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ON A QUESTION CONCERNING THE COHEN'S THEOREM

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Abstract. Let R be a commutative ring with identity, and let M be an R-module. The Cohen's theorem is the classic result that a ring is Noetherian if and only if its prime ideals are finitely generated. Parkash and Kour obtained a new version of Cohen's theorem for modules, which states that a finitely generated R-module M is Noetherian if and only if for every prime ideal p of R with $Ann(M) \subseteq p$, there exists a finitely generated submodule N of M such that $pM \subseteq N \subseteq M(p)$, where $M(p) = \{x \in M | sx \in pM \text{ for some } s \in R \backslash p\}$. In this paper, we prove this result for some classes of modules.

1. Introduction

Throughout this paper, R denotes a commutative ring with identity and all modules are unitary. Let N and K be two submodules of an R-module M. Then the *colon ideal of N into K* is defined to be $(N :_R K) = \{r \in R : rK \subseteq N\}$. Particularly, we use $Ann_R(M)$ instead of $(0 :_R M)$ and $(N :_R m)$ instead of $(N :_R Rm)$, where Rm is the cyclic submodule of M generated by an element $m \in M$. A submodule P of an R-module M is called *prime or p-prime* if $P \neq M$ and for $p = (P :_R M)$, whenever $re \in P$ for $r \in R$ and $e \in M$, we have $r \in p$ or $e \in P$ (see [8]). If Q is a maximal submodule of M, then Q is a prime submodule and $(Q :_R M) = m$ is a maximal ideal of R. In this case, we say Q is an *m-maximal* submodule of M (see [9]). If $p \in Spec(R)$ (resp. $m \in Max(R)$), then $Spec_p(M)$ (resp. $Max_m(M)$)

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is the set of all p-prime (resp. m-maximal) submodules of M (see [11, 9]). Also $M(p) = S_p(pM) = \{x \in M | sx \in pM \textit{ for some } s \in R \backslash p\}$ is the contraction of pM_p in M (see [10]). Early in 1950, Cohen showed that a ring R is Noetherian if and only if every prime ideal of R is finitely generated (see [6, Theorem 2]). In 1994, Smith proved that for a finitely generated module M, the following statements are equivalent (see [14]).

(1) M is Noetherian;
(2) pM is finitely generated for each prime ideal p of $V(Ann_R(M)$;
(3) $M(p)$ is finitely generated for each prime ideal p of $V(Ann_R(M))$.

In 2021, Parkash and Kour generalized the Smith's result on finitely generated modules as follows (see [13]). Let M be a finitely generated R-module, then the following are equivalent.

(1) M is Noetherian;
(2) For every prime ideal p of $V(Ann_R(M))$, there exists a finitely generated submodule N of M such that $pM \subseteq N \subseteq M(p)$.

In [13], there is a natural question which says that whether for finitely generated module M, the following statements are equivalent.

(1) M is Noetherian;
(2) For every prime ideal p of $V(Ann_R(M))$, there exists a finitely generated submodule N of M such that $(N :_R M) = p$.

Parkash and Kour have given a negative answer to this question in [13, Example 2.4]. We will give a positive answer to the above question under some conditions (see Theorem 2.2).

2. Main results

Remark 2.1. Let M be an R-module.

(a) M is said to be X-*injective* if $|Spec_p(M)| \leq 1$ for every prime ideal p of R (see [4, Definition 3.2]).
(b) M is said to be a *multiplication (weak multiplication)* module if for every submodule (prime submodule) N of M there exists an ideal I of R such that $N = IM$ (see [7, 5]).
(c) Consider the finitely generated $\mathbb{Z}$-module $M = \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i}$, where p_i's are distinct positive prime integers. Then $Spec_{\mathbb{Z}}(M) = \bigcup_{i=1}^{n} Spec_{(p_i\mathbb{Z})}(M) = \bigcup_{i=1}^{n} \{p_iM\}$. This implies that M is an X-injective $\mathbb{Z}$-module by part (a).

Theorem 2.2. *Let M be an X-injective R-module. Then the following are equivalent.*

(a) *M is Noetherian;*

(b) *For every prime ideal* p *of* $V(Ann_R(M))$, *there exists a finitely generated submodule* N *of* M *such that* $(N :_R M) = p$.

Proof. $((b) \Rightarrow (a))$. Let p be a maximal ideal of R and $p \in V(Ann_R(M))$. Then by hypothesis, there exists a finitely generated submodule N of M such that $(N :_R M) = p$. This implies that $pM \subseteq N$ and Hence $pM \neq M$. So that $(pM :_R M) = p$. Since p is a maximal ideal of R, pM is a p-prime submodule of M by [8, Proposition 2]. Now by [3, Proposition 3.3], there is a maximal submodule H of M such that $(H :_R M) = p$. Since p is maximal ideal, then by [8, Proposition 4], H and N are prime submodules of M. On the other hand, M is X-injective, so that $H = N$ by [4, Lemma 3.1]. It follows that H is a finitely generated submodule of M. But M/H is cyclic and hence M is finitely generated. Now we assume that M is not a Noetherian R-module. Then there exists a proper submodule K of M such that it is not finitely generated. Set $\Sigma = \{L \subseteq M \mid L \textit{ is a non-finitely generated submodule of } M\}$. Firstly, $\Sigma \neq \emptyset$, because $K \in \Sigma$. Secondly, if $\{L_i\}_{i \in I}$ is a chain of elements of Σ, then $\bigcup_{i \in I} L_i$ is non-finitely generated. Now by Zorne lemma, Σ has a maximal element. Let L be a maximal element of Σ. Hence L is a prime submodule of M by [8, Proposition 9]. Set $(L :_R M) = q$. Therefore by hypothesis, we have $(L' :_R M) = q$ for some finitely generated submodule L' of M. Hence $(L :_R M)M = (L' :_R M)M$. But by [4, Corollary 3.12], M is multiplication. So that we have $L = (L :_R M)M = (L' :_R M)M = L'$. This means that L is finitely generated, which is a contradiction.
$((a) \Rightarrow (b))$. This follows from [10, Corollary 3.8]. □

Corollary 2.3. *Let* M *be an* R*-module, and suppose that one of the following hold:*

(1) M *is a multiplication module;*
(2) M *is a weak multiplication module;*
(3) M *is a locally cyclic module.*

Then the following are equivalent.

(a) M *is Noetherian;*
(b) *For every prime ideal* p *of* $V(Ann_R(M))$, *there exists a finitely generated submodule* N *of* M *such that* $(N :_R M) = p$.

Proof. This is an immediate result of Theorem 2.2 by [4, Proposition 3.3 and 3.10]. □

The following example shows that the condition "M is X-injective" of Theorem 2.2 can not be dropped.

Example 2.4. Let F be a field and $R = F[[x_1, x_2, ...]]$ be the power series ring over F with intermediates $x_1, x_2, ...$ and $I =< x_1^2, x_2^2, ... >$ and $J =< x_2, x_3, ... >$ be two ideals of R. Set $M = \frac{R}{I} \times \frac{\frac{R}{I}}{\frac{J}{I}}$. Then we have the following.

(a) $Spec(R/I) = Max(R/I) = \{\frac{p}{I}\}$, where $p =< x_1, x_2, ... >$.
(b) M is a finitely generated $\frac{R}{I}$-module.
(c) $N = \frac{R}{I} \times \frac{\frac{p}{I}}{\frac{J}{I}}$ is a finitely generated $\frac{R}{I}$-submodule of M and $(N :_{\frac{R}{I}} M) = \frac{p}{I}$.
(d) $\frac{p}{I}M = \frac{p}{I} \times \frac{\frac{p}{I}}{\frac{J}{I}}$ is a non-finitely generated $\frac{R}{I}$-submodule of M and $\frac{p}{I}M :_{\frac{R}{I}} M) = \frac{p}{I}$.
(e) M is not a Noetherian $\frac{R}{I}$-module and it is not an X-injective $\frac{R}{I}$-module by part (c), (d), [8, Proposition 2] and [4, Lemma 3.1].

Remark 2.5. Let M be an R-module. Then

(a) M is said to be *Max-injective* if $|Max_p(M)| \leq 1$ for every maximal ideal p of R (see [1, 12]).
(b) M is said to be *Max-weak multiplication* R-module if either $Max(M)) = \emptyset$ or $Max(M)) \neq \emptyset$ and for every maximal submodule P of M, $P = IM$ for some ideal I of R (see [2]).

In [12], it is proved that these two classes (Max-injective and Max-weak multiplication) of modules are the same. Since every X-injective module is a Max-injective module, it seems possible to generalize Theorem 2.2 for Max-injective modules. Therefore, it is natural to ask the following question.

Question. Let M be a Max-injective R-module and let for every prime ideal p of $V(Ann_R(M))$, there exists a finitely generated submodule N of M such that $(N :_R M) = p$. Is M a Noetherian R-module?

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