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Φ-Connes amenability of l^1-Munn algebras

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ABSTRACT

Generalizing the notion of Connes amenability for LM(A, P, m, n), we study Φ- Connes amenability of LM(A, P, m, n) that Φ is a character on LM(A, P, m, n). Among the other things, we study when Φ- Connes amenability of LM(A, P, m, n) is equivalent to φ- Connes amenability of A where φ is the unique character on A associated to Φ. We apply this results to semigroup algebras.

1. Introduction

The concept of amenability first appeared in connection with paradoxical decompositions. The Banach–Tarski paradox shows that the sphere can be partitioned into finitely many sets which may be rotated and reassembled to form two copies of the original sphere. It follows that there is no finitely additive measure on the family of all subsets of the sphere which extends the familiar area measure on Borel sets. In 1972, Barry Johnson introduced the cohomological notion of an amenable Banach algebra [1]. Amenable Banach algebras have since proved themselves to be widely applicable in modern analysis. In many instances the classical concept of amenability is, however, too strong. For this reason, by relaxing some of the constrains in the definition of amenability, new concepts have been introduced. B. E. Johnson, R. V. Kadison and J. Ringrose introduced a notion of amenability for Von Neumann algebras which modifies Johnson's original definition for general Banach algebras. This notion of amenability was later dubbed Connes amenability. In [2], Runde extended the notion of Connes amenability to dual Banach algebras.

Let A be a Banach algebra. A Banach algebra A is said to be dual if there is a closed submodule A_* of A^* such that A = $(A_*)^*$. For a locally compact group G, the group algebra $l^1(G)$ that is

$$\{a = (\alpha_g)_{g\epsilon G};\ \|a\| = \Sigma_{g\, \epsilon\epsilon G} |\alpha_g| < \infty\}$$

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and the measure algebra $M(G)$ that is the algebra of all bounded, regular, complex-valued measures on G equipped with the total variation norm $\|\mu\|= |\mu(G)|$ are two examples of dual Banach algebras. A dual Banach A-bimodule E is called normal Banach A-bimodule if for each $x \in E$, the module maps $a \to a. x$ and $a \to x. a$ ($a \in$ A) are weak*-continuous. Dual Banach algebra A is called Connes amenable, if for every normal Banach A-bimodule E, every weak*-continuous derivation D : A $\to E$ is inner. In [3], Runde showed that G is amenable if and only if $M(G)$ is Connes amenable. In particular, $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable.

In [4], Eslamzadeh introduced l^1-Munn algebras. He used these algebras to characterize amenable semigroup algebras. A special case of these algebras was introduced by Munn [5]. l^1-Munn algebras has been studied in some texts. Eslamzadeh in [6] and [7] investigated the structure of l^1-Munn algebras. Duncan and Paterson used the l^1-Munn algebras to study of semigroup algebras of completely simple semigroups [8]. Recently it is shown that bounded Hochschild (co)-homology of l^1 -Munn Banach algebras are isomorphic to those of the underlying Banach algebra A when the related sandwich matrix is invertible over Inv(A) [9] and in [10] Soroushmehr investigated weak amenability of certain classes of commutative semigroup algebras.

In [11], the authors have introduced the φ-version of Connes amenability of dual Banach algebra A that φ is a homomorphism from A onto $\mathbb{C}$ that lies in predual A_*. We study the Runde's theorem for the case of semigroup algebra. We do further study of the Connes amenability of l^1-Munn algebra, in particular their σwc-virtual diagonal [12]. In this paper, we study φ-Connes amenability of l^1-Munn algebras. We use the l^1-Munn algebras to study of φ-Connes amenability of semigroup algebras. In order to do this, we follow the argument of [4] and [13].

1 Main Results

Let A be a unital Banach algebra, let I and J be nonempty sets and $P = (p_{ij}) \in M_{J\times I}$ (A) be such that $\|P\|_\infty = sup\{ \|p_{ji}\| : j \in J, i\in I \leq 1\}$. The set $M_{I\times J}$ (A) of all $I \times J$ matrices $a = (a_{ij})$ on A with l^1-norm and the product A⊙B=APB, (A, B $\in$ $M_{I\times J}$(A)) is a Banach algebra that is called l^1-Munn algebra on A with sandwich matrix P. It is denoted by LM(A, P, I, J) [4]. If index sets I and J are finite with $|I| = m$ and $|J| = n$, then we use the notation LM(A, P, m, n). Suppose ξ_{ij} is denoted the element of $M_{I\times J}(\mathbb{C})$ with 1 in (i, j)th place and 0 elsewhere. Throughout we use the notations of [4]. It is known that A** is equipped with the first and second Arens product. Indeed for each $F = (f_{ij}) \in$ LM(A, P, m, n)* and $A = (a_{ij})$, $X = (x_{ij}) \in$ LM(A, P, m, n) and M=(m_{ij}), N=$(n_{ij}) \in$ LM(A, P, m, n)**

$$\begin{aligned}\langle (F\odot A)_{ij}, x_{ij}\rangle &= \langle F, A\odot X\xi_{ij}\rangle = \Sigma_{k=1}^m\Sigma_{l=1}^n\langle F, a_{kl}p_{li}x_{ij}\rangle \\ &= \Sigma_{k=1}^m\Sigma_{l=1}^n\langle f_{kj}, a_{kl}p_{li}x_{ij}\rangle = \langle \Sigma_{k=1}^m\Sigma_{l=1}^n f_{kj}a_{kl}p_{li}, x_{ij}\rangle\end{aligned}$$

and then

$$\begin{aligned}\langle (M\odot F)_{ij}, a_{ij}\rangle &= \langle M, F\odot A\xi_{ij}\rangle = \Sigma_{r=1}^m\Sigma_{s=1}^n\langle m_{rs}, f_{is}a_{ij}p_{jr}\rangle \\ &= \Sigma_{r=1}^m\Sigma_{s=1}^n\langle a_{ij}p_{jr}, m_{rs}f_{is}\rangle = \langle \Sigma_{r=1}^m\Sigma_{s=1}^n p_{jr}m_{rs}f_{is}, a_{ij}\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\langle (N\odot M)_{ij}, f_{ij}\rangle &= \langle N, M\odot F\xi_{ij}\rangle = \Sigma_{r=1}^m\Sigma_{l=1}^n\langle n_{il}, p_{lr}m_{rj}f_{ij}\rangle \\ &= \Sigma_{r=1}^m\Sigma_{s=1}^n\langle n_{il}p_{lr}m_{rj}, f_{ij}\rangle.\end{aligned}$$

If I and J are finite, and then by [[7] Lemma 3.2], LM(A, $P, I, J)^{**}$ is topologically algebra isomorphic to LM(A^{**}, P, I, J). We define $\Gamma\ : M_{I\times J}(A_*) \to M_{I\times J}$ (A) by $< (\Gamma(f))_{ij}\ ,\ a\ \xi_{ij} > \ \to < (f_{ij}), a\ \xi_{ij} >$, then $M_{I\times J}$ (A) is a dual space with predual $M_{I\times J}(A_*)$. It is clear that multiplication in LM(A, P, I, J) is separately weak*-continuous and from Proposition 1.2 in [2], LM(A, P, I, J) is a dual Banach algebra. The character space of A is denoted by Δ (A). We may write Δ (LM(A, P, I, J)) $\subset \Delta$(A). In [14], it is shown that if $\Phi \in \Delta$(LM(A, P, I, J)) and $P = (p_{ji})$ be a sandwich matrix such that $\{p_{ji}: i \in I, j \in J\} \cap Inv(A)\neq \emptyset$ and $p_{j_0 i_0} \in \{p_{ji}:\ i \in I,\ j \in J\} \cap Inv$(A), then $\varphi \in \Delta$(A) with $\varphi(a)= \Phi(p_{(j_0 i_0)}{}^{-1}\ a\xi_{(i_0 j_0)})$ is a unique character on A that $\varphi(p_{ji}\, p_{lk}) = \varphi(p_{jk}\, p_{li})$ ($j, l \in J$, $i, k \in I$)and for each $N = (n_{ij}) \in$ LM(A, P, I, J), Φ is defined by $\Phi(N) = \Sigma_{i\in I,\, j\in J}\, \varphi(n_{ij})\varphi(p_{ji})$.

Let A be a Banach algebra. Let X be a subspace of A^* and φ be a character, that is a homomorphism from A onto $\mathbb{C}$ and lies in X. A linear functional $m \in X^{**}$ is called a mean on X if $< m, \varphi> = 1$ and $<m, f.\ a> = \varphi(a) <m, f>$ for all $a \in$ A and $f \in$ X. In [15], Lau, Kanuith and Pym have introduced the concept of φ-amenability of Banach algebra A. A Banach algebra A is called φ-amenable if there exists a left invariant φ-mean on A^*. The concept of φ-amenability of Banach algebras is characterized in terms of cohomological properties of Banach algebras. A Banach algebra A is φ-amenable if and only if for any Banach A-bimodule X, with left action $a.\ x = \varphi(a)x$ ($a \in$ A, $x \in X$), each derivation D : A $\to X^*$ is inner [15]. In [11], we introduce the concept of φ-Connes amenability of dual Banach algebras. A dual Banach algebra A is called φ-Connes amenable if for every normal Banach A-bimodule X, with left action $a.\ x = \varphi(a)x$ ($a \in$ A, $x \in X$), every bounded weak*-continuous derivation D : A $\to$ X is inner. We show that dual Banach algebra A with predual A_* is φ-Connes amenable if and only if there exists a left invariant φ-mean on A_* [Theorem 2.1 [11]].

Theorem 2.1. *Let* LM(A, P, m, n) *be a* l^1*-Munn algebra on Banach algebra* A *with sandwich matrix* P *such that* $\{p_{ji};\ 1 \le i \le m,\ 1 \le j \le n\} \cap Inv(A) \neq \emptyset$. *Let* Φ *be a character on* LM(A, P, m, n) *that* $\Phi \in \Delta$(LM(A, P, m, n)) $\cap$ LM(A_*, P, m, n) *with* $\varphi(p_{ji}p_{lk}) = \varphi(p_{jk}p_{li})$ ($1 \le j, l \le n, 1 \le i, k \le m$) *and* $\Phi(N) = \sum_{i=1}^{m}\sum_{j=1}^{n}\varphi(n_{ij})\varphi(p_{ji})$ *for each* $N = (n_{ij}) \in$ LM(A, P, m, n). *Then:*

(i) *If* LM(A, P, m, n) *is* Φ*-Connes amenable, then* A *is* φ*-Connes amenable* .

(ii) *If* A *is* φ*-Connes amenable, then there exist finite index I and J with* $|I| = m$ *and* $|J| = n$ *such that* l^1*-Munn algebra* LM(A, P, m, n) *is* Φ*-Connes amenable.*

Proof. Suppose that LM(A, P, m, n) is Φ-Connes amenable. By Theorem 2.3 in [11], there exists a linear functional $M = (m_{ij}) \in$ LM(A, $P, m, n)^{**}$ such that $<M, \Phi>= 1$ and $<M, F.\ A>= \Phi\ (A) <M,F>$ for each $A = (a_{ij}) \in$ LM(A, P, m, n) and $F = (f_{ij}) \in$ LM(A_*,P, m, n). By [Lemma 3.2 [7]], LM(A, $P, m, n)^{**}$ = LM(A^{**}, P, m, n), then we put $M = (m_{ij})$ that $m_{ij} \in A^{**}$. This follows that

$$\Sigma_{r=1}^{m}\Sigma_{s=1}^{n}\langle m_{rs}, f_{is}a_{ij}p_{jr}\rangle = \Sigma_{r=1}^{m}\Sigma_{s=1}^{n}\phi(a_{ij})\phi(p_{jr})\langle m_{rs}, f_{is}\rangle. \quad (1)$$

On the other hand there exists a net $\{M_\alpha\}_\alpha \in$ LM(A, P, m, n) such that M_α converges to M in the weak*-topology. Since $\Phi \in$ LM(A_*, P, m, n), then

$$\Sigma_{i=1}^{m}\Sigma_{j=1}^{n}m_{ij}(\phi)\phi(p_{ji}) = 1$$

Note that there exist $1 \le i_0 \le m$ and $1 \le j_0 \le n$ such that $\varphi\ (p_{j_0 i_0}) \neq 0$. For each $a \in$ A, in (1) put $A = a\xi_{i_0 j_0}$.

$\Sigma_{r=1}^{m}\langle m_{rj_0}.f_{i_0j_0}, ap_{j_0r}\rangle = \phi(a)\phi(p_{j_0i_0})\langle m_{i_0j_0}, f_{i_0j_0}\rangle$.

Now put $A = \xi_{i_0j_0}$.

$\Sigma_{r=1}^{m}\langle m_{rj_0}.f_{i_0j_0}, p_{j_0r}\rangle = \phi(p_{j_0i_0})\langle m_{i_0j_0}, f_{i_0j_0}\rangle$.

Then

$\Sigma_{r=1}^{m}\langle m_{rj_0}.f_{i_0j_0}, ap_{j_0r}\rangle = \phi(p_{j_0i_0})\langle a.m_{i_0j_0}, f_{i_0j_0}\rangle$.

This implies that

$$\varphi(p_{j_0i_o}) < a.m_{i_0j_o}, f_{i_0j_o} > - \varphi(p_{j_0i_0})\varphi(a) < m_{i_0j_o}, f_{i_0j_o} >$$
$$= \varphi(p_{j_0i_o}) < a.\, m_{i_0j_o},\ f_{i_0j_o} > \ - \textstyle\sum_{r=1}^{m}. < m_{rj_o}.f_{i_0j_o}, ap_{j_or} >$$
$$- \quad \varphi(p_{j_0i_o})\varphi(a) < m_{i_0j_o}, f_{i_0j_o} > + \textstyle\sum_{r=1}^{m}. < m_{rj_o}.f_{i_0j_o}, ap_{j_or} > = 0.$$

This means that $m_0 = \frac{m_{i_0j_0}}{\varphi(m_{i_0j_0})}$ is a φ-invariant mean on predual $A_* \subset A^*$ and so A is φ-Connes amenable [Theorem 2.1[11]].

Let A be φ-Connes amenable and $m \in A^{**}$ be a left invariant φ-mean on predual A_*. Put $M = (m_{1j})$ that $m_{1j} = m$ $(1 \le j \le n)$. Then M is a left invariant Φ-mean on predual LM(A_*, P, 1, n) and then LM(A, P, 1, n) is Φ-Connes amenable. Indeed, for each $F = (f_{1j}) \in$ LM(A_*, P, 1, n) and $A = (a_{1j}) \in$ LM(A, P, 1, n)

$$\begin{aligned}\langle (A\odot M)_{1j}, F_{1j}\rangle &= \langle (A.P.M)_{1j}, F_{1j}\rangle = \langle (A.P)_{11}(M)_{1j}, F_{1j}\rangle\\ &= \langle (\Sigma_{l=1}^{m} a_{1l}p_{l1}).m_{1j}, f_{1j}\rangle = \Sigma_{l=1}^{m}\phi(a_{1l}p_{l1})\langle m_{1j}, f_{1j}\rangle\\ &= \Sigma_{l=1}^{m}\phi(a_{1l})\phi(p_{l1})\langle m_{1j}, f_{1j}\rangle = \Phi(A)\langle (M)_{1j}, F_{1j}\rangle\end{aligned}$$

Theorem 2.2. Let LM(A, P, m, n) be a l^1-Munn algebra on Banach algebra A with sandwich matrix P such that $\{p_{ji};\ 1 \le i \le m, 1 \le j \le n\} \cap \mathrm{Inv}(A) \ne \emptyset$. Let $p_{j_0i_0} \in \{p_{ji};\ 1 \le i \le m,\ 1 \le j \le n\} \cap \mathrm{Inv}(A)$. Let Φ be a character on LM(A, P, m, n) that $\Phi \in \Delta(\mathrm{LM}(A,P, m, n)) \cap \mathrm{LM}(A_*,P, m, n)$ with $\varphi(p_{ji}\, p_{lk}) = \varphi(p_{jk}\, p_{li})$ $(1 \le j, l \le n,\ 1 \le i, k \le m)$ and $\Phi(N) = \Sigma_{i,j}\, \varphi(n_{ij})\varphi(p_{ji})$ for each $N = (n_{ij}) \in$ LM(A, P, m, n). Let A be φ-Connes amenable and let E be a normal A-bimodule with left module action
$[a_{ij}].\ x = \Phi([a_{ij}])\ x([a_{ij}] \in \mathrm{LM}(A, P, m, n),\ x \in E)$.

Then for every bounded weak-continuous derivation* $D^{\sim}$: LM(A, P, m, n) → E*, the restriction map* $D^{\sim}|_{A\xi_{i_0j_0}}$ *is inner.*

Proof. Let E be a normal Banach LM(A, P, m, n)-module with the left module action

$[a_{ij}].\ x = \Phi([a_{ij}])\ x,\ ([a_{ij}] \in LM(A, P, m, n),\ x \in E)$.

Clearly E is a normal A-bimodule with the following module action:

$$a\triangle x = \Phi(p_{j_0i_0}^{-1}a\xi_{i_0j_0})x, \quad x\triangle a = x.(p_{j_0i_0}^{-1}a\xi_{i_0j_0})$$

Now suppose that $D^{\sim}$: LM(A ,P, m, n) → E is a weak*-continuous bounded derivation and define D : A → E with $D(a) = D^{\sim}(p_{j_0i_0}{}^{-1}\ a\xi_{i_0j_0})$. Since $D^{\sim}$ is weak*-continuous, then D is weak*-continuous. For every a, $b \in$ A,

$$\begin{aligned} D(ab) &= \tilde{D}(p_{j_0i_0}^{-1}ab\xi_{i_0j_0}) = \tilde{D}(p_{j_0i_0}^{-1}a\xi_{i_0j_0}p_{j_0i_0}^{-1}b\xi_{i_0j_0}) \\ &= \tilde{D}(p_{j_0i_0}^{-1}a\xi_{i_0j_0})p_{j_0i_0}^{-1}b\xi_{ij} + p_{j_0i_0}^{-1}a\xi_{i_0j_0}\tilde{D}(p_{j_0i_0}^{-1}b\xi_{i_0j_0}) \\ &= \tilde{D}(p_{j_0i_0}{}^{-1}\, a\xi_{i_0j_0})\Delta b + \Phi(p_{j_0i_0}^{-1}\, a\xi_{i_0j_0})\tilde{D}(p_{j_0i_0}{}^{-1}a\xi_{i_0j_0}) = D(a)\Delta b + \varphi(a)D(b). \end{aligned}$$

This means that D is a bounded weak*-continuous derivation and then there exists $x_0 \in E$ such that $D(a) = \varphi(a)x_0 - x_0\Delta a$.

Therefore for each $a \in$ A

$\tilde{D}(a\xi_{i_0j_0}) = D(p_{j_0i_0}a) = \varphi(p_{j_0i_0}a)x_0 - x_0 \Delta p_{j_0i_0}a = \Phi(a\,\xi_{i_0j_0})\,x_0 - x_0.\ a\xi_{i_0j_0}$.

This completes the proof.

Like Connes amenability, the notion of a virtual diagonal adapts naturally to the context of general dual Banach algebras. Let E be a Banach A-bimodule. An element $x \in E$ is called weak*-weakly continuous if the module maps $a \to a.\ x$ and $a \to x.\ a$ ($a \in$ A) are weak*-weak continuous. The collection of all weak*-weakly continuous elements of E is denoted by $\sigma wc(E)$. It is shown that, $\sigma wc(E)^*$ is normal [16]. Let π : A⨂A → A be the multiplication map. From Corollary 4.6 in [16], π^* maps A_* into $\sigma wc((A\otimes A)^*)$. Consequently, π^{**} drops to a homomorphism $\pi_{\sigma wc}$: $\sigma wc((A\otimes A)^*)^* \to$ A. An element $M \in \sigma wc((A\otimes A)^*)^*$ is called a σwc-virtual diagonal for A, if $M.u = u.M$ and $u.\pi_{\sigma wc}(M) = u$ for every $u \in$ A. In [16], Runde showed that A is Connes amenable if and only if there is a σwc-virtual diagonal for A. An element $M \in \sigma wc((A\otimes A)^*)^*$ is called a φ-σwc-virtual diagonal for A, if $M.\ u = \varphi(u)\, M$ and $<\varphi, \pi_{\sigma wc}(M)> = 1$ for every $u \in$ A. It is shown that dual Banach algbra A is φ-Connes amenable if and only if there exists a φ-σwc-virtual diagonal for A [11].

Theorem 2.3. Let A be a Banach algebra. Let LM(A, P, I, J) be a unital dual Banach algebra. Then if LM(A, P, I, J) has a σwc-diagonal, then A has a σwc-diagonal.

Proof. Since LM(A, P, I, J) is a unital Banach algebra, then index sets I and J are finite and P is invertible, and so LM(A, P, I, J) is isometrically algebra isomorphic to $M_m\otimes$A [Lemma 3.7 [4]]. Let M be a σwc-diagonal for LM(A, P, I, J), then from [Lemma 3.2 [4]] we may write

$M = \Sigma^m_{i,j=1} (\xi_{ij} \otimes a_{ij}) \otimes \Sigma^m_{r,l=1} (\xi_{rl} \otimes b_{rl})$.

Let $c \in$ A, now write $c = \Sigma^m_{s,t=1}\ \xi_{st} \otimes c_{st}$. We have

$c= c.\ \pi_{\sigma wc}(M)= \Sigma^m_{s,t=1}\ \ \xi_{st} \otimes c_{st}.\ \Sigma^m_{i,j,l=1}\ \ (\xi_{il} \otimes a_{ij}\, b_{rl})$

$\quad = \Sigma^m_{i,j,s,l=1}\ \ (\xi_{sl} \otimes c_{si}\, a_{ij}\, b_{rl})$.

Then

$\Sigma^m_{i,j,s,t=1}\xi_{st} \otimes (c_{st} - c_{si}a_{ij}b_{rt}) = 0\ (2)$.

Hence

$c.\ M = \Sigma^m_{s,t=1}\ \ (\xi_{st} \otimes c_{st}).(\Sigma^m_{i,j=1}\ \ (\xi_{ij} \otimes a_{ij}) \otimes \Sigma^m_{r,l=1}\ \ (\xi_{rl} \otimes b_{rl}))$

$\quad =(\Sigma^m_{i,j=1} (\xi_{ij} \otimes a_{ij}) \otimes \Sigma^m_{r,l=1}\ \ (\xi_{rl} \otimes b_{rl})).\ \Sigma^m_{s,t=1}\ \ (\xi_{st} \otimes c_{st}) = M.\ c.$

Therefore

$$\Sigma^{m}_{s,t,r,j,l=1} \ (\xi_{sj} \otimes c_{st}a_{tj}) \otimes (\xi_{rl} \otimes b_{rl}) = \Sigma^{m}_{i,j,r,l,t=1} \ (\xi_{ij} \otimes a_{ij}) \otimes (\xi_{rt} \otimes b_{rl}\, c_{lt}).$$

By applying ψ, we have

$$\Sigma^{m}_{i,t,r,l,j=1} \ (\xi_{ij} \otimes \xi_{rl}) \otimes (c_{it}\, a_{tj} \otimes b_{rl}) = \Sigma^{m}_{i,j,r,l,t=1} \ (\xi_{ij} \otimes \xi_{rt}) \otimes (a_{ij} \otimes b_{rl}\, c_{lt}).$$

Suppose $c = \ \Sigma^{m}_{s,t=1}\, \xi_{st} \otimes c_{st}$ that $c_{11} = c$, $c_{st} = 0$ if $s= 1$ or $t = 1$. Then

$$\Sigma^{m}_{r,j=1} \ (\xi_{1j} \otimes \xi_{r1}) \otimes (c_{11}\, a_{1j} \otimes b_{r1}) = \Sigma^{m}_{r,j=1} \ (\xi_{1j} \otimes \xi_{r1}) \otimes (a_{1j} \otimes b_{r1}c_{11}). \qquad (3)$$

Define

$\theta : ((M_m \otimes M_m) \otimes (A \otimes A)) \rightarrow A$

$\theta\, (\Sigma^{m}_{i,j,r,l=1} \ (\xi_{ij} \otimes \xi_{rl}) \otimes (a_{ij} \otimes b_{rl})) = \Sigma^{m}_{i,j,r,l=1} \ a_{ij}\, b_{rl}.$

It is easy to see that ψ and θ are weak*-continuous. Now consider

$\lambda = \theta O \psi : (\mathcal{M}_m \widehat{\otimes} \mathcal{A}) \widehat{\otimes} (\mathcal{M}_m \widehat{\otimes} \mathcal{A}) \rightarrow \mathcal{A}$

From Lemma 4.9 in [16] λ^* maps A_* into σwc $(((M_m \otimes A) \otimes (M_m \otimes A))^*)$ and so $(\lambda^*|A_*)^*$ maps $\sigma wc(((M_m \otimes A) \otimes (M_m \otimes A))^*)^*$ into A. We apply θ on (3) and we get

$$\Sigma^{m}_{r,j=1} \ c_{11}a_{1j}b_{r1} = \Sigma^{m}_{r,j=1} \ a_{1j}b_{r1}c_{11} \qquad (4).$$

Put $M_1 = \Sigma^{m}_{r,j=1}(\xi_{1j} \otimes a_{1j}) \otimes (\xi_{r1} \otimes b_{r1})$ and $M^{\square} = \lambda(M_1)$. We obtain from (2) and (4),

$$M^{\square}.\ c = \Sigma^{m}_{r,j=1}\, a_{1j}b_{r1}.c_{11} = \Sigma^{m}_{r,j=1} \ c_{11}a_{1j}b_{r1} = c.\ M^{\square}\ ,$$

and

$$c.\pi_{\sigma wc}(M^{\square}) = \Sigma^{m}_{r,j=1} \ c_{11}a_{1j}b_{r1} = c. \square$$

In Theorem 2.3, sandwich matrix P is invertible and Munn algebra LM(A, P, I, J) is isomorphic to the matrix algebra M_n(A). It is known that M_n(A) does not have any character when $n > 1$.

Corollary 2.4. Let A be a Banach algebra. Let LM(A, P, I, J) be a unital dual Banach algebra and let LM(A, P, I, J) has a σwc-diagonal, then for each $\varphi \in \Delta(A) \cap A_*$, A has a φ-σwc-diagonal.

Proof. Since LM(A, P, I, J) has a σwc-diagonal, then by Theorem 2.3, A has a σwc-diagonal. By [Theorem 4.8 [16]], A is Connes amenable. Then for each $\varphi \in \Delta(A) \cap A_*$, A is φ-Connes amenable. Now by [Theorem 2.11 [11]], A has a φ-σwc-diagonal.□

In [8], Eslamzadeh showed that if P is regular matrix, then a nonzero functional $\Phi \in$ LM(A, $P, I, J)^*$ is a character if and only if there is a unique non-zero character φ of A associated to Φ via $\varphi(a) = \Phi(p_{j_0 i_0}{}^{-1}\, a \xi_{i_0 j_0})$. It is shown that if P is regular matrix and has a zero entry, then LM(A, P, I, J) has no nonzero character.

Example 2.5. let G be a group and consider l^1-Munn algebra LM($l^1(G)$, P, 2, 2). Then the three possible forms of regular sandwich matrices P in Munn algebra LM($l^1(G)$, P, 2, 2) are

$$P_1 = \begin{pmatrix} \delta_e & 0 \\ 0 & \delta_e \end{pmatrix} \; P_2 = \begin{pmatrix} \delta_e & 0 \\ \delta_e & \delta_e \end{pmatrix} \; P_3 = \begin{pmatrix} \delta_e & \delta_e \\ \delta_e & \delta_e \end{pmatrix}$$

Two matrices P_1, P_2 are invertible and P_3 is not invertible. Then there is not any nonzero character on Munn algebras LM($l^1(G)$, P_1, 2 ,2) and LM($l^1(G)$, P_2, 2, 2). Since P_3 is regular matrix that has not zero entry, then each character Φ on LM($l^1(G)$, P, 2, 2) is the form
$\Phi([a_{ij}]) = \Sigma^2_{i=1}\Sigma^2_{j=1}\ \varphi(a_{ij})\varphi(p_{ji})\ (a_{ij} \in l_1(G))$.

For a semigroup S and $s \in S$, we define maps L_s, $R_s : S \to S$ by $L_s(t) = st$, $R_s(t) = ts$, $t \in S$. If for each $s \in S$, R_s and L_s are finite-to-one maps, then we say that S is weakly cancellative. Before turning our results, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ [17].

Let G be a group, I and J be arbitrary nonempty sets and $G^0 = G \cup \{0\}$. Let $P_G = (a_{ij}) \in M_{J\times I}(G)$. For $x \in G$, let $(x)_{ij}$ be the element of $M_{I\times J}(G^0)$ with x in $(i,j)^{th}$ place and 0 elsewhere. As a set, S consists of collection of all these matrices $(x)_{ij}$. Multiplication in S is given by the formula
$(x)_{ij}(y)_{kl} = (x\, a_{jk}\, y)_{il}\ (x, y \in G,\ i, k \in I,\ j, l \in J)$.

We write S = M(G, P, I, J). S is called Rees matrix semigroup with sandwich matrix P. Similarly, we have the semigroup $M^0(G, P, I, J)$ where the elements of this semigroup are those of M(G, P, I, J) togheter with the element 0 so that 0 is a matrix with 0 everywhere and $P_G = (a_{ij}) \in M_{J\times I}(G^0)$ [18].

Let G be a locally compact group and consider group algebra $l^1(G)$. In the discrete group case G and $\varphi \in \Delta(l^1(G)) \cap c_0(G)$, we show that there is a left invariant φ-mean on predual $c_0(G)$ if and only if there is a left invariant φ-mean on $l^1(G)^*$ [9]. In the following we study this for Rees matrix semigroup algebras.

Theorem 2.6. Let G be a group and S = M(G, P, m, n) be a weakly cancellative Rees matrix semigroup with a zero over G. Then $\Phi \in \Delta(l^1(S)) \cap c_0(S)$ if and only if there is a character $\varphi \in \Delta(l^1(G)) \cap c_0(G)$ such that for each $f \in l^1(S)$,
$\Phi(f) = \Sigma_{i=1}^m \Sigma_{j=1}^n \phi(f_{ij})\phi(p_{ji})$
and $\varphi(p_{ji}\, p_{lk}) = \varphi(p_{li}\ p_{jk})$ $(1 \le j, l \le n, 1 \le i, k \le m)$.

Proof. Since $\Phi \in \Delta(l^1(S)) \cap c_0(S)$, we get $\Phi(\delta_0) = 0$. Write $\Phi: \frac{l^1(S)}{\mathbb{C}\delta_0} \to C, \boldsymbol{\Phi}^{\sim}(f + \delta_0) = \boldsymbol{\Phi}(f)$. Then Φ induces a character on $\frac{l^1(S)}{\mathbb{C}\delta_0}$. It is known that $\frac{l^1(S)}{\mathbb{C}\delta_0} = LM(l^1(G), P, m, n)$. This means that Φ is a character on l^1-Munn algebra LM($l^1(G)$, P, m, n). Then by [Theorem 2.1 [14]], there is a character φ on $l^1(G)$ with $\varphi(f_0) = \Phi(p_{ij}^{-1} f_0 \xi_{ij})$ and

$$\Phi(f) = \Sigma_{i=1}^m \Sigma_{j=1}^n \phi(f_{ij})\phi(p_{ji})$$

Note that $c_0(G)$ is a closed submodule of $l^1(G)^*$. Since P is considered as a matrix over $l^1(G)$ and for each $f_0 \in l^1(G)$, $<\Phi p^{-}{}_{ij}{}^1, f_0\xi_{ij}> = <\Phi, p^{-}{}_{ij}{}^1 f_0 \xi_{ij}>$, then $\varphi \in \Delta(l^1(G)) \cap c_0(G)$. □

Theorem 2.7. Let G be a group. Let S = M(G, P, m, n) be a weakly cancellative Rees matrix semigroup with a zero over G. If $\Phi \epsilon \Delta(l^1(S)) \cap c_0(S)$ and $l^1(S)$ is Φ-Connes amenable, then there is a character $\varphi \epsilon \Delta(l^1(G)) \cap c_0(G)$ such that $l^1(G)$ is φ-Connes amenable.

Proof. Since $\Phi(\delta_0) = 0$, then each $\Phi \epsilon \Delta(l^1(S)) \cap c_0(S)$ can be extended to a character on $\frac{l^1(S)}{\mathbb{C}\delta_0}$. By [Theorem 3.4 [11]], $\frac{l^1(S)}{\mathbb{C}\delta_0}$ is $\boldsymbol{\Phi}^{\sim}$-Connes amenable. It is known that $\frac{l^1(S)}{\mathbb{C}\delta_0} = LM(l^1(G), P, m, n)$, then l^1-Munn algebra $LM(l^1(G), P, m, n)$ is $\Phi^{\sim}$- Connes amenable. By Theorem 2.6 there is a character φ $\epsilon\Delta(l^1(G)) \cap c_0(G)$ such that for each $f \epsilon l^1(S)$,

$$\tilde{\Phi}(f) = \Sigma_{i=1}^{m} \Sigma_{j=1}^{n} \phi(f_{ij})\phi(p_{ji})$$

and $\varphi(p_{ji}\, p_{lk}) = \varphi(p_{li}\, p_{jk})$ $(1 \le j, l \le n, 1 \le i, k \le m)$. Therefore Theorem 2.1 implies that $l^1(G)$ is φ-Connes amenable.□

Theorem 2.8. Let G be a group. The following are equivalent:

(i) There exists finite index I and J such that for semigroup S = M(G, P, I, J), there exists a left invariant Φ-mean on $l^1(S)^*$.

(ii) There exists finite index I and J such that for semigroup S = M(G ,P, I, J), there exists a left invariant Φ-mean on $c_0(S)$.

Proof. Let $\Phi \epsilon \Delta(l^1(S)) \cap c_0(S)$ and $l^1(S)$ be Φ-Connes amenable. From Theorem 2.7, there is a character $\varphi \epsilon\Delta(l^1(G)) \cap c_0(G)$ such that $l^1(G)$ is φ-Connes amenable and from [Corollary 2.9 [11]] $l^1(G)$ is φ-amenable. Therefore Theorem 2.1 implies that there exist finite index I and J with $|I| = m$ and $|J| = n$ that $LM(l^1(G), P, m, n)$ is Φ-amenable. Now since $\frac{l^1(S)}{\mathbb{C}\delta_0} = LM(l^1(G), P, m, n)$, then [Proposition 3.5 [15]] implies that $l^1(S)$ is Φ-amenable. This means that there exists a left invariant Φ-mean on $l^1(S)^*$. □

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