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A GRAPH ASSOCIATED TO SPECTRUM OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring. In this paper, by using algebraic properties of R, we study the Hase digraph of prime ideals of R.

1. Introduction

Throughout this paper all rings are commutative with non-zero identity and all prime ideals have finite height. For a ring R, also we denote the sets of prime ideals, maximal ideals, and minimal prime ideals of R by $\mathrm{Spec}(R)$, $\mathrm{Max}(R)$, and $\mathrm{Min}(R)$, respectively.

Throughout the rest of this paper, R will denote a commutative ring. There are many papers on assigning a graph to a ring, see for example [4], [3], [5], [7] and [9]. Also, the Hase diagram of ideals of R is defined. In this paper, among the other things, we study some combinatorial property of the Hase digraph of prime ideals of R. Hence, for an arbitrary commutative ring R, we define the Spec-graph of R, which is denoted by $\mathcal{S}(R)$, as follows: The vertex set of $\mathcal{S}(R)$ is $\mathrm{Spec}(R)$ and, for two distinct vertices $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathrm{Spec}(R)$, $\mathfrak{p}$ is adjacent to $\mathfrak{q}$ if and only if the inclusions $\mathfrak{p} \subsetneqq \mathfrak{q}$ or $\mathfrak{q} \subsetneqq \mathfrak{p}$ is saturated.

Whenever the inclusion $\mathfrak{p} \subsetneqq \mathfrak{q}$ is saturated, we say that $\mathfrak{q}$ is a cover of $\mathfrak{p}$ and write $\mathfrak{p} \looparrowright \mathfrak{q}$. Hence, for two distinct prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R, $\mathfrak{p}$ is adjacent to $\mathfrak{q}$ if and only if $\mathfrak{p} \looparrowright \mathfrak{q}$ or $\mathfrak{q} \looparrowright \mathfrak{p}$.

Clearly a vertex $\mathfrak{p}$ in $\mathcal{S}(R)$ is singular if and only if $\mathfrak{p}$ is both maximal and minimal prime ideal of R. Hence, the Spec-graph $\mathcal{S}(R)$ is totally

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disconnected if and only if $\mathrm{Min}(R) = \mathrm{Max}(R)$. Also, If R is an Artinian ring, then $\mathcal{S}(R)$ is totally disconnected.

Now we recall some definitions and notations on graphs. We use the standard terminology of graphs in [4] and [6], and commutative algebra in [1] and [8]. The distance between two distinct vertices a and b in a graph G, denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a,b) := \infty$. The diameter of G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a,b) : a$ and b are distinct vertices of $G\}$. A graph G is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. We say that G is empty if no two vertices of G are adjacent. For a vertex x in G, we denote the set of all adjacent vertices to x, by $\mathrm{N}_G(x)$ and also the size of $\mathrm{N}_G(x)$ is called the degree or valence of x in G, and we denote it by $v_G(x)$. A vertex x is an isolated, if $v_G(x) = 0$. An independent set of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. The independence number of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. For a graph G, let $\chi(G)$ denote the chromatic number of the graph G, i.e., the minimal number of colours which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colours. A clique of a graph is its complete subgraph and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number of G.

2. SPEC-GRAPH AND RING EXTENSION

For a subset T of $\mathrm{Spec}(R)$, we denote the induced sub-graph of $\mathcal{S}(R)$ with vertex set T by $\mathcal{S}(R)_T$.

Theorem 2.1. (i) *Let I be an ideal of R. Then $\mathcal{S}(R/I) \cong \mathcal{S}(R)_{V(I)}$, where $V(I) := \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq I \}$.*

(ii) *Let S be a multiplicatively closed subset of R. Then $\mathcal{S}(S^{-1}R) \cong \mathcal{S}(R)_\rho$, where $\rho := \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap R = \emptyset \}$.*

Proof. (i) Consider the map $\varphi : \mathcal{S}(R/I) \longrightarrow \mathcal{S}(R)$ given by $\varphi(\mathfrak{p}/I) = \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathfrak{p} \supseteq I$. Clearly φ induces an isomorphism between two graphs $\mathcal{S}(R/I)$ and $\mathcal{S}(R)_{V(I)}$.

(ii) Suppose that $f : R \longrightarrow S^{-1}R$ is the natural homomorphism given by $f(r) = r/1$ for all $r \in R$. Clearly, for every arbitrary distinct elements $\mathfrak{p}$ and $\mathfrak{q}$ of ρ, $\mathfrak{p} \nleftrightarrow \mathfrak{q}$ if and only if $\mathfrak{p}^e \nleftrightarrow \mathfrak{q}^e$, where $\mathfrak{p}^e$ is the extension of prime ideal $\mathfrak{p}$ under f. This means that the map $\psi : \mathcal{S}(R)_\rho \longrightarrow \mathcal{S}(S^{-1}R)$ with $\psi(\mathfrak{p}) = \mathfrak{p}^e$ for all $\mathfrak{p} \in \rho$, is a graph isomorphism. $\square$

Corollary 2.2. *If $f : R \longrightarrow R'$ is a ring epimorphism, then $\mathcal{S}(R')$ is an isomorphism to an induced sub-graph of $\mathcal{S}(R)$.*

Proof. By using the first isomorphism theorem for rings, $R/\mathrm{Ker}(f) \cong R'$. The result now immediately follows from Theorem 2.1 (i). $\square$

Remark 2.3. *Let R' be an extension ring of R and $\mathfrak{q} \in \mathrm{Spec}(R)$. A prime ideal Q of R' lying over $\mathfrak{q}$ if $Q \cap R = \mathfrak{q}$. In this situation we say that lying over theorem holds. Also we denote the set of all prime ideals of R' lying over $\mathfrak{q}$, by $\mathrm{O}(\mathfrak{q})$. Note that if R' is an integral extension of R, then, by [8, Theorem 9.3] lying over theorem holds; thus $\mathrm{O}(\mathfrak{q})$ is not empty for all prime ideal $\mathfrak{q}$ of R. Clearly for every $Q \in \mathrm{Spec}(R')$, we have that $Q \in \mathrm{O}(Q \cap R)$.*

Lemma 2.4. *Let R' be an integral extension of R. Then $\mathcal{S}(R')_{\mathrm{O}(\mathfrak{q})}$ is totally disconnected for all $\mathfrak{q} \in \mathrm{Spec}(R)$.*

Proof. Suppose that Q_1 and Q_2 are two distinct vertices in $\mathrm{O}(\mathfrak{q})$ such that $Q_1 \looparrowright Q_2$ for some $\mathfrak{q} \in \mathrm{Spec}(R)$. Then $Q_1 \subseteq Q_2$, and so $Q_1 \cap R = Q_2 \cap R = \mathfrak{q}$. Hence, by [8, Theorem 9.3(ii)], we have $Q_1 = Q_2$, which is impossible. Hence the graph $\mathcal{S}(R')_{\mathrm{O}(\mathfrak{q})}$ is totally disconnected for all $\mathfrak{q} \in \mathrm{Spec}(R)$. $\square$

Proposition 2.5. *Let R' be an integral extension of R and there exist $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ such that $\mathfrak{p}$ is adjacent to $\mathfrak{q}$ in $\mathcal{S}(R)$. Then there exists an edge between a vertex in $\mathrm{O}(\mathfrak{p})$ and a vertex in $\mathrm{O}(\mathfrak{q})$ in $\mathcal{S}(R')$.*

Proof. Without loss of generality, we may assume that $\mathfrak{p} \looparrowright \mathfrak{q}$ in $\mathcal{S}(R)$. Now, by lying over theorem, there exists $P \in \mathrm{O}(\mathfrak{p})$. Also, by the Going Up Theorem (cf.[1, page 67]), there is $Q \in \mathrm{O}(\mathfrak{q})$ that $P \subsetneqq Q$. We only need to show that the inclusion $P \subsetneqq Q$ is saturated. Assume in contrary that there exists a prime ideal H of R' with $P \subsetneqq H \subsetneqq Q$. Thus $\mathfrak{p} = P \cap R \subseteq H \cap R \subseteq Q \cap R = \mathfrak{q}$. Hence $\mathfrak{p} = H \cap R$ or $\mathfrak{q} = H \cap R$. Again by applying [8, Theorem 9.3(ii)]we have that $H = P$ or$H = Q$, which is the required contradiction. Therefore $P \looparrowright Q$, and so the vertices P and Q are adjacent in $\mathcal{S}(R')$. $\square$

Let R' be an integral extension of R. We define a graph $\mathcal{H}(R', R)$ as a simple graph with vertex set $\{\mathrm{O}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\ \}$ and two distinct vertices $\mathrm{O}(\mathfrak{p})$ and $\mathrm{O}(\mathfrak{q})$ are adjacent if and only if there is an adjacency in $\mathcal{S}(R)$ between a vertex in $\mathrm{O}(\mathfrak{p})$ and a vertex in $\mathrm{O}(\mathfrak{q})$. Clearly $\{\mathrm{O}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}$ is a partition of $\mathrm{Spec}(R')$, since $\mathrm{O}(\mathfrak{p}) \cap \mathrm{O}(\mathfrak{q}) = \varnothing$, for distinct prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ in R. Moreover, in view of Remark 2.3, $\mathrm{Spec}(R') = \cup_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{O}(\mathfrak{p})$.

Theorem 2.6. *With the above notation, $\mathcal{S}(R)$ is isomorphic to an spanning sub-graph of $\mathcal{H}(R,R')$.*

Proof. Consider the mapping $\varphi : \mathcal{S}(R) \longrightarrow \mathcal{H}(R,R')$ given by $\varphi(\mathfrak{p}) = \mathrm{O}(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. In view of Proposition 2.5, φ is a one to one homomorphism of graphs. Moreover, the image of this homomorphism is an spanning sub-graph of $\mathcal{H}(R,R')$. $\square$

Remarks 2.7. *Let R' be an integral extension of R. Then we have the following facts.*

(i) *Suppose that the graph $\mathcal{S}(R)$ is connected. Then it is easy to see that the graph $\mathcal{H}(R,R')$ has a connected component at least in size of $\mathcal{S}(R)$.*
(ii) *The graphs $\mathcal{S}(R)$ and $\mathcal{H}(R,R')$ are isomorphic, if $|\mathrm{O}(\mathfrak{p})| = 1$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.*

3. connectedness of Spec-graph

In this section, we will study the connectedness of graph $\mathcal{S}(R)$. So we investigate the case that $\mathcal{S}(R)$ has not isolated vertex. Thus we assume that the set $\mathrm{Min}(R) \cap \mathrm{Max}(R)$ is empty.

Proposition 3.1. *Suppose that $\mathcal{S}(R)$ is disconnected. Then each vertex in any connected component is co-prime with vertices lie in other connected components.*

Proof. Suppose that C_1 and C_2 are two different connected components of $\mathcal{S}(R)$ and that $\mathfrak{p} \in C_1$ and $\mathfrak{q} \in C_2$. We must show that $\mathfrak{p}$ and $\mathfrak{q}$ are co-prime. Assume in contrary that $\mathfrak{p} + \mathfrak{q} \neq R$. Then there is $\mathfrak{r} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} + \mathfrak{q} \subseteq \mathfrak{r}$. Since $\mathrm{ht}(\mathfrak{r})$ is finite, we can obtain two saturated chains $\mathfrak{p} = \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_t = \mathfrak{r}$ and $\mathfrak{q} = \mathfrak{q}_0 \subsetneqq \mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_s = \mathfrak{r}$. This means that there are paths from $\mathfrak{p}$ to $\mathfrak{r}$ and from $\mathfrak{q}$ to $\mathfrak{r}$. Hence $\mathfrak{p}, \mathfrak{r} \in C_1$ and $\mathfrak{q}, \mathfrak{r} \in C_2$ which is the required contradiction. Thus $\mathfrak{p}$ and $\mathfrak{q}$ are co-prime. $\square$

Proposition 3.2. *A minimal (respectively, maximal) element in a connected component of $\mathcal{S}(R)$ is a minimal (respectively, maximal) element in* $\mathrm{Spec}(R)$.

Proof. We prove the claim for minimal elements. By similar arguments one can establish the result for maximal elements. So suppose that C is a connected component of $\mathcal{S}(R)$ and that $\mathfrak{p} \in C$ is a minimal element in C. Assume in contrary that $\mathfrak{p}$ is not minimal in $\mathrm{Spec}(R)$. Thus there exists a prime ideal $\mathfrak{q}$ of R such that $\mathfrak{q} \subsetneqq \mathfrak{p}$. Hence we can obtain a saturated chain of prime ideals from $\mathfrak{q}$ to $\mathfrak{p}$. This means that there is a path between $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathcal{S}(R)$, and so $\mathfrak{q} \in \mathcal{S}(R)$ which is the required contradiction. $\square$

Suppose that $\mathrm{nil}(R)$ is the nil-radical of R. The following theorem shows that in Spec-graph $\mathcal{S}(R)$, we may replace R by $R/\mathrm{nil}(R)$. So in the rest of the paper we assume that R is reduced.

Theorem 3.3. *We have the following isomorphism.*

$$\mathcal{S}(R) \cong \mathcal{S}(R/\mathrm{nil}(R))$$

Proof. Clearly $\mathrm{nil}(R) = \bigcap_{\mathfrak{p}\in \mathrm{Spec}(R)} \mathfrak{p}$. So for all $\mathfrak{p} \in \mathrm{Spec}(R)$, we have $\mathfrak{p} \supseteq \mathrm{nil}(R)$. This means that the natural homomorphism $\varphi : R \longrightarrow R/\mathrm{nil}(R)$ provide a graph isomorphism $\mathcal{S}(R) \cong \mathcal{S}(R/\mathrm{nil}(R))$. $\square$

Theorem 3.4. *Suppose that S is another non-zero commutative ring. Then $\mathcal{S}(R \times S)$ is disconnected.*

Proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R and S, respectively. Then $R \times \mathfrak{q}, \mathfrak{p} \times S \in \mathrm{Spec}(R \times S)$. If there exists a path between two vertices $R \times \mathfrak{q}$ and $\mathfrak{p} \times S$, then we have a path as follows: $\mathfrak{p} \times S = P_0 - P_1 - \cdots - P_t = R \times \mathfrak{q}$. Since $P_i = \mathfrak{p}_i \times S$, for some prime ideal $\mathfrak{p}_i$ of R $1 \leq i \leq t$. It follows that $\mathfrak{q} = S$, which is impossible. This means that two vertices $R \times \mathfrak{q}$ and $\mathfrak{p} \times S$ lie in two different connected components of $\mathcal{S}(R \times S)$, and so $\mathcal{S}(R \times S)$ is disconnected. $\square$

We use $G \oplus H$ to denote disjoint union of two graphs G and H.

Theorem 3.5. $\mathcal{S}(R \times S) \cong \mathcal{S}(R) \oplus \mathcal{S}(S)$

Proof. Consider the mapping $\varphi : \mathcal{S}(R \times S) \longrightarrow \mathcal{S}(R) \oplus \mathcal{S}(S)$ given by $\varphi(R \times \mathfrak{q}) = \mathfrak{q}$ if $\mathfrak{q} \in \mathrm{Spec}(S)$ and $\varphi(\mathfrak{p} \times S) = \mathfrak{q}$ if $\mathfrak{p} \in \mathrm{Spec}(R)$. By definition φ is one to one and onto. Also it is clear that φ is a isomorphism between two graphs, because the adjacency $\mathfrak{p} \times S \looparrowright \mathfrak{p}' \times S$ in $\mathcal{S}(R \times S)$ implies that $\mathfrak{p} \looparrowright \mathfrak{p}'$ in $\mathcal{S}(R)$. Also $R \times \mathfrak{q} \looparrowright R \times \mathfrak{q}'$ in $\mathcal{S}(R \times S)$ implies that $\mathfrak{q} \looparrowright \mathfrak{q}'$ in $\mathcal{S}(S)$. Moreover, it is easy to see that φ^{-1} is also a homomorphism between graphs. Thus $\mathcal{S}(R \times S)$ and $\mathcal{S}(R) \oplus \mathcal{S}(S)$ are isomorphic. $\square$

Corollary 3.6. *Suppose that α and β are the numbers of connected components of Spec-graph $\mathcal{S}(R)$ and $\mathcal{S}(S)$ of non-zero rings R and S, respectively. Then the number of connected component of $\mathcal{S}(R \times S)$ is equal to $\alpha + \beta$.*

Proof. It is immediate result of Theorem 3.5. $\square$

Theorem 3.7. *Suppose that $\mathrm{Min}(R)$ is a finite set. The following statements are equivalent:*

(i) $\mathcal{S}(R)$ *is disconnected.*
(ii) $R \cong A_1 \times A_2$ *for some non-zero rings* A_1 *and* A_2*.*
(iii) R *has non-trivial idempotent.*

Proof. (1)⇒(2) Let C be a connected component of $\mathcal{S}(R)$ and C^c be the complement of C. By our assumption we have that $C^c \neq \varnothing$. Now, consider two ideals

$$I := \bigcap_{\substack{\mathfrak{q}\in \mathrm{Min}(R)\\ \mathfrak{q}\in C}} \mathfrak{q} \qquad \text{and} \qquad J := \bigcap_{\substack{\mathfrak{q}\in \mathrm{Min}(R)\\ \mathfrak{q}\in C^c}} \mathfrak{q}$$

of R. Since R is reduced, we have that $I \cap J = 0$. We will show that I and J are co-prime. Assume in contrary that $I + J \neq R$. So there is $\mathfrak{r} \in \mathrm{Spec}(R)$ such that $I + J \subseteq \mathfrak{r}$. Now, two inclusions $I \subseteq \mathfrak{r}$ and $J \subseteq \mathfrak{r}$ imply that there are $\mathfrak{p} \in \mathrm{Min}(R) \cap C$ and $\mathfrak{q} \in \mathrm{Min}(R) \cap C^c$ such that $\mathfrak{p} \subseteq \mathfrak{r}$ and $\mathfrak{q} \subseteq \mathfrak{r}$. Thus there is a path joining $\mathfrak{p}$ to $\mathfrak{q}$ passing through $\mathfrak{r}$. This means that $\mathfrak{p}$ and $\mathfrak{q}$ lie in same a connected component, whereas $\mathfrak{p} \in C$ and $\mathfrak{q} \in C^c$. This is the required contradiction. Therefore $I + J = R$, and so $R \cong R/I \times R/J$.

(2)⇒(3) Clearly, the element $(0, 1) \in A_1 \times A_2$ is an idempotent .

(3)⇒(2) Let $e \in R$ be a non-trivial idempotent. Since $\mathrm{Ann}(e) \cap \mathrm{Ann}(1-e) = 0$ and $\mathrm{Ann}(e)+\mathrm{Ann}(1-e) = 0$, we have $R \cong R/\mathrm{Ann}(e)\times R/\mathrm{Ann}(1-e)$,

(2)⇒(1) It immediately follows from Theorem 3.4. □

Remark 3.8. *In view of Theorem 3.7, if* R *is a ring with finite number of minimal prime ideals, then the Zariski topology on spec*(R) *is connected if and only if the graph* $\mathcal{S}(R)$ *is connected.*

Corollary 3.9. *Suppose that* $\mathrm{Min}(R)$ *is a finite set. Then the number of connected components of* $\mathcal{S}(R)$ *is equal to number of factors in a decomposition of* R *to Cartesian product of indecomposable rings.*

Proof. Suppose that $C_1, \dots, C_k$ are connected components of $\mathcal{S}(R)$. If $k = 1$, then $\mathcal{S}(R)$ is connected, so by Theorem 3.7 R is indecomposable. Hence we assume that $k \neq 1$. For $i =, 1 \dots, k$, we set

$$I_i := \bigcap_{\mathfrak{q}\in \mathrm{Min}(R)\cap C_i} \mathfrak{q}.$$

By a method which is similar to that we used in Theorem 3.2, we have that $I_i + I_j = R$ for all $1 \leq i, j \leq k$ with $i \neq j$. Moreover $\cap_{i=1}^k I_i = 0$. Hence $R \cong R/I_1 \times \cdots \times R/I_k$. Now, by Theorem 3.5, $\mathcal{S}(R) \cong \mathcal{S}(R/I_1) \oplus \cdots \oplus \mathcal{S}(R/I_k)$. On the other hand, by applying Corollary 3.6 and that $\mathcal{S}(R)$ has k connected component, the graph $\mathcal{S}(R/I_i)$ is connected, for $i = 1, \dots, k$. Thus, in view of Theorem 3.7,

R/I_i is indecomposable, for $i = 1, \ldots, k$. In other words, k is equal to number of factors in a decomposition of R to Cartesian product of indecomposable rings. $\square$

Lemma 3.10. *Let R be a reduced ring and $f(x)$ be an idempotent element in $R[x]$. Then* $\deg(f(x)) = 0$.

Proof. Assume in contrary that $\deg(f(x)) \neq 0$ and that $a_k \neq 0$ is the leading coefficient of $f(x)$. Since $f(x)$ is idempotent, we have that $a_k^2 = 0$. But, by our assumption R is reduced. Thus $a_k = 0$, which is contradiction. $\square$

Lemma 3.11. *We have* $|\mathrm{Min}(R)| = |\mathrm{Min}(R[x])|$.

Proof. Consider the map $\varphi : \mathrm{Min}(R[x]) \longrightarrow \mathrm{Min}(R)$ given by $\varphi(P) = P \cap R$ for all P in $\mathrm{Spec}(R[x])$. Assume that P is minimal prime ideal in $R[x]$. We claim that $P \cap R$ is a minimal prime ideal in $\mathrm{Spec}(R)$. To do this, let $\mathfrak{q} \subseteq P \cap R$, for some prime ideal $\mathfrak{q}$ in R. Then we have that $\mathfrak{q}[x] \subseteq (P \cap R)[x] \subseteq P$. Thus, by the minimality assumption of P, we have $\mathfrak{q}[x] = P$. Therefore $\mathfrak{q} = P \cap R$. By a similar way, if $\mathfrak{p}$ is a minimal prime ideal in $\mathrm{Spec}(R)$, then $\mathfrak{p}[x]$ is minimal prime ideal in $\mathrm{Spec}(R[x])$. Thus $\mathfrak{p}[x] \cap R = \mathfrak{p}$. This implies that φ is onto. On the other hand, φ is one to one, because whenever $P, Q \in \mathrm{Min}(R[X])$ then we have $P \cap R = Q \cap R$. So we have $(P \cap R)[x] \subseteq P$ and $(Q \cap R)[x] \subseteq Q$, but $(P \cap R)[x]$ and $(Q \cap R)[x]$ are prime ideals in $\mathrm{Spec}(R[x])$. Now, the minimality of P, Q implies that $P = (P \cap R)[x]$ and $Q = (Q \cap R)[x]$. Therefore $P = Q$. $\square$

Theorem 3.12. *Suppose that* $\mathrm{Min}(R)$ *is a finite set. Then* $\mathcal{S}(R[x])$ *is connected if and only if* $\mathcal{S}(R)$ *is connected.*

Proof. The result follows from lemmas 3.10, 3.11 and then, Theorem 3.7. $\square$

4. DEGREE OF VERTICES AND TREE CONDITION

In this section all rings assumed to be Noetherian. Recall that, if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of R, with non saturated inclusion $\mathfrak{p} \subsetneqq \mathfrak{q}$, then there are infinite number of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$.

Lemma 4.1.
(i) *Let R be an integral domain. Then* $\mathrm{v}_{\mathcal{S}(R)}(0)$ *is equal to number of prime ideals $\mathfrak{p}$ with* $\mathrm{ht}(\mathfrak{p}) = 1$.
(ii) *Let $(R, \mathfrak{m})$ be a local ring. Then* $\dim(R) = 0$ *if and only if* $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{m}) = 0$.

In view of Theorem 2.1, $\mathcal{S}(R_{\mathfrak{p}})$ and $\mathcal{S}(R/\mathfrak{p})$ can be considered as sub-graphs of $\mathcal{S}(R)$. These two sub-graphs admit a unique vertex in common that we denote by $\mathfrak{p}$.

Theorem 4.2. *Let $\mathfrak{p}$ be a prime ideal in R. Then*

$$\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = \mathrm{v}_{\mathcal{S}(R/\mathfrak{p})}(0) + \mathrm{v}_{\mathcal{S}(R_\mathfrak{p})}(\mathfrak{p}R_\mathfrak{p}).$$

Proof. Let $\mathrm{N}_{\mathcal{S}(R)}(\mathfrak{p})$ be set of all vertices in $\mathcal{S}(R)$ which are adjacent to $\mathfrak{p}$. Put

$$\mathrm{A} := \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{p} \looparrowright \mathfrak{q}\}$$

and

$$\mathrm{B} := \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \looparrowright \mathfrak{p}\}.$$

Clearly $\mathrm{N}_{\mathcal{S}(R)}(\mathfrak{p}) = \mathrm{A} \cup \mathrm{B}$ and $\mathrm{valence}_{\mathcal{S}(R)}(\mathfrak{p}) = |\mathrm{N}_{\mathcal{S}(R)}(\mathfrak{p})| = |\mathrm{A}| + |\mathrm{B}|$. On the other hand, by Lemma 4.1, $\mathrm{v}_{\mathcal{S}(R/\mathfrak{p})}(0) = |\mathrm{A}|$ and $\mathrm{v}_{\mathcal{S}(R_\mathfrak{p})}(\mathfrak{p}R_\mathfrak{p}) = |\mathrm{B}|$. □

Corollary 4.3. *Let $\mathfrak{p}$ be a prime ideal in R. Then the following statements hold:*

(i) $\mathfrak{p} \in \mathrm{Min}(R)$ *if and only if* $\mathrm{v}_{\mathcal{S}(R_\mathfrak{p})}(\mathfrak{p}R_\mathfrak{p}) = 0$.
(ii) $\mathfrak{p} \in \mathrm{Max}(R)$ *if and only if* $\mathrm{v}_{\mathcal{S}(R/\mathfrak{p})}(0) = 0$.

Proof. We only prove the first statement. The second one is similar.

Let $\mathfrak{p}$ be a minimal prime ideal of R. Thus by using the notations in the proof of Theorem 4.2, we have that $|\mathrm{B}| = 0$. This implies that $\mathrm{v}_{\mathcal{S}(R_\mathfrak{p})}(\mathfrak{p}R_\mathfrak{p}) = 0$.
Conversely, if $\mathrm{v}_{\mathcal{S}(R_\mathfrak{p})}(\mathfrak{p}R_\mathfrak{p}) = 0$, then $\mathrm{B} = \varnothing$, and so $|\mathrm{B}| = 0$. Thus $\mathrm{ht}(\mathfrak{p}) = 0$, and hence $\mathfrak{p} \in \mathrm{Min}(R)$. □

Theorem 4.4. *Let $\mathfrak{p}$ be a prime ideal of R. Then the followings statements hold:*

(i) *If* $\dim(R/\mathfrak{p}) \geq 2$ *or* $\dim(R_\mathfrak{p}) \geq 2$, *then* $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = \infty$.
(ii) *If* $\dim(R/\mathfrak{p}) = \dim(R_\mathfrak{p}) = 1$, *then* $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = |\mathrm{Min}(R_\mathfrak{p})| + |\mathrm{Max}(R/\mathfrak{p})|$. *Moreover* $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = \infty$ *if and only if* $|\mathrm{Max}(R/\mathfrak{p})| = \infty$.
(iii) *If* $\dim(R/\mathfrak{p}) = 1$ *and* $\dim(R_\mathfrak{p}) = 0$, *then* $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = |\mathrm{Max}(R/\mathfrak{p})|$.
(iv) *If* $\dim(R/\mathfrak{p}) = 0$ *and* $\dim(R_\mathfrak{p}) = 1$, *then* $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = |\mathrm{Min}(R_\mathfrak{p})|$.

Proof. (i) If $\dim(R/\mathfrak{p}) \geq 2$, then there exists a saturated chain $\mathfrak{p} \subsetneqq \mathfrak{p}' \subsetneqq \mathfrak{p}''$ of prime ideals in R. As we mentioned at the beginning of this section, there exist infinitely many prime ideals $\mathfrak{p}'$ of R with $\mathfrak{p} \subsetneqq \mathfrak{p}' \subsetneqq \mathfrak{p}''$, and so $\mathfrak{p}$ has not finite valence in $\mathcal{S}(R)$. Similarly, one can show that the inequality $\dim(R_\mathfrak{p}) \geq 2$ implies that $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = \infty$.

(ii) Suppose that $\dim(R/\mathfrak{p}) = \dim(R_\mathfrak{p}) = 1$, and $\mathfrak{q}$ is a prime ideal of R such that $\mathfrak{q} \looparrowright \mathfrak{p}$. Hence $\mathrm{ht}_{R_\mathfrak{p}}(\mathfrak{q}R_\mathfrak{p}) = 0$. This means that $\mathfrak{q}R_\mathfrak{p}$ is a minimal prime ideal of $R_\mathfrak{p}$. Moreover, all minimal prime ideals of $R_\mathfrak{p}$ are adjacent to $\mathfrak{p}R_\mathfrak{p}$. Hence, $\mathrm{v}_{\mathcal{S}(R_\mathfrak{p})}(\mathfrak{p}R_\mathfrak{p}) = |\mathrm{Min}(R_\mathfrak{p})|$. On the other hand, $\dim(R/\mathfrak{p}) = 1$. Thus, there exists a one to one correspondence between elements in the set $\mathrm{Max}(R/\mathfrak{p})$ and prime ideal of $R/\mathfrak{p}$ of height one. Therefore, in view of Lemma 4.1, $\mathrm{v}_{\mathcal{S}(R/\mathfrak{p})}(0) = |\mathrm{Max}(R/\mathfrak{p})|$. Now the result follows from Corollary 4.3. Moreover, since a Noetherin ring admits only finitely many minimal prime ideals, we have that $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = \infty$ if and only if $|\mathrm{Max}(R/\mathfrak{p})| = \infty$.

(iii) Suppose that $\dim(R_{\mathfrak{p}}) = 0$. Thus $\mathfrak{p}$ is a minimal prime ideal of R and so $\mathfrak{p}$ is an isolated vertex in $\mathcal{S}(R_{\mathfrak{p}})$. This means that $\mathrm{v}_{\mathcal{S}(R_{\mathfrak{p}})}(\mathfrak{p}R_{\mathfrak{p}}) = 0$. Thus, by Theorem 4.2, we have $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = \mathrm{v}_{\mathcal{S}(R/\mathfrak{p})}(0)$. Now, the claim follows from a method similar to that we use in the proof of part (ii).

(iv) Suppose that $\dim(R/\mathfrak{p}) = 0$. Thus $\mathfrak{p}$ is a maximal ideal of R. Moreover $\mathfrak{p}$ is an isolated vertex in $\mathcal{S}(R/\mathfrak{p})$. Thus $\mathrm{v}_{\mathcal{S}(R/\mathfrak{p})}(0) = 0$. Now, by Theorem 4.2, we have that $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = \mathrm{v}_{\mathcal{S}(R_{\mathfrak{p}})}(\mathfrak{p}R_{\mathfrak{p}})$. On the other hand, a vertex in $\mathcal{S}(R_{\mathfrak{p}})$ is adjacent to $\mathfrak{p}$ if and only if $\mathfrak{p}$ is a minimal prime ideal of R. Thus, $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = |\mathrm{Min}(R_{\mathfrak{p}})|$. □

Corollary 4.5. *All vertices in $\mathcal{S}(R)$ have finite valency if and only if $\dim(R) \leq 1$ and $|\mathrm{Max}(R)|$ is finite.*

Proof. Suppose that $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. So, by Theorem 4.4(i), one can easily check that $\dim(R_{\mathfrak{p}}) \leq 1$ for all prime ideals $\mathfrak{p}$ of R. This implies that $\dim(R) = 1$. On the other hand, $|\mathrm{Min}(R)|$ is finite and every adjacent vertices to minimal primes are in $\mathrm{Max}(R)$. This implies that $|\mathrm{Max}(R)|$ is finite.

The converse is clear, because $\mathcal{S}(R)$ is a finite graph. □

Theorem 4.6. *Let $\mathfrak{p}$ be a prime ideal of R. If $\mathrm{v}_{\mathcal{S}(R)}(\mathfrak{p}) = 1$, then $\mathfrak{p}$ together with a vertex that is adjacent to it are minimal or maximal ideals of R.*

Proof. In view of Theorem 4.2, we have that $\mathrm{v}_{\mathcal{S}(R/\mathfrak{p})}(0) + \mathrm{v}_{\mathcal{S}(R_{\mathfrak{p}})}(\mathfrak{p}R_{\mathfrak{p}}) = 1$, and so $\mathrm{v}_{\mathcal{S}(R/\mathfrak{p})}(0) = 0$ or $\mathrm{v}_{\mathcal{S}(R_{\mathfrak{p}})}(\mathfrak{p}R_{\mathfrak{p}}) = 0$, this means that $\mathfrak{p} \in \mathrm{Min}(R)$ or $\mathfrak{p} \in \mathrm{Max}(R)$.

Now, suppose that $\mathfrak{q}$ is an adjacent vertex to $\mathfrak{p}$. So we have the following cases.

Case 1: Suppose that $\mathfrak{p}$ is a maximal ideal of R. Then in view of Theorem 4.4, $\dim(R_{\mathfrak{p}}) = 1$, and hence $\mathrm{ht}(\mathfrak{p}) = 1$. Thus $\mathrm{ht}(\mathfrak{q}) = 0$, which implies that $\mathfrak{q}$ is minimal prime ideal of R.

Case 2: Suppose that $\mathfrak{p}$ is a minimal prime ideal of R. Again in view of Theorem 4.4, we have that $\dim(R/\mathfrak{p}) = 1$. Therefore $\mathfrak{q}/\mathfrak{p}$ is a maximal ideal of $R/\mathfrak{p}$. This means that $\mathfrak{q}$ is a maximal ideal of R. □

Theorem 4.7. *The following statements hold:*

(i) *Let R be an integral domain. Then $\mathcal{S}(R)$ is a tree if and only if $\mathcal{S}(R_{\mathfrak{p}})$ is a tree for all non zero prime ideal $\mathfrak{p}$ of R. Moreover, if $\mathcal{S}(R)$ is a tree, then $\dim(R) = 1$ and $\mathcal{S}(R)$ is a star graph.*

(ii) *Let R be local. Then $\mathcal{S}(R)$ is a tree if and only if $\mathcal{S}(R/\mathfrak{p})$ is a tree for all non-zero prime ideal $\mathfrak{p}$ of R. Moreover, if $\mathcal{S}(R)$ is a tree, then $\dim(R) = 1$ and $\mathcal{S}(R)$ is a star graph.*

Proof. We prove only the statement (i). The second one is similar.

Hence suppose that R is an integral domain and that $\mathcal{S}(R)$ is a tree. By Theorem 2.1, it is easy to see that $\mathcal{S}(R_{\mathfrak{p}})$ is a tree for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Conversely, suppose that $\mathcal{S}(R_{\mathfrak{p}})$ is a tree for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Assume in contrary that $\mathcal{S}(R)$ is not a tree. Hence $\mathcal{S}(R)$ contains a cycle say C. Let $\mathfrak{p}$ be a maximal vertex in C with respect to inclusion. So there exist two distinct non-zero prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in C, such that $\mathfrak{p}_1 \hookrightarrow \mathfrak{p}$ and $\mathfrak{p}_2 \hookrightarrow \mathfrak{p}$. This implies that there exists a cycle in $\mathcal{S}(R_{\mathfrak{p}})$ passing trough vertices $0, \mathfrak{p}_1 R_{\mathfrak{p}}, \mathfrak{p}_2 R_{\mathfrak{p}}$ and $\mathfrak{p} R_{\mathfrak{p}}$. Thus $\mathcal{S}(R_{\mathfrak{p}})$ is not tree which is the required contradiction.

For the second assertion, in the case that $\dim(R) \geq 2$, as we mentioned in the beginning of this section, $\mathcal{S}(R)$ admits a cycle, which is impossible. Thus $\dim(R) = 1$. Hence all non-zero vertices in $\mathrm{Spec}(R)$ are maximal ideals of R and also they are adjacent to zero vertex. This means that $\mathcal{S}(R)$ is a star graph with center zero. □

5. Bipartition conditions on $\mathcal{S}(R)$

First of all we establish our notation. For each $i \in \mathbb{N}_0$ with $i \leq \dim(R)$, put

$$\mathrm{Spec}^i(R) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}(\mathfrak{p}) = i\}.$$

We define a simple graph denoted by $\mathcal{S}'(R)$ with vertex set $\{\mathrm{Spec}^i(R)\}_{i=1}^{\dim(R)}$, and two distinct vertices $\mathrm{Spec}^i(R)$ and $\mathrm{Spec}^j(R)$ are adjacent if and only if there is an adjacency in $\mathcal{S}(R)$, between a vertex in $\mathrm{Spec}^i(R)$ and a vertex in $\mathrm{Spec}^j(R)$. Fore two graphs $\mathcal{S}(R)$ and $\mathcal{S}'(R)$ we have the following remark.

Remark 5.1. (i) $\mathrm{Spec}^i(R)$ *is an independent set in* $\mathcal{S}(R)$*, for* $i = 0, 1, \ldots, \dim(R)$.

(ii) $|\mathcal{S}'(R)| = \dim(R)$.

(iii) *Two vertices* $\mathrm{Spec}^i(R)$ *and* $\mathrm{Spec}^{i+1}(R)$ *of* $\mathcal{S}'(R)$ *are adjacent in* $\mathcal{S}'(R)$*, for* $i = 0, \ldots, \dim(R) - 1$. *Hence, the graph* $\mathcal{S}'(R)$ *is a tree if and only if it is a path.*

We say that the condition $(*)$ holds in R, if for given prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R with $\mathfrak{p} \subseteq \mathfrak{q}$ satisfy the equality $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}) - \mathrm{ht}(\mathfrak{p})$.

Recall that a commutative ring R is Catenary, if the length of saturated chains between any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R with $\mathfrak{p} \subseteq \mathfrak{q}$, is equal to $\mathrm{ht}(\mathfrak{q}/\mathfrak{p})$. By [8, page 84] , the condition $(*)$ holds in Catenary domains. Also one can easily see that the condition $(*)$ holds in the rings of dimension one.

Theorem 5.2. *The graph $\mathcal{S}'(R)$ is bipartite if and only if $\mathcal{S}(R)$ is a bipartite graph such that all minimal prime ideals of R lie in a same part.*

Proof. Suppose that $\mathcal{S}'(R)$ is a bipartite graph and thatA and B are two parts of $\mathcal{S}'(R)$. Without loss of the generality, we may assume that $\mathrm{Spec}^0(R) \in \mathrm{A}$. This implies that the even height vertices of $\mathcal{S}'(R)$ lie in A. Therefore B contains only the odd vertices. Now, put

$$A' := \bigcup_{i \text{ is even}} \mathrm{Spec}^i(R) \qquad\qquad B' := \bigcup_{i \text{ is odd}} \mathrm{Spec}^i(R)$$

then it is easy to see that $\mathcal{S}(R)$ is bipartite graph with parts A' and B'. Moreover, it is clear that $\mathrm{Min}(R) = \mathrm{Spec}^0(R)$ lie in the part A'.

The converse statement follows from Remark 5.1. □

Corollary 5.3. *Let R be a local ring or an integral domain. If $\mathcal{S}(R)$ is bipartite, then $\mathcal{S}'(R)$ is bipartite.*

Proof. By Theorem 5.2, we only need to prove the claim in the case that $(R, \mathfrak{m})$ is local. Again by Theorem 5.2, it is enough to show that all minimal prime ideals of R lie in same part. Put $n := \mathrm{ht}(\mathfrak{m})$. By Remark 5.1, if n is odd then every minimal prime ideal of R and $\mathfrak{m}$ lie in same part; otherwise they lie in different parts. □

Theorem 5.4. *The condition $(*)$ holds in R if and only if $\mathcal{S}'(R)$ is a path.*

Proof. Suppose that the condition $(*)$ holds in R and that $\mathfrak{p}$ and $\mathfrak{q}$ two prime ideals of R such that $\mathfrak{p} \hookrightarrow \mathfrak{q}$ in $\mathcal{S}(R)$. Thus by condition $(*)$, $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + 1$. This implies that, two vertices $\mathrm{Spec}^i(R)$ and $\mathrm{Spec}^j(R)$ are adjacent in $\mathcal{S}'(R)$ if only if $|i - j| = 1$. Thus $\mathcal{S}'(R)$ has no cycle, and so by Remark 5.1(iii), $\mathcal{S}'(R)$ is a path.

Conversely, suppose that $\mathcal{S}'(R)$ is a path. For any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R, with $\mathfrak{p} \subseteq \mathfrak{q}$, we must show that $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + 1$, where $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = r$. Let $\mathrm{ht}(\mathfrak{p}) = k$. So, there exists a maximal saturated chain between $\mathfrak{p}$ and $\mathfrak{q}$ of length r as follows.

$$\mathfrak{p} \hookrightarrow \mathfrak{p}_1 \hookrightarrow \cdots \hookrightarrow \mathfrak{p}_r = \mathfrak{q}$$

Since $\mathcal{S}'(R)$ is a path, thus $\mathrm{ht}(\mathfrak{p}_i) = \mathrm{ht}(\mathfrak{p}_{i-1}) + 1$. Consequently, $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + r$ as desired. □

The following corollary immediately follows by Theorems 5.2 and 5.4.

Corollary 5.5. *If the condition* $(*)$ *holds in* R*, then* $\mathcal{S}(R)$ *is a bipartite graph.*

Theorem 5.6. *If the graph* $\mathcal{S}'(R)$ *is a path or cycle, then* R *is Catenary.*

Proof. We prove the claim for the case that $\mathcal{S}'(R)$ is a path. The proof for cycles is similar. So suppose that $\mathcal{S}'(R)$ is the following path.

$\mathrm{Spec}^0(R) \looparrowright \mathrm{Spec}^1(R) \looparrowright \cdots \looparrowright \mathrm{Spec}^{i-1}(R) \looparrowright \mathrm{Spec}^i(R) \looparrowright \mathrm{Spec}^{i+1}(R) \looparrowright \ldots.$

Now, Suppose that $\mathfrak{p}$ and $\mathfrak{q}$ are two prime ideals of R with $\mathfrak{p} \subseteq \mathfrak{q}$. Put $k := \mathrm{ht}(\mathfrak{q}/\mathfrak{p})$ and $r := \mathrm{ht}(\mathfrak{p})$. If there exists only one saturated chain between $\mathfrak{p}$ and $\mathfrak{q}$, then we have nothings to prove. Hence assume that there exist at least two saturated chains of prime ideals from $\mathfrak{p}$ to $\mathfrak{q}$ as follows

$$\mathfrak{p} \looparrowright \mathfrak{p}_1 \looparrowright \cdots \looparrowright \mathfrak{p}_k = \mathfrak{q}, \qquad \mathfrak{p} \looparrowright \mathfrak{q}_1 \looparrowright \cdots \looparrowright \mathfrak{q}_s = \mathfrak{q}$$

We need only to show that $k = s$. Since $\mathcal{S}'(R)$ is a path, $\mathfrak{p}_1, \mathfrak{q}_1 \in \mathrm{Spec}^{r+1}(R)$. By continuing this process, one can conclude that $h = s$. Thus R is catenary. □

At the end of this paper we provide some results about independent number and chromatic number of graphs $\mathcal{S}(R)$ and $\mathcal{S}'(R)$

Theorem 5.7. *If* R *is a Noetherian ring, then*

$$\alpha(\mathcal{S}(R)) = \max\{|\mathrm{Spec}^0(R)|, |\mathrm{Spec}^1(R)|\}$$

Proof. First of all, note that $\mathrm{Spec}^0(R)$ and $\mathrm{Spec}^1(R)$ are two independent sets of vertices of $\mathcal{S}(R)$. If $\dim(R) = 1$, then, by Corollary 5.5, $\mathcal{S}(R)$ is a bipartite graph. Thus $\mathcal{S}(R) = \mathrm{Spec}^0(R) \cup \mathrm{Spec}^1(R)$. So $\alpha(\mathcal{S}(R)) = \max\{|\mathrm{Spec}^0(R)|, |\mathrm{Spec}^1(R)|\}$. Now, if $\dim(R) \geq 2$, then as we mentioned at the beginning of Section 4, $\mathrm{Spec}^1(R)$ is an infinite set. Thus, in this case we have $\alpha(\mathcal{S}(R)) = \infty = \max\{|\mathrm{Spec}^0(R)|, |\mathrm{Spec}^1(R)|\}$. □

Theorem 5.8. *Let* $\{\mathrm{Spec}^{i_k}(R)\}_{k=1}^t$ *be an independent set of vertices in graph* $\mathcal{S}'(R)$ *where* $\alpha(\mathcal{S}'(R)) = t$*. Then* $\alpha(\mathcal{S}(R)) \geqslant \sum_{k=1}^t |\mathrm{Spec}^{i_k}(R)|$.

Proof. Clearly, $\bigcup_{k=1}^t \mathrm{Spec}^{i_k}(R)$ is a disjoin union of independent subsets of vertices in $\mathcal{S}(R)$. Hence, $\bigcup_{k=1}^t \mathrm{Spec}^{i_k}(R)$ is an independent set in $\mathcal{S}(R)$. Thus $\alpha(\mathcal{S}(R)) \geq |\bigcup_{k=1}^t \mathrm{Spec}^{i_k}(R)| = \sum_{k=1}^t |\mathrm{Spec}^{i_k}(R)|$. □

Theorem 5.9. $\chi(\mathcal{S}(R)) \leqslant \chi(\mathcal{S}'(R))$.

Proof. Let $\chi(\mathcal{S}'(R)) = t$. So there is a proper colouring map $f : \mathcal{S}'(R) \longrightarrow \{1, \ldots, k\}$. Since, for $i = 1, \ldots, t$, $\mathrm{Spec}^i(R)$ is an independent set of vertices in $\mathcal{S}(R)$, the map $\widehat{f} : \mathcal{S}(R) \longrightarrow \{1, \ldots, k\}$ given by $\widehat{f}(\mathfrak{p}) = f(\mathrm{Spec}^i(R))$ for $\mathfrak{p} \in \mathrm{Spec}^i(R)$, is a proper colouring of vertices in $\mathcal{S}(R)$. Thus, $\chi(\mathcal{S}(R)) \leqslant \chi(\mathcal{S}'(R))$. $\square$

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