Journal of Mathematical Modeling
Vol. 11, No. 3, 2023, pp. 491–505. Research Article

JMM

Taylor's formula for general quantum calculus

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Abstract. Let $I \subseteq \mathbb{R}$ be an interval and $\beta : I \to I$ a strictly increasing continuous function with a unique fixed point $s_0 \in I$ satisfying $(t - s_0)(\beta(t) - t) \leq 0$ for all $t \in I$. Hamza et al. introduced the general quantum difference operator D_β by $D_\beta f(t) := \frac{f(\beta(t))-f(t)}{\beta(t)-t}$ if $t \neq s_0$ and $D_\beta f(t) := f'(s_0)$ if $t = s_0$. In this paper, we establish results concerning Taylor's formula associated with D_β. For this, we define two types of monomials and then present our main results. The obtained results are new in the literature and are useful for further research in the field.

Keywords: Quantum calculus, quantum difference operator, β-derivative, β-integral, Taylor's formula, monomials.
AMS Subject Classification 2010: 39A13, 39A70, 41A58, 47B39.

1 Introduction

The standard calculus (differential and integral) is based on the notion of limit, where the concepts of derivative and integral are defined in terms of limit. But, there is calculus which works without the notion of limit. In fact, this type of calculus were existed and used already by Leonard Euler and Carl Gustov Jacobi in their studies. The modern name of such calculus is 'quantum calculus'. The main advantage of quantum calculus is that it deals with nondifferentiable functions also. Therefore, this calculus is very much suitable to deal with any physical phenomena which are described by equations involving nondifferentiable functions. Indeed, since the early twentieth century, this type of calculus received significant attention due to its effective utility in several mathematical areas such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, the theory of relativity, quantum computing, probability distributions, etc. Quantum calculus is based on the idea of finite difference. Depending on the re-scaling of finite differences, there are several different dialects of quantum calculus. Among them, classical difference calculus, the Jackson q-calculus, and the Hahn

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Received: 25 February 2023/ Revised: 24 April 2023/ Accepted: 22 May 2023
DOI: 10.22124/jmm.2023.23936.2139

quantum calculus are significant and more widely used quantum calculi. The classical difference calculus is based on the difference operator D_ω, which is defined by

$$D_\omega f(t) := \frac{f(t+\omega)-f(t)}{\omega}, \quad t \in \mathbb{R},$$

where ω is a nonzero fixed number. The calculus associated with D_ω is well-studied. There is ample of literature available for the ordinary difference calculus, see, for example [14, 18, 25, 26, 28]. The Jackson q-calculus is based on the q-difference operator, which is defined by

$$D_q f(t) := \begin{cases} \dfrac{f(qt)-f(t)}{t(q-1)} & \text{if } t \neq 0, \\ f'(0) & \text{if } t = 0, \end{cases} \tag{1}$$

where $q \in (0,1)$ is a fixed number and the function f is defined on a q-geometric set $G \subseteq \mathbb{R}$ such that $qt \in G$ for $t \in G$. Annaby et al. [4] established Taylor series and interpolation results for q-difference operators. For further studies about q-difference operators, readers can refer [1, 15, 23]. The Hahn quantum calculus is based on the (q,ω)-difference operator, which is defined by

$$D_{q,\omega} f(t) := \begin{cases} \dfrac{f(qt+\omega)-f(t)}{tq+\omega-t} & \text{if } t \neq \omega_0, \\ f'(\omega_0) & \text{if } t = \omega_0, \end{cases} \tag{2}$$

where $q \in (0,1)$ and ω_0 are fixed such that $\omega_0 = \frac{\omega}{1-q}$ and the function f is defined on a q-geometric set $G \subseteq \mathbb{R}$. Oraby and Hamza [29] established the Taylor theory associated with (q,ω)-difference operator. The operator $D_{q,\omega}$ is applicable in various fields like, during the process of construction of families of orthogonal polynomials and in the investigation of some approximation and optimization problems, see [2, 11–13, 34], also in symmetric calculus [5–7] and variational calculus [27]. Annaby et al. [3] constructed the inverse of the operator $D_{q,\omega}$ and investigated rigorously some of its properties. It was found that this inverse also generalizes both Nörlund sums and the Jackson q-integrals.

Recently, in 2015, Hamza et al. [19] introduced a general quantum difference operator D_β defined by

$$D_\beta f(t) := \begin{cases} \dfrac{f(\beta(t))-t}{\beta(t)-t}, & \text{if } \beta(t) \neq t, \\ f'(s_0), & \text{if } \beta(t) = t, \end{cases}$$

provided that $f'(s_0)$ exists. Here, the function f is defined on an interval $I \subseteq \mathbb{R}$ and the operator $\beta : I \to I$ is specified later in Section 2. This operator D_β unifies and generalizes the above three difference operators. The calculus associated with D_β is known as 'general quantum calculus' [19]. Since 2015, several researchers have studied this topic and investigated various new results. For instance, the exponential functions introduced in [17], the Laplace transform is introduced in [31], the convolution theorem is proved in [32], the problems of variational calculus are studied in [8], directional derivative is defined in [24], some inequalities are established in [20], certain results concerning fixed points are established in [20, 30], Leibniz's rule and Fubini's theorem are established in [22], properties of some function spaces are studied in [9], and the Sturm–Liouville problems are studied in [10]. Also, some difference equations associated with general quantum calculus are studied [16, 21, 33]. Consequently, within a short period of time, a rigorous analysis of the calculus associated with D_β was constructed. But still there are some basic concepts which are not studied in the context of β-difference operator D_β.

In this paper, proceeding with the calculus associated with D_β, we deduce Taylor's formula for general quantum calculus. This theory will be helpful in the development of further rigorous analysis of the calculus associated with D_β. We organize this paper as follows. After the introduction Section, in Section 2, we give essential concepts, definitions, and theorems from the β-quantum calculus. This will help the reader of this paper to understand the paper easily. In Section 3, we first define two monomials associated with D_β and prove essential results. Taylor's formulae associated with two monomials are proved in Section 4. Finally, the conclusion of the paper is included in Section 5.

2 β-quantum calculus essentials

To understand the present paper, in this section, we introduce essential concepts, definitions, and theorems from the β-quantum calculus.

Throughout the paper, let $I \subseteq \mathbb{R}$ be an interval and $\beta: I \to I$ is a strictly increasing continuous function which has a unique fixed point $s_0 \in I$ and satisfies the following condition:

$$(t - s_0)(\beta(t) - t) \leq 0 \text{ for all } t \in I. \tag{3}$$

Note that the equality in (3) holds only when $t = s_0$.

For $t \in I$ and $k \in \mathbb{N}_0$, we denote $\beta^k(t) := \underbrace{(\beta \circ \beta \circ \ldots \circ \beta)}_{k \text{ times}}(t)$, $\beta^{-k}(t) := \underbrace{(\beta^{-1} \circ \beta^{-1} \circ \ldots \circ \beta^{-1})}_{k \text{ times}}(t)$, and $\beta^0(t) = t$.

Lemma 1 ([19, Theorem 2.1])**.** *The following statements are true.*

1. *The sequence of functions $\{\beta^k(t)\}_{k\in\mathbb{N}_0}$ converges uniformly to the constant function $\widehat{\beta}(t) := s_0$ on every compact interval $J \subseteq I$ containing s_0.*
2. *The series of functions $\sum_{k=0}^{\infty} |\beta^k(t) - \beta^{k+1}(t)|$ is uniformly convergent to $|t - s_0|$ on every compact interval $J \subseteq I$ containing s_0.*

Definition 1 ([19, β-quantum difference operator])**.** *For a function $f: I \to \mathbb{R}$, the β-quantum difference operator D_β is defined by*

$$D_\beta f(t) := \begin{cases} \dfrac{f(\beta(t)) - t}{\beta(t) - t} & \text{if } t \neq s_0, \\ f'(s_0) & \text{if } t = s_0, \end{cases}$$

provided that $f'(s_0)$ exists. The number $D_\beta f(t)$ is called as the β-derivative of f at $t \in I$. If $f'(s_0)$ exists, then f is said to be β-differentiable on I.

Remark 1. *The β-quantum difference operator has the following properties.*

1. *The operator D_β is linear.*
2. *If f is β-differentiable at t, then $f(\beta(t)) = f(t) + (\beta(t) - t)D_\beta f(t)$.*

3. *If f is β-differentiable, then f is continuous at s_0.*

Theorem 1 ([19, Theorem 2.6])**.** *Assume that $f,g\colon I\to\mathbb{R}$ are β-differentiable functions at $t\in I$. Then, we have the following.*

(i) *The product $fg\colon I\to\mathbb{R}$ is β-differentiable at t and*

$$D_\beta(fg)(t)=(D_\beta f(t))g(t)+f(\beta(t))D_\beta g(t)=(D_\beta f(t))g(\beta(t))+f(t)D_\beta g(t).$$

(ii) *The function f/g is β-differentiable at t and*

$$D_\beta(f/g)(t)=\frac{(D_\beta f(t))g(t)-f(t)D_\beta g(t)}{g(t)g(\beta(t))},\quad g(t)g(\beta(t))\neq 0.$$

Definition 2 ([19, Definition 3.4])**.** *Let $f\colon I\to\mathbb{R}$ and $a,b\in I$. The β-integral of f from a to b is defined by*

$$\int_a^b f(t)d_\beta t:=\int_{s_0}^b f(t)d_\beta t-\int_{s_0}^a f(t)d_\beta t,$$

where

$$\int_{s_0}^x f(t)d_\beta t=\sum_{k=0}^{\infty}(\beta^k(x)-\beta^{k+1}(x))f(\beta^k(x)),\quad x\in I,$$

provided that the series on right-hand side converges at $x=a$ and $x=b$. The function f is said to be β-integrable on I if the series converges at a and b for all $a,b\in I$.

Note 1. *If $f\colon I\to\mathbb{R}$ is continuous at $s_0\in I$, then f is β-integrable on I.*

Theorem 2 ([19, Theorem 3.10])**.** *Assume that $f,g\colon I\to\mathbb{R}$ are β-differentiable functions on I such that $D_\beta f$ and $D_\beta g$ are both continuous at s_0. Then, for $a,b\in I$*

$$\int_a^b f(t)D_\beta g(t)d_\beta t=f(b)g(b)-f(a)g(a)-\int_a^b(D_\beta f(t))g(\beta(t))d_\beta t.$$

3 Generalized quantum monomials

First, we introduce the generalized quantum monomials as follows.

Definition 3. *Let $I\subseteq\mathbb{R}$. For $x,y\in I$, we define the generalized quantum monomials $h_k\colon I\times I\to\mathbb{R}$, $k\in\mathbb{N}_0$, recursively by*

$$\begin{aligned}h_0(x,y)&=1,\\ h_k(x,y)&=\int_y^x h_{k-1}(z,y)d_\beta z,\quad k\in\mathbb{N},\quad x,y\in I.\end{aligned}\tag{4}$$

Remark 2. *For $x,y\in I$ and $k\in\mathbb{N}$, we have*

$$D_{\beta x}h_k(x,y)=h_{k-1}(x,y).\tag{5}$$

First, we prove the following useful theorem.

Theorem 3. *Let $h\colon I\times I\to\mathbb{R}$ be continuous and β-differentiable with respect to its first variable. Define $g\colon I\to\mathbb{R}$ by*

$$g(x):=\int_{s_0}^{x}h(x,s)d_\beta s,\quad x\in I.$$

Then

$$D_{\beta x}g(x)=h(\beta(x),x)+\int_{s_0}^{x}D_{\beta x}h(x,s)d_\beta s,\quad x\in I. \tag{6}$$

Proof. First, suppose $x\neq s_0$. Then, using the definition of function g and [19, Corollary 3.7, p. 14], we have

$$\begin{aligned}g(\beta(x))-g(x)&=\int_{s_0}^{\beta(x)}h(\beta(x),s)d_\beta s-\int_{s_0}^{x}h(x,s)d_\beta s\\&=\int_{s_0}^{x}(h(\beta(x),s)-h(x,s))d_\beta s+(\beta(x)-x)h(\beta(x),x).\end{aligned}$$

Using this, we can write

$$D_{\beta x}g(x)=\frac{g(\beta(x))-g(x)}{\beta(x)-x}=h(\beta(x),x)+\int_{s_0}^{x}D_{\beta x}h(x,s)d_\beta s.$$

Now, we take $x=s_0$. Then for any $\eta\neq 0$, $s_0+\eta\in I$, we have

$$\begin{aligned}g(s_0+\eta)&=\sum_{k=0}^{\infty}(\beta^k(s_0+\eta)-\beta^{k+1}(s_0+\eta))h(s_0+\eta,\beta^k(s_0+\eta))\\&=\sum_{k=0}^{\infty}(\beta^k(s_0+\eta)-\beta^{k+1}(s_0+\eta))(h(s_0+\eta,\beta^k(s_0+\eta))-h(s_0,s_0))\\&\quad+\sum_{k=0}^{\infty}(\beta^k(s_0+\eta)-\beta^{k+1}(s_0+\eta))h(s_0,s_0).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{g(s_0+\eta)}{\eta}&=\sum_{k=0}^{\infty}\frac{\beta^k(s_0+\eta)-\beta^{k+1}(s_0+\eta)}{\eta}(h(s_0+\eta,\beta^k(s_0+\eta))-h(s_0,s_0))\\&\quad+\sum_{k=0}^{\infty}\frac{\beta^k(s_0+\eta)-\beta^{k+1}(s_0+\eta))}{\eta}h(s_0,s_0).\end{aligned}$$

Now, in view of Lemma 1, we obtain that

$$\lim_{\eta\to 0}\frac{g(s_0+\eta)}{\eta}=h(s_0,s_0),$$

and thus, we get (6). This completes the proof. □

Theorem 4. *For the generalized quantum monomials $h_k\colon I\times I\to\mathbb{R}$, $k\in\mathbb{N}_0$, defined in (4), we have*

$$h_{k+m+1}(x,s_0)=\int_{s_0}^{x}h_k(x,\beta(y))h_m(y,s_0)d_\beta y,\quad x\in I,\quad m\in\mathbb{N}_0. \tag{7}$$

Proof. Let

$$g(x)=\int_{s_0}^{x} h_k(x,\beta(y))h_m(y,s_0)d_\beta y, \quad x\in I.$$

Then, in view of Theorem 3, we can write

$$D_\beta g(x)=h_k(\beta(x),\beta(x))h_m(x,s_0)+\int_{s_0}^{x} D_{\beta x}h_k(x,\beta(y))h_m(y,s_0)d_\beta y.$$

But, using (5), we have $D_{\beta x}h_k(x,\beta(y))=h_{k-1}(x,\beta(y))$. This yields

$$D_\beta g(x)=\int_{s_0}^{x} h_{k-1}(x,\beta(y))h_m(y,s_0)d_\beta y.$$

Performing this procedure recursively, we obtain $D_\beta^k g(x)=h_{m+1}(x,s_0),\ x\in I$. Hence,

$$D_\beta^{k-1}g(x)-D_\beta^{k-1}g(s_0)=\int_{s_0}^{x} h_{m+1}(y,s_0)d_\beta y=h_{m+2}(x,s_0), \quad x\in I.$$

That is, $D_\beta^{k-1}g(x)=h_{m+2}(x,s_0),\ x\in I$. Continuing this process, we get $D_\beta g(x)=h_{k+m}(x,s_0),\ x\in I$, whereupon

$$g(x)-g(s_0)=\int_{s_0}^{x} h_{k+m}(y,s_0)d_\beta y=h_{k+m+1}(x,s_0), \quad x\in I.$$

That is, for $x\in I$, $g(x)=h_{k+m+1}(x,s_0)$, which gives (7). This completes the proof. □

Theorem 5. *For the generalized quantum monomials $h_k\colon I\times I\to\mathbb{R}$, $k\in\mathbb{N}_0$, defined in (4), we have*

$$h_k(x,s_0)\geq\frac{(x-s_0)^k}{k!}, \quad k\in\mathbb{N}, \tag{8}$$

where $x\in I$ and $x\geq s_0$.

Proof. Define $g\colon I\to\mathbb{R}$ by

$$g(x)=(x-s_0)^{k+1}, \quad k\in\mathbb{N} \text{ and } x\in I \text{ with } x\geq s_0.$$

Then, we have

$$D_\beta g(x)=\sum_{j=0}^{k}(\beta(x)-s_0)^{k-j}(x-s_0)^j, \quad x\in I \text{ with } x\geq s_0. \tag{9}$$

Really, if $x=s_0$, then (9) is evident. Let $x\neq s_0$. Then

$$\begin{aligned} D_\beta g(x)&=\frac{g(\beta(x))-g(x)}{\beta(x)-x}\\ &=\frac{1}{\beta(x)-x}\left((\beta(x)-s_0)^{k+1}-(x-s_0)^{k+1}\right)\\ &=\frac{1}{\beta(x)-x}(\beta(x)-x)\sum_{j=0}^{k}(\beta(x)-s_0)^{k-j}(x-s_0)^j\\ &=\sum_{j=0}^{k}(\beta(x)-s_0)^{k-j}(x-s_0)^j, \end{aligned}$$

i.e., we get (9). Note that the inequality (8) holds for $k = 0, 1$. Assume that the inequality (8) holds for some $k \in \mathbb{N}$. We will prove the inequality (8) for $k+1$. We have

$$\begin{aligned} h_{k+1}(x,s_0) &= \int_{s_0}^{x} h_k(z,s_0)d_\beta z \\ &\overset{(8)}{\geq} \frac{1}{k!}\int_{s_0}^{x}(z-s_0)^k d_\beta z \\ &= \frac{1}{(k+1)!}\int_{s_0}^{x}\sum_{j=0}^{k}(z-s_0)^k d_\beta z \\ &= \frac{1}{(k+1)!}\int_{s_0}^{x}\sum_{j=0}^{k}(z-s_0)^{k-j}(z-s_0)^j d_\beta z \\ &\overset{(3)}{\geq} \frac{1}{(k+1)!}\int_{s_0}^{x}\sum_{j=0}^{k}(\beta(z)-s_0)^{k-j}(z-s_0)^j d_\beta z \\ &\overset{(9)}{=} \frac{1}{(k+1)!}\int_{s_0}^{x} D_\beta g(z)d_\beta z \\ &= \frac{(x-s_0)^{k+1}}{(k+1)!}. \end{aligned}$$

Thus, $h_{k+1}(x,s_0) \geq \frac{(x-s_0)^{k+1}}{(k+1)!}$ for all $x \in I$ with $x \geq s_0$. By the principal of the mathematical induction, it follows that the inequality (8) holds for any $k \in \mathbb{N}$. This completes the proof. $\square$

Theorem 6. *Let $h_n\colon I \times I \to \mathbb{R}$ be the generalized quantum monomials defined in (4). Then, for each $n \in \mathbb{N}$, $h_n(\cdot,\cdot)$ satisfy the relationship*

$$h_n(\beta^k(t),t) = 0 \text{ for all } t \in I \text{ and for all } k \in \mathbb{N} \text{ such that } 0 \leq k \leq n-1.$$

Proof. Let $n \in \mathbb{N}$ be arbitrarily chosen. Then,

$$h_n(\beta^0(t),t) = h_n(t,t) = \int_t^t h_{n-1}(z,t)d_\beta z = 0.$$

Now, assume that

$$h_{n-1}(\beta^k(t),t) = 0 \text{ and } h_n(\beta^k(t),t) = 0,$$

for some $k \in \{0,\dots,n-2\}$. We shall prove that $h_n(\beta^{k+1}(t),t) = 0$. We have, by Definition 1,

$$D_\beta h_n(\beta^k(t),t) = \frac{h_n(\beta^{k+1}(t),t) - h_n(\beta^k(t),t)}{\beta(t)-t}.$$

Rearranging the terms, we obtain

$$\begin{aligned} h_n(\beta^{k+1}(t),t) &= h_n(\beta^k(t),t) + (\beta(t)-t)D_\beta h_n(\beta^k(t),t) \\ &= h_n(\beta^k(t),t) + (\beta(t)-t)h_{n-1}(\beta^k(t),t) = 0. \end{aligned}$$

That is, $h_n(\beta^{k+1}(t),t) = 0$. Thus, $h_n(\beta^k(t),t) = 0$ for all $k \in \mathbb{N}$ with $0 \leq k \leq n-1$. This completes the proof. $\square$

Before moving ahead, we introduce another generalized quantum monomials which are used henceforth in the paper.

Definition 4. *Let $s,t \in I$. We define the generalized quantum monomials $g_n\colon I \times I \to \mathbb{R}$, $n \in \mathbb{N}$, recursively by*

$$\begin{aligned} g_0(t,s) &= 1, \\ g_n(t,s) &= \int_s^t g_{n-1}(\beta(z),s)d_\beta z. \end{aligned} \tag{10}$$

Remark 3. *For $s,t \in I$ and $n \in \mathbb{N}$, we have*

$$D_{\beta t} g_n(t,s) = g_{n-1}(\beta(t),s).$$

4 The Taylor Formula

In this section, we shall prove our main results.

Theorem 7. *Let $m \in \mathbb{N}$ and suppose that $f\colon I \to \mathbb{R}$ is m-times β-differentiable at $\beta^{m-1}(t)$. Then*

$$f(t) = \sum_{k=0}^{m-1} (-1)^k D_\beta^k f(\beta^{m-1-k}(t)) h_k(\beta^{m-1}(t),t), \quad t \in I, \tag{11}$$

where $h_k\colon I \times I \to \mathbb{R}$, $k \in \mathbb{N}_0$, are generalized quantum monomials defined in Definition 3.

Proof. We shall make use of the principle of the mathematical induction. The proof is given in the following steps.

The equation (11) is true when $m = 1$. Since

$$\sum_{k=0}^{0} (-1)^k D_\beta^k f(\beta^{0-k}(t)) h_k(\beta^0(t),t) = (-1)^0 f(t) h_0(t,t) = f(t), \quad t \in I.$$

Now, assume that (11) is true for some $m \in \mathbb{N}$. That is,

$$f(t) = \sum_{k=0}^{m-1} (-1)^k D_\beta^k f(\beta^{m-1-k}(t)) h_k(\beta^{m-1}(t),t), \quad t \in I, \text{ for some } m \in \mathbb{N}. \tag{12}$$

We will prove (11) for $m+1$. That is,

$$f(t) = \sum_{k=0}^{m} (-1)^k D_\beta^k f(\beta^{m-k}(t)) h_k(\beta^m(t),t), \quad t \in I. \tag{13}$$

Using Definition 3 and Remark 2, we have

$$D_\beta h_k(\beta^{m-1}(t),t) = \frac{h_k(\beta^m(t),t) - h_k(\beta^{m-1}(t),t)}{\beta(t)-t} = h_{k-1}(\beta^{m-1}(t),t), \quad t \in I.$$

Rearranging the terms, we write

$$h_k(\beta^m(t),t) = h_k(\beta^{m-1}(t),t) + (\beta(t)-t) h_{k-1}(\beta^{m-1}(t),t), \quad t \in I. \tag{14}$$

By Theorem 6, we have

$$h_m(\beta^{m-1}(t),t) = 0, \quad t \in I. \tag{15}$$

Next, for $k \in \{0,\ldots,m-1\}$, we have

$$D_\beta^{k+1} f\left(\beta^{m-k-1}(t)\right) = D_\beta\left(D_\beta^k f\left(\beta^{m-k-1}(t)\right)\right) = \frac{D_\beta^k f\left(\beta^{m-k}(t)\right) - D_\beta^k f\left(\beta^{m-k-1}(t)\right)}{\beta(t)-t}, \quad t \in I,$$

whereupon

$$(\beta(t)-t)D_\beta^{k+1} f\left(\beta^{m-k-1}(t)\right) = D_\beta^k f\left(\beta^{m-k}(t)\right) - D_\beta^k f\left(\beta^{m-k-1}(t)\right), \quad t \in I,$$

and

$$D_\beta^k f\left(\beta^{m-k}(t)\right) - (\beta(t)-t)D_\beta^{k+1} f\left(\beta^{m-k-1}(t)\right) = D_\beta^k f\left(\beta^{m-k-1}(t)\right), \quad t \in I. \tag{16}$$

Therefore, for $t \in I$, we get

$$\begin{aligned}
&\sum_{k=0}^{m}(-1)^k D_\beta^k f(\beta^{m-k}(t))h_k(\beta^m(t),t)\\
&\qquad = f(\beta^m(t)) + \sum_{k=1}^{m}(-1)^k D_\beta^k f(\beta^{m-k}(t))h_k(\beta^m(t),t)\\
&\qquad \overset{(14)}{=} f(\beta^m(t)) + \sum_{k=1}^{m}(-1)^k D_\beta^k f(\beta^{m-k}(t))h_k(\beta^{m-1}(t),t)\\
&\qquad\quad + (\beta(t)-t)\sum_{k=1}^{m}(-1)^k D_\beta^k f(\beta^{m-k}(t))h_{k-1}(\beta^{m-1}(t),t)\\
&\qquad = f(\beta^m(t)) + \sum_{k=1}^{m-1}(-1)^k D_\beta^k f(\beta^{m-k}(t))h_k(\beta^{m-1}(t),t)\\
&\qquad\quad + (-1)^m D_\beta^m f(t)h_m(\beta^{m-1}(t),t)\\
&\qquad\quad + (\beta(t)-t)\sum_{k=0}^{m-1}(-1)^{k+1} D_\beta^{k+1} f(\beta^{m-k-1}(t))h_k(\beta^{m-1}(t),t)\\
&\qquad \overset{(15)}{=} \sum_{k=0}^{m-1}(-1)^k D_\beta^k f(\beta^{m-k}(t))h_k(\beta^{m-1}(t),t)\\
&\qquad\quad - (\beta(t)-t)\sum_{k=0}^{m-1}(-1)^k D_\beta^{k+1} f(\beta^{m-k-1}(t))h_k(\beta^{m-1}(t),t)\\
&\qquad = \sum_{k=0}^{m-1}(-1)^k\left(D_\beta^k f(\beta^{m-k}(t)) - (\beta(t)-t)D_\beta^{k+1} f(\beta^{m-k-1}(t))\right)h_k(\beta^{m-1}(t),t)\\
&\qquad \overset{(16)}{=} \sum_{k=0}^{m-1}(-1)^k D_\beta^k f\left(\beta^{m-k-1}(t)\right)h_k(\beta^{m-1}(t),t)\\
&\qquad \overset{(12)}{=} f(t).
\end{aligned}$$

That is,

$$\sum_{k=0}^{m}(-1)^k D_\beta^k f(\beta^{m-k}(t))h_k(\beta^m(t),t) = f(t), \quad t\in I.$$

Hence, (13) holds. This completes the proof. □

Theorem 8. *Let $n\in\mathbb{N}$. If $f\colon I\to\mathbb{R}$ is n-times β-differentiable and $p_k\colon I\to\mathbb{R}$, $0\le k\le n-1$, are β-differentiable at $t\in I$ with*

$$D_\beta p_{k+1}(t) = p_k(t), \quad 0\le k\le n-2, \quad n\in\mathbb{N}\setminus\{1\},$$

then

$$D_\beta\left(\sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{-k}(\cdot)\right)p_k(\cdot)\right)(t) = (-1)^{n-1}D_\beta^n f\left(\beta^{-(n-1)}(t)\right)p_{n-1}(t) + f(\beta(t))D_\beta p_0(t), \quad t\in I.$$

Proof. We have

$$\begin{aligned}
&D_\beta\left(\sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{-k}(\cdot)\right)p_k(\cdot)\right)(t)\\
&= \sum_{k=0}^{n-1}(-1)^k D_\beta\left(D_\beta^k f\left(\beta^{-k}(\cdot)\right)p_k(\cdot)\right)(t)\\
&= \sum_{k=0}^{n-1}(-1)^k\left[D_\beta^{k+1} f\left(\beta^{-k}(t)\right)p_k(t) + D_\beta^k f\left(\beta^{1-k}(t)\right)D_\beta p_k(t)\right]\\
&= \sum_{k=0}^{n-1}(-1)^k D_\beta^{k+1} f\left(\beta^{-k}(t)\right)p_k(t) + \sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{1-k}(t)\right)D_\beta p_k(t)\\
&= \sum_{k=0}^{n-2}(-1)^k D_\beta^{k+1} f\left(\beta^{-k}(t)\right)p_k(t) + (-1)^{n-1}D_\beta^n f\left(\beta^{-(n-1)}(t)\right)p_{n-1}(t)\\
&\quad + \sum_{k=1}^{n-1}(-1)^k D_\beta^k f\left(\beta^{1-k}(t)\right)D_\beta p_k(t) + f(\beta(t))D_\beta p_0(t)\\
&= \sum_{k=0}^{n-2}(-1)^k D_\beta^{k+1} f\left(\beta^{-k}(t)\right)p_k(t) - \sum_{k=0}^{n-2}(-1)^k D_\beta^{k+1} f\left(\beta^{-k}(t)\right)p_k(t)\\
&\quad + (-1)^{n-1}D_\beta^n f\left(\beta^{-(n-1)}(t)\right)p_{n-1}(t) + f(\beta(t))D_\beta p_0(t)\\
&= (-1)^{n-1}D_\beta^n f\left(\beta^{-(n-1)}(t)\right)p_{n-1}(t) + f(\beta(t))D_\beta p_0(t).
\end{aligned}$$

That is, for $t\in I$

$$D_\beta\left(\sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{-k}(\cdot)\right)p_k(\cdot)\right)(t) = (-1)^{n-1}D_\beta^n f\left(\beta^{-(n-1)}(t)\right)p_{n-1}(t) + f(\beta(t))D_\beta p_0(t).$$

This completes the proof. □

Now, below we prove the Taylor formula involving the generalized quantum monomials h_k defined in (4).

Theorem 9 (The Taylor Formula)**.** *Let $n \in \mathbb{N}$. Suppose that $f\colon I \to \mathbb{R}$ is n-times β-differentiable on I and let $a,t \in I$. Then*

$$f(t) = \sum_{k=0}^{n-1} (-1)^k D_\beta^k f\left(\beta^{-k}(a)\right) h_k(a,t) + (-1)^{n-1} \int_a^{\beta^{n-1}(t)} D_\beta^n f\left(\beta^{-(n-1)}(\tau)\right) h_{n-1}(\tau,t) d_\beta \tau, \tag{17}$$

where $h_k\colon I \times I \to \mathbb{R}$, $k \in \mathbb{N}_0$, are the generalized quantum monomials defined in Definition 3.

Proof. First, in view of Theorem 8, we get

$$D_\beta\left(\sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{-k}(\cdot)\right) h_k(\cdot,t)\right)(\tau) = (-1)^{n-1} D_\beta^n f\left(\beta^{-(n-1)}(\tau)\right) h_{n-1}(\tau,t) + f(\beta(\tau)) D_\beta h_0(\tau,t).$$

That is, for $\tau, t \in I$

$$D_\beta\left(\sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{-k}(\cdot)\right) h_k(\cdot,t)\right)(\tau) = (-1)^{n-1} D_\beta^n f\left(\beta^{-(n-1)}(\tau)\right) h_{n-1}(\tau,t).$$

Now, integrating the last equation from a to $\beta^{n-1}(t)$, we get

$$\begin{aligned}\int_a^{\beta^{n-1}(t)} & D_\beta\left(\sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{-k}(\cdot)\right) h_k(\cdot,t)\right)(\tau) d_\beta \tau \\ &= (-1)^{n-1} \int_a^{\beta^{n-1}(t)} D_\beta^n f\left(\beta^{-(n-1)}(\tau)\right) h_{n-1}(\tau,t) d_\beta \tau, \quad t \in I,\end{aligned}$$

which yields

$$\begin{aligned}\sum_{k=0}^{n-1} & (-1)^k D_\beta^k f\left(\beta^{n-1-k}(t)\right) h_k(\beta^{n-1}(t),t) - \sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{-k}(a)\right) h_k(a,t) \\ &= (-1)^{n-1} \int_a^{\beta^{n-1}(t)} D_\beta^n f\left(\beta^{-(n-1)}(\tau)\right) h_{n-1}(\tau,t) d_\beta \tau, \quad t \in I.\end{aligned}$$

Now, in view of Theorem 7, we obtain

$$\begin{aligned}f(t) & - \sum_{k=0}^{n-1}(-1)^k D_\beta^k f\left(\beta^{-k}(a)\right) h_k(a,t) \\ &= (-1)^{n-1} \int_a^{\beta^{n-1}(t)} D_\beta^n f\left(\beta^{-(n-1)}(\tau)\right) h_{n-1}(\tau,t) d_\beta \tau, \quad t \in I,\end{aligned}$$

which gives (17), and this completes the proof. □

In the next theorem, we obtain the relation between generalized quantum monomials h_n and g_n, $n \in \mathbb{N}$, defined, respectively, in Definitions 3 and 4.

Theorem 10. *The functions $h_n\colon I\times I\to\mathbb{R}$ and $g_n\colon I\times I\to\mathbb{R}$ defined, respectively, in (4) and (10), satisfies the relationship*

$$h_n(t,s)=(-1)^n g_n(s,t) \text{ for all } n\in\mathbb{N} \text{ and } s,t\in I. \tag{18}$$

Proof. Take $n\in\mathbb{N}$ arbitrarily and define $f\colon I\to\mathbb{R}$ by

$$f(\tau)=g_n(\tau,s),\quad \tau,s\in I. \tag{19}$$

Then, in view of Remark 3, we obtain $D_\beta f(\tau)=g_{n-1}(\beta(\tau),s)$. Performing this repeatedly, we get $D_\beta^k f(\tau)=g_{n-k}(\beta^k(\tau),s)$, for $0\le k\le n-1$, and

$$\begin{aligned} D_\beta^n f(\tau) &= g_0(\beta^n(\tau),s)\\ &\overset{(10)}{=} 1,\quad \tau,s\in I. \end{aligned} \tag{20}$$

Hence,

$$\begin{aligned} D_\beta^k f(\beta^{-k}(s)) &= g_{n-k}(\beta^k(\beta^{-k}(s)),s)\\ &= g_{n-k}(s,s)\\ &= 0,\quad 0\le k\le n-1. \end{aligned} \tag{21}$$

Now, applying Theorem 9 for $a=s$, we get

$$\begin{aligned} f(t) &= \sum_{k=0}^{n}(-1)^k D_\beta^k f(\beta^{-k}(s))h_k(s,t)+(-1)^n\int_s^{\beta^n(t)} D_\beta^{n+1}f(\beta^{-n}(\tau))h_{n+1}(\tau,t)d_\beta\tau\\ &= \sum_{k=0}^{n-1}(-1)^k D_\beta^k f(\beta^{-k}(s))h_k(s,t)+(-1)^n D_\beta^n f(\beta^{-n}(s))h_n(s,t)\\ &\quad +(-1)^n\int_s^{\beta^n(t)} D_\beta^{n+1}f(\beta^{-n}(\tau))h_{n+1}(\tau,t)d_\beta\tau\\ &\overset{(21)}{=} (-1)^n\left[D_\beta^n f(\beta^{-n}(s))h_n(s,t)+\int_s^{\beta^n(t)} D_\beta^{n+1}f(\beta^{-n}(\tau))h_{n+1}(\tau,t)d_\beta\tau\right]. \end{aligned} \tag{22}$$

Using Theorem 2, we write

$$\begin{aligned} &D_\beta^n f(\beta^{-n}(s))h_n(s,t)+\int_s^{\beta^n(t)} D_\beta^{n+1}f(\beta^{-n}(\tau))h_{n+1}(\tau,t)d_\beta\tau\\ &\qquad = h_{n+1}(\beta^n(t),t)D_\beta^n f(\beta^{-n}(\beta^n(t)))-\int_s^{\beta^n(t)} D_\beta h_{n+1}(\tau,t)D_\beta^n f(\beta^{-n}(\beta^n(t)))d_\beta\tau\\ &\qquad = h_{n+1}(\beta^n(t),t)D_\beta^n f(t)-\int_s^{\beta^n(t)} D_\beta h_{n+1}(\tau,t)D_\beta^n f(t))d_\beta\tau\\ &\qquad \overset{(20)}{=} h_{n+1}(\beta^n(t),t)-\int_s^{\beta^n(t)} D_\beta h_{n+1}(\tau,t)d_\beta\tau\\ &\qquad = h_{n+1}(\beta^n(t),t)-\int_s^{\beta^n(t)} h_n(\tau,t)d_\beta\tau\\ &\qquad = h_{n+1}(\beta^n(t),t)-h_{n+1}(\beta^n(t),t)+h_n(s,t)\\ &\qquad = h_n(s,t). \end{aligned}$$

Now, (22) becomes $f(t) = (-1)^n h_n(s,t)$. Thus, (19) gives $g_n(t,s) = (-1)^n h_n(s,t)$ for all $n \in \mathbb{N}$ and $s,t \in I$. This completes the proof. $\square$

Now, using the relation between generalized quantum monomials h_n and g_n obtained in Theorem 10, we get the following Taylor formula.

Theorem 11 (The Taylor Formula)**.** *Let $n \in \mathbb{N}$. Suppose that $f\colon I \to \mathbb{R}$ is n-times β-differentiable on I and let $a,t \in I$. Then*

$$f(t) = \sum_{k=0}^{n-1} D_\beta^k f\left(\beta^{-k}(a)\right) g_k(t,a) + \int_a^{\beta^{n-1}(t)} D_\beta^n f\left(\beta^{-(n-1)}(\tau)\right) g_{n-1}(t,\tau) d_\beta \tau,$$

where $g_k\colon I \times I \to \mathbb{R}$, $k \in \mathbb{N}_0$, are the generalized quantum monomials defined in Definition 10.

5 Conclusion

This article is devoted in establishing Taylor's formula associated with the β-quantum calculus introduced by Hamza et al. [19]. Since the β-quantum calculus unifies and generalizes the classical difference calculus, the Jackson q-calculus and the Hahn quantum calculus, the results obtained in the context of β-quantum calculus are more general. These results will be useful in the investigation of more details about the theory of β-quantum calculus.

Acknowledgements

The authors are thankful to the referees for their insightful and constructive comments towards the improvement of the quality of the paper.

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