slowly raise the amount of anti-predator μ in the interval $[0.01, 0.04]$. It is noted that the dynamical behaviour of model (3) near E^* switches from unstable limit cycle to stable through Hopf-bifurcation, which is shown in Figure 6 and Figure 7 respectively. Figure 6 shows that phase portrait of the solutions of system (3) near E^*, for $\mu = 0.01$, demonstrates the unstable limit cycle. Figure 7 shows the phase portrait of system (3) for $\mu = 0.04$, clearly demostrates the stable steady state of E^*. It follows that when the value of the anti-predator μ passes a threshold value $\mu^* = 0.03$, the dynamics convert from an unstable to a stable state through Hopf-bifurcation.

Case 2 : We fix the anti-predator level at $\mu = 0.01$ and slowly reduce the value of q. By decreasing the value of q system (3) switches to stable steady state around the equilibrium point E^*, which is shown in Figures.8, 9 and 10. Figure 8, 9 and 10 show the phase portraits of system (3), for q= 0.99, q= 0.88 and q= 0.87, respectively. From Figure 10, we clearly observe that when the value of q reduces to 0.87, the system switches to stable steady state.

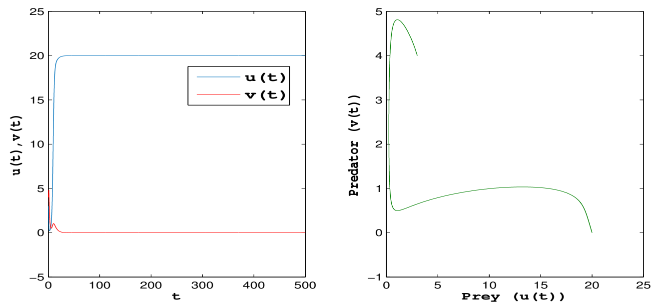

Figure 7: Numerical values of $u(t)$, $v(t)$ of system (3) for $q = 1$ and $\mu = 0.04$.

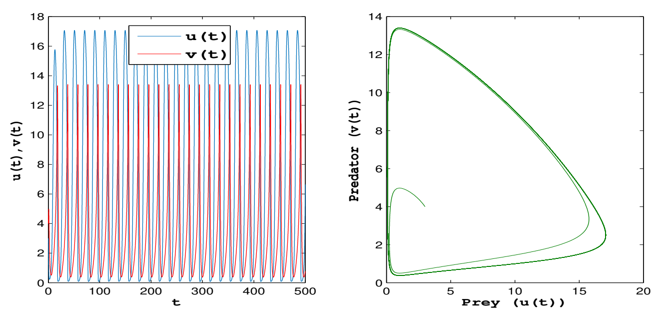

Figure 8: Numerical values of $u(t)$, $v(t)$ of system (3) for $\mu = 0.01$ and $q = 0.99$.

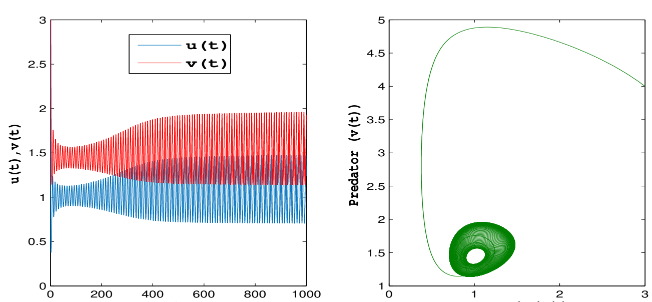

Figure 9: Numerical values of $u(t)$, $v(t)$ of system (3) for $\mu = 0.01$ and $q = 0.88$.

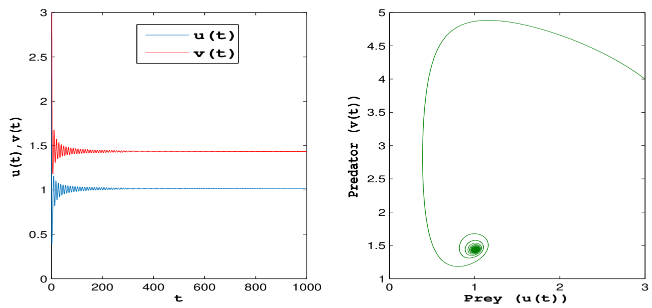

Figure 10: Numerical values of $u(t)$, $v(t)$ of system (3) for $\mu = 0.01$ and $q = 0.87$.

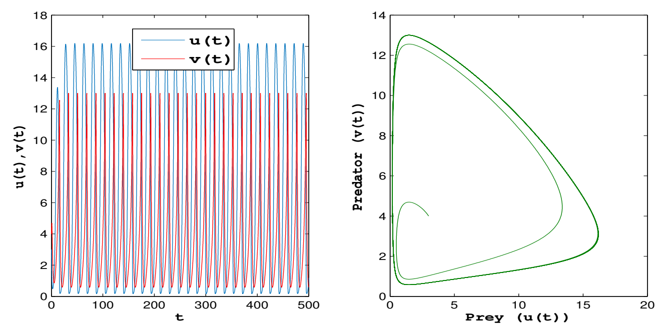

Figure 11: Numerical values of $u(t)$, $v(t)$ of system (3) for $\mu = 0.01, \theta = 0.3$ and $q = 1$.

8.3 Fractional-order model system in the presence of both anti-predator and prey refuge

In the presence of both anti-predator and prey refuge in system (3), we fix the parametric values as $\mu = 0.01$ and $\theta = 0.3$ for anti-predator term and prey refuge term respectively and varying the order of q in the range $(0,1)$. It is observable that system (3) obtains stability at E^* as a result of parameter q decrease continuously. Figures11, 12, 13 and 14 show the phase portrait, for q= 1, q= 0.99, q= 0.88 and q= 0.87, respectively and Figure14 clearly shows that when the order of q reduces to 0.8, the system slowly switches to stable state.

8.4 Hopf bifurcation for the parameter μ

In this case we take the values $\alpha = 1,\ \beta = 0.05,\ \gamma = 0.5,\ \eta = 1.2,\ \delta = 0.8,\ \theta = 0.3$ and $q = 0.87$. We fix the the values of μ in the range $0.01 < \mu < 0.02$ from Figures 15 and 16. Now μ^* is numerically calculated such that $0.87\frac{\pi}{2} - \tan^{-1}\left(\frac{v(\mu^*)}{u(\mu^*)}\right) = 0$ and $\frac{df(\mu)}{d\mu}|_{\mu=\mu^*} \neq 0$ which yeilds $\mu^* \approx 0.018$. Therefore, we get $\lambda_{1,2} = 0.129052 \pm i\, 0.674162$. It is noted that the equilibrium point E^* is locally asymptotically stable for $\mu < \mu^*$, which is shown in Figures 15 and 16.

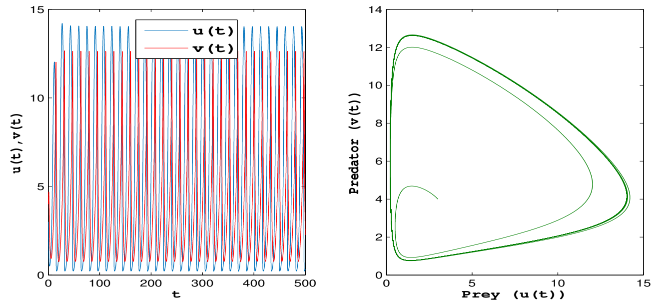

Figure 12: Numerical values of $u(t)$, $v(t)$ of system (3) for $\mu = 0.01, \theta = 0.3$ and $q = 0.99$.

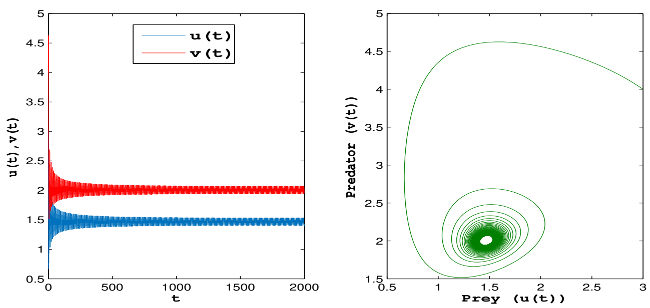

Figure 13: Numerical values of $u(t)$, $v(t)$ of system (3) for $\mu = 0.01, \theta = 0.3$ and $q = 0.88$.

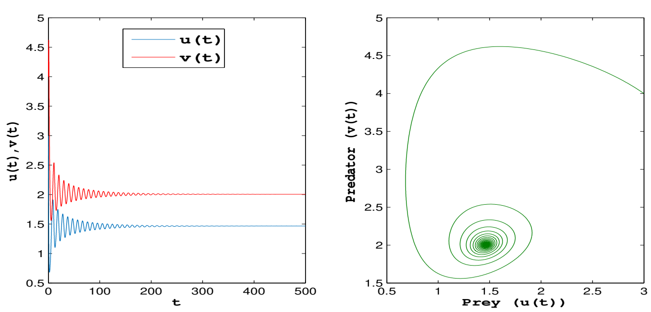

Figure 14: Numerical values of $u(t)$, $v(t)$ of system (3) for $\mu = 0.01, \theta = 0.3$ and $q = 0.87$.

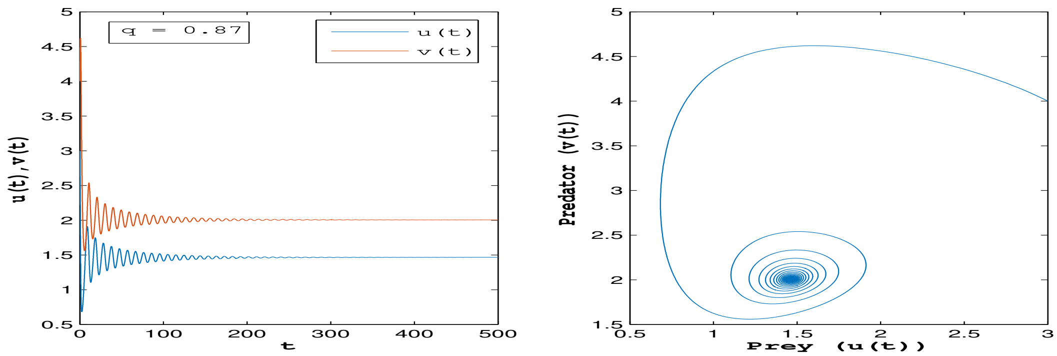

Figure 15: The trajectory and phase portrait of fractional-order system (3) for $\mu = 0.01$.

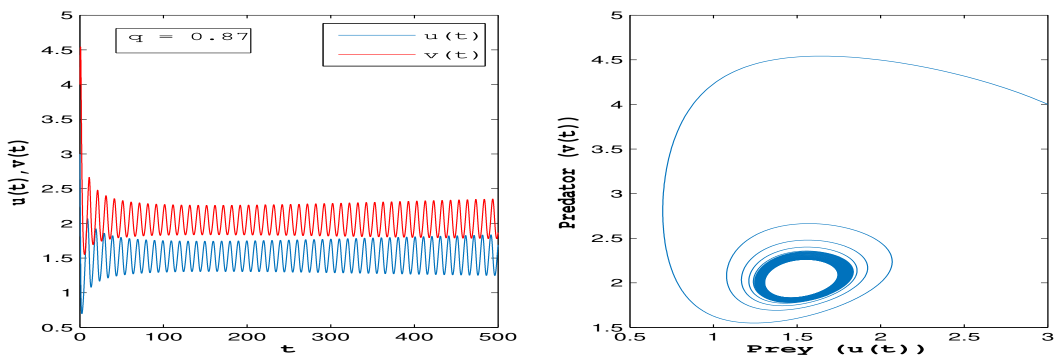

Figure 16: The trajectory and phase portrait of fractional-order system (3) for $\mu = 0.02$.

9 Conclusions

In this paper, we have investigated dynamical behaviour of fractional-order system (3). We first showed that the system possesses the existence, uniqueness, non-negaivity and boundedness of the solutions. The stability conditions of the predator-extinction equilibrium point and the co-existence equilibrium point have been established. Moreover, the global stability of the equilibrium point E^* of the fractional-order system (3) with certain Lyapunov functions are derived. The emergence of Hopf bifurcation are obtained for a fractional-order predator-prey model incorporating a prey refuge and anti-predator behaviour. In order to verify the theoretical findings and show how prey refuge and anti-predator species play significant roles in modifying the coexistence of prey species and predator species, numerical simulations are finally performed. It is found through theoretical study and a few numerical simulations that the parameter q, θ and μ significantly affects each population density. Also, we presented the numerical discussion for three cases such as absence of prey refuge in dynamical system, absence of anti-predator in dynamical system and presence of both anti-predator and prey refuge in dynamical system. Additionally, we found that the solution of our model is stable for the fractional-order model when $\mu = 0.01$ and $\theta = 0.3$, but unstable for the integer-order case.

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