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REMARKS ON THE SUM OF ELEMENT ORDERS OF NON-GROUP SEMIGROUPS

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Abstract. The invariant $\psi(G)$, the *sum of element orders* of a finite group G will be generalized and defined for the finite non-group semigroups in this paper. We give an appropriate definition for the order of elements of a semigroup. As well as in the groups we denote the sum of element orders of a non-group semigroup S, which may possess the zero element and/ or the identity element, by $\psi(S)$. The non-group monogenic semigroup will be denoted by $C_{n,r}$ where $2 \leq r \leq n$. In characterizing the semigroups $C_{n,r}$ we give a suitable upper bound and a lower bound for $\psi(C_{n,r})$, and then investigate the sum of element orders of the semi-direct product and the wreath product of two semigroups of this type. A natural question concerning this invariant may be posed as "For a finite non-group semigroup S and the group G with the same presentation as the semigroup, is $\psi(S)$ equal to $\psi(G)$ approximately?" We answer this question in part by giving classes of non-group semigroups, involving an odd prime p and satisfying $\lim_{p\to\infty} \frac{\psi(S)}{\psi(G)} = 1$. As a result of this study, we attain the sum of element orders of a wide class of cyclic groups, as well.

1. Introduction

The sum of element orders of a finite group G denoted by $\psi(G)$ is defined to be $\psi(G) = \sum_{g\in G} o(g)$. This numerical invariant was studied for finite groups during the years. As a short chronological report on the interesting results of $\psi(G)$, one may consult [1, 6, 8, 9, 11, 15],

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for example. Obviously, this definition is not perfectly adequate for a non-group semigroup (or monoid) because of the lack of inverse elements and probable existence of zero element, identity element and the idempotent elements, in general. So, we present the following suitable definition:

Definition 1.1. For a finite non-group semigroup (or monoid) S, the order of an element $x \in S$ is defined to be:

$$o(x) = \begin{cases} 1, & \text{if } x^2 = x, \\ n, & \text{otherwise,} \end{cases}$$

where r and $n - r + 1$ are the least integers satisfying the relator $x^{n+1} = x^r$ and called the index and the period of x, respectively.

Throughout this paper, every semigroup is finite and non-group, which may possess the zero element, the identity element and idempotent elements. Our notation is merely standard. As usual, the cyclic group of order n will be denoted by C_n. We follow [5, 7] for the preliminaries on the semigroup theory and recall the following applicable definition.

Definition 1.2. For a semigroup S without the identity element, S^1 is the semigroup $S \cup \{1\}$ where 1 is the identity adjoint. Three equivalence relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$, called the Green's relations are defined on S as $a\mathcal{L}b \Leftrightarrow S^1a = S^1b$, $a\mathcal{R}b \Leftrightarrow aS^1 = bS^1$ and $a\mathcal{J}b \Leftrightarrow S^1aS^1 = S^1bS^1$, respectively. The equivalence classes are indeed, $L_x = \{y \mid x\mathcal{L}y\}$, $R_x = \{y \mid x\mathcal{R}y\}$ and $J_x = \{y \mid x\mathcal{J}y\}$, respectively.

Note that, there are two more Green's relations $\mathcal{H}$ and $\mathcal{D}$ defined with respect to the above relations. The Green's equivalence relations are indeed congruence relations as well. Note that, all of the five Green's relations of a finite commutative semigroup coincide. Also, we follow [10, 14, 2] and recall the notion of a *presentation* $\pi = \langle X \mid R \rangle$ of a set of formal generators X and a set of relators R where $\langle X \mid R \rangle$ is defined appropriately for finitely generated groups, semigroups and monoids. As usual, a semigroup S presented by π will be denoted by $S = Sg(\pi)$ and a group G presented by π, by $G = gp(\pi)$. Evidently, G is a homomorphism image of S. For more information on the presentation of algebraic structures, one may consider the mentioned articles.

As a preliminary result of the Definition 1.1, we deduce that for a finite band S, evidently $\psi(S) = |S|$ (a difference between the semigroups and groups, where for a finite group G the equality $\psi(G) = |G|$ is hold if and only if G is the trivial group).

2. The semigroup $C_{n,r}$

For an integer $n \geq 2$ and any integer $2 \leq r \leq n$ let $S = C_{n,r} = \langle a \mid a^{n+1} = a^r \rangle$. Obviously, if $2 \leq r \leq n$, $C_{n,r}$ is a non-group semigroup. Indeed, there are $n-1$ non-isomorphic monogenic semigroups of order $n \geq 2$ in contrast the fact that, there is a unique cyclic group of a given order n up to isomorphism. Our main results in this section are:

Theorem 2.1. *For every positive integers r and n where $n \geq 11$ and $2 \leq r \leq n$, let $S = C_{n,r}$. Then,*

(i). *$4n \leq \psi(S) \leq n^2 - n - 2$, if n is even,*
(ii). *$3n \leq \psi(S) \leq n^2 - 3n - 1$, if n is odd.*

For the initial values $n = 2, 3, \ldots, 10$ and every r $(2 \leq r \leq n)$ we will give the explicit values of $\psi(C_{n,r})$ at the end of this section. We'll also specify all of the monogenic semigroups of order at most 10 which are non-isomorphic but are of the same order and with the same sum of element orders.

Corollary 2.2. *For every odd integer $\alpha \geq 3$ let $n = 2^{\alpha}$. Then, $\psi(C_{n,2}) = n^2 - 2n + 3$ and $\psi(C_{n-1}) = n^2 - 3n + 3$.*

Some preliminary results are needed to prove the assertions. The first lemma concerns a property of numbers in which the notation $[x]$ is used for the integer part of a real number x.

Lemma 2.3. *Let $n \geq 3$ be an integer. For every integer i, $(2 \leq i \leq n)$, there exits an integer j such that $i[\frac{n}{i}] = n - j$. Moreover, any proper divisor k of n satisfies $k \leq \frac{n}{2}$ (or $k \leq \frac{n+1}{2}$) if n is even (or odd).*

Proof. By the definition of $[x]$, $[\frac{n}{i}] \leq \frac{n}{i} < [\frac{n}{i}] + 1$. Hence, $i[\frac{n}{i}] \leq n < i[\frac{n}{i}] + i$, which yields in turn one of the inequalities $i[\frac{n}{i}] \leq n < i[\frac{n}{i}] + j$ where, $j = 0, 1, \ldots, i-1$. So, $i[\frac{n}{i}] = n - j$. The last part may be checked easily by a contradiction method. □

The next lemma identifies the idempotent elements of the semigroup $C_{n,r}$ for possible values of r and n.

Lemma 2.4. *Consider the semigroup $C_{n,r}$ where $n \geq 8$ and $2 \leq r \leq n$. Then,*

(i). *for every r, where $2 \leq r \leq [\frac{n}{2}]$, a^{n-r+1} is an idempotent element,*
(ii). *if $r = [\frac{n}{2}] + 1$, then a^r (or a^n) is an idempotent element if n is odd (or even),*
(iii). *if $r = [\frac{n}{2}] + 2$, then a^{n-1} (or a^{n-2}) is an idempotent element if n is odd (or even).*

Proof. The condition in (i) yields that $n-2r \geq 0$, and then by letting $e = a^{n-r+1}$ we get

$$e^2 = a^{2(n-r+1)} = a^{n+1+(n-2r+1)} = a^{r+(n-2r+1)} = e.$$

The assertions (ii) and (iii) may be verified in a similar way. For instance, if $r = [\frac{n}{2}]+1$, then for any odd integer n, $r = \frac{n+1}{2}$ and $a^{2r} = a^{n+1} = a^r$. □

The following lemma computes $\psi(L_{a^r})$ and is a key lemma in proving the main results.

Lemma 2.5. *For a given integer $n \geq 8$ if $2 \leq r \leq [\frac{n}{2}]-2$, then $\psi(L_{a^r}) = 1+2\alpha$, where*

$$\alpha = \begin{cases} \displaystyle\sum_{i=r}^{\frac{n-r}{2}} o(a^i) + \sum_{i=n-r+2}^{n-r} o(a^i), & \text{if } n-r \text{ is even,} \\ \displaystyle\sum_{i=r}^{\frac{n-r-1}{2}} o(a^i) + \sum_{i=n-r+2}^{n-r} o(a^i) + \frac{1}{2} o(a^{\frac{n-r+1}{2}}), & \text{if } n-r \text{ is odd.} \end{cases}$$

Proof. By Lemma 2.4(i), $e = a^{n-r+1}$ is an idempotent element. Since L_{a^r} is a group, then e is the identity element of this group. We may easily check that the inverse of any element $a^i \in L_{a^r}$ is equal to:

$$(a^i)^{-1} = \begin{cases} a^{n-r+1-i}, & \text{if } r \leq i \leq n-2r+1, \\ a^{2n-2r+2-i}, & \text{if } n-2r+2 \leq i \leq n-r. \end{cases}$$

So,

$$o(a^i) = \begin{cases} o(a^{n-r+1-i}), & \text{if } r \leq i \leq n-2r+1, \\ o(a^{2n-2r+2-i}), & \text{if } n-2r+2 \leq i \leq n-r. \end{cases}$$

We consider two cases in partitioning L_{a^r}.
Case 1: ($n-r$ is even). $L_{a^r} = A_1 \cup A_1' \cup A_2 \cup \{a^{n-r-1}\} \cup A_2'$, where

$$A_1 = \{a^i \mid r \leq i \leq \frac{n-r}{2}\}, \quad A_1' = \{a^i \mid \frac{n-r+2}{2} \leq i \leq n-2r+1\},$$

$$A_2 = \{a^i \mid n-2r+2 \leq i \leq n-r\}, \quad A_2' = \{a^i \mid n-r+2 \leq i \leq n\}.$$

Case 2: ($n-r$ is odd). $L_{a^r} = A_3 \cup \{a^{\frac{n-r+1}{2}}\} \cup A_3' \cup A_2 \cup \{a^{n-r-1}\} \cup A_2'$, where

$$A_3 = \{a^i \mid r \leq i \leq \tfrac{n-r-1}{2}\}, \quad A_3' = \{a^i \mid \tfrac{n-r+3}{2} \leq i \leq n-2r+1\}.$$

Consequently, $\displaystyle\sum_{x \in A_i} o(x) = \sum_{x \in A_i'} o(x)$, for $i = 1,2,3$. Since $o(e) = 1$, the results follow at once. □

A semigroup S is said to be nilpotent if $|S^m| = 1$, for some integer $m \geq 2$, where $S^i = \{s_1.s_2 \dots s_i \mid s_1, s_2, \dots, s_i \in S\}$. The least integer m is called the *nilpotency rank* of S. The following lemma studies the behavour of the elements of a non-group nilpotent semigroup.

Lemma 2.6. *The semigroup $C_{n,n} = \langle a \mid a^{n+1} = a^n \rangle$ satisfies the following properties, for every integer $n \geq 8$.*

(i). *$C_{n,n}$ is a non-group nilpotent semigroup with the zero element a^n.*

(ii). *For every i where $2 \leq i \leq n-1$, $o(a^i) = [\frac{n}{i}]$ (or $o(a^i) = [\frac{n}{i}]+1$) if $i \mid n$ (or $i \nmid n$).*

(iii). *For even n, $o(a^2) = \frac{n}{2}$ and for every i where $\frac{n}{2} \leq i \leq n-1$, $o(a^i) = 2$. Moreover, $o(a^i) > \frac{n}{i} - 1$ if $3 \leq i \leq \frac{n}{2} - 1$.*

(iv). *For odd n, $o(a^2) = \frac{n+1}{2}$ and for every i where $\frac{n+1}{2} \leq i \leq n-1$, $o(a^i) = 2$. Moreover, $o(a^i) > \frac{n}{i} - 1$ if $3 \leq i \leq \frac{n-1}{2}$.*

Proof. Let $S = C_{n,n}$. Since $a^{n+1} = a^n$, we get $S \supset S^2 \supset \cdots \supset S^{n-1} \supset S^n = \{a^n\}$. The element a^n is the zero element of S, for the relator $a^k a^n = a^n a^k = a^{n+k} = a^n$ holds for every positive integer k. To prove (ii), if $i \mid n$ then the relator

$$(a^i)^{[\frac{n}{i}]+1} = a^{i(\frac{n}{i}+1)} = a^{n+i} = a^n = a^{i(\frac{n}{i})} = (a^i)^{[\frac{n}{i}]}$$

proves that $o(a^i) = [\frac{n}{i}]$. For the values of i satisfying $i \nmid n$ proof is similar. The assertions (iii) and (iv) may be verified immediately by using (ii), the Lemma 2.4 and considering the well-known properties of the integer part function $[x]$. □

Lemma 2.7. *For every integer $n \geq 8$,*

$$\psi(C_{n,2}) = \begin{cases} 5n - 3 + 2 \times \sum_{i=3}^{\frac{n-2}{2}} o(a^i), & \text{if } n \text{ is even,} \\ 4n + 2 \times \sum_{i=3}^{\frac{n-3}{2}} o(a^i), & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The relator $a^{n+1} = a^2$ yields the relator $a^{kn} = a^k$, for every positive integer $k > 1$. Hence, $o(a^{n-2}) = n-1$ and $o(a^{n-1}) = 1$ hold in both cases. Suppose that n is even. Since $a^{2n} = a^2$, then $o(a^2) = n-1$ in this case. We now use the Lemma 2.6 and substitute for $o(a)$, $o(a^2)$,

$o(a^{n-2})$ and $o(a^{n-1})$. Then,

$$\begin{aligned}\psi(C_{n,2}) &= o(a)+2o(a^2)+2o(a^{n-2})+o(a^{n-1})+2\sum_{i=3}^{\frac{n-2}{2}} o(a^i)\\ &= n+4(n-1)+1+2\sum_{i=3}^{\frac{n-2}{2}} o(a^i)\\ &= 5n-3+2\sum_{i=3}^{\frac{n-2}{2}} o(a^i).\end{aligned}$$

For odd values of n we get that $o(a^2)=\frac{n-1}{2}$ and $o(a^{\frac{n-1}{2}})=2$. As in the first case, the Lemma 2.6 gives us:

$$\begin{aligned}\psi(C_{n,2}) &= o(a)+2o(a^{n-2})+2\times o(a^{n-1})+o(a^{\frac{n-1}{2}})+2\sum_{i=2}^{\frac{n-3}{2}} o(a^i)\\ &= n+2(n-1)+1+2+2(\frac{n-1}{2})+2\sum_{i=3}^{\frac{n-3}{2}} o(a^i)\\ &= 4n+2\sum_{i=3}^{\frac{n-3}{2}} o(a^i).\end{aligned}$$

□

Proof of Theorem 2.1. The semigroup S may be decomposed as $\{a\}\cup\{a^2\}\cup\cdots\cup\{a^{r-1}\}\cup\{a^r,\ldots,a^n\}$. So, the left Green's classes are $L_a, L_{a^2},\ldots,L_{a^{r-1}}, L_{a^r}$ where, $L_a=\{a,a^2,\ldots,a^n\}$ and

$$\begin{aligned}&L_{a^i}=\{a^i,a^{i+1},\ldots,a^n\},(2\le i\le r-1),\\ &L_{a^r}=L_{a^{r+1}}=\cdots=L_{a^n}=\{a^r,\ldots,a^n\}.\end{aligned}$$

Consequently, $L_a\supset L_{a^2}\supset\cdots\supset L_{a^{n-1}}\supset L_{a^n}$. Hence, $\psi(C_{n,n})\le\psi(S)\le\psi(C_{n,2})$.

To complete the proof we use the Lemmas 2.6 and 2.7. First of all note that the relator $a^{kn}=a^k$ holds in the semigroup $C_{n,n}$ for every positive integer $k\ge 2$ and then, for each n (odd or even) $o(a)=n$ and

$o(a^n) = 1$. If n is even the Lemma 2.6(iii) gives us:

$$\begin{aligned}\psi(C_{n,n}) &= 1 + \frac{n}{2} + \sum_{i=3}^{\frac{n}{2}-1} o(a^i) + \sum_{i=\frac{n}{2}}^{n-1} o(a^i) + n,\\ &= \frac{3n+2}{2} + \sum_{i=3}^{\frac{n}{2}-1} o(a^i) + \sum_{i=\frac{n}{2}}^{n-1} 2,\\ &= \frac{5n+2}{2} + 2\frac{n}{2} + \sum_{i=3}^{\frac{n}{2}-1} o(a^i),\\ &= \frac{7n+2}{2} + \sum_{i=3}^{\frac{n}{2}-1} o(a^i).\end{aligned}$$

Since,

$$\sum_{i=3}^{\frac{n}{2}-1} o(a^i) > (\frac{n}{3} - 1) + (\frac{n}{4} - 1) + \cdots + (\frac{n}{\frac{n}{2}-1} - 1) > (\frac{n}{2} - 3)(\frac{n}{\frac{n}{2}-1} - 1)$$

then $\psi(C_{n,n}) > \frac{4(n^2-2n-2)}{n-2}$ which in turn yields

$$\psi(C_{n,n}) \geq [\frac{4(n^2-2n-2)}{n-2}] + 1 = [4n - \frac{8}{n-2}] + 1 = 4n - 1 + 1 = 4n.$$

Similarly, the Lemma 2.6(iv) yields the following result in the case when n is an odd integer.

$$\begin{aligned}\psi(C_{n,n}) &= 1 + \frac{n+1}{2} + \sum_{i=3}^{\frac{n-1}{2}} o(a^i) + \sum_{i=\frac{n+1}{2}}^{n-1} o(a^i) + n,\\ &= \frac{3n+3}{2} + \sum_{i=3}^{\frac{n-1}{2}} o(a^i) + \sum_{i=\frac{n+1}{2}}^{n-1} 2,\\ &= \frac{5n+1}{2} + \sum_{i=3}^{\frac{n-1}{2}} o(a^i).\end{aligned}$$

As well as in the last case,

$$\sum_{i=3}^{\frac{n-1}{2}} o(a^i) > (\frac{n-5}{2})(\frac{n}{\frac{n-1}{2}} - 1)$$

Hence, $\psi(C_{n,n}) > \frac{3n^2-4n-3}{n-1}$. So,

$$\psi(C_{n,n}) \geq [\frac{3n^2-4n-3}{n-1}]+1 = [3n - \frac{4}{n-1}]+1 = 3n-1+1 = 3n.$$

On the other hand, since $o(a^i) < n$ then the Lemma 2.7 gives us:

$$\psi(C_{n,2}) = \begin{cases} 5n-3+2\sum_{i=3}^{\frac{n-2}{2}} o(a^i) < 5n-3+2n(\frac{n-2}{2}-2), & \text{if } n \text{ is even} \\ \\ 4n+2\sum_{i=3}^{\frac{n-3}{2}} o(a^i) < 4n+2n(\frac{n-3}{2}-2), & \text{if } n \text{ is odd.} \end{cases}$$

Consequently,

$$\psi(C_{n,2}) < \begin{cases} n^2-n-3, & \text{if } n \text{ is even} \\ n^2-3n, & \text{if } n \text{ is odd,} \end{cases}$$

and the proof is completed. □

Proof of Corollary 2.2. We have $\psi(C_{n,2}) = 5n-3+2\sum_{i=3}^{\frac{n-2}{2}} o(a^i)$, by Lemma 2.7. On the other hand,

$$(a^i)^n = a^{i(n+1-1)} = a^{i(n+1)-i} = a^{2i-i} = a^i.$$

So, $o(a^i) = n-1$, for each $3 \leq i \leq \frac{n-2}{2}$. This yields in turn

$$\psi(C_{n,2}) = 5n-3+(n-1)(n-6) = n^2-2n+3.$$

Finally, $\psi(C_{n-1}) = n^2-3n+3$ is a result of $C_{n,2} = \{a\} \cup C_{n-1}$, because of $o(a) = n$. □

As a complement of the Theorem 2.1 the sum of element orders of all monogenic semigroups of order less than 11 are computed here by an almost tedious hand calculation and using GAP [16]. The results are gathered in the following table. As indicated in this table there are six pairs of monogenic semigroups such that the semigroups of each pair are non-isomorphic with the same order and of the same sum of element order. Indeed, $\psi(C_{3,2}) = \psi(C_{3,3}) = 6$, $\psi(C_{4,3}) = \psi(C_{4,4}) = 9$, $\psi(C_{5,2}) = \psi(C_{5,3}) = 16$, $\psi(C_{5,4}) = \psi(C_{5,5}) = 13$, $\psi(C_{7,6}) = \psi(C_{7,7}) = 21$, $\psi(C_{8,5}) = \psi(C_{8,6}) = 30$.

TABLE 1.

(n,r)	$\psi(C_{n,r})$	(n,r)	$\psi(C_{n,r})$	(n,r)	$\psi(C_{n,r})$
$(2,2)$	3	$(7,2)$	28	$(9,4)$	37
$(3,2)*$	6	$(7,3)$	34	$(9,5)$	49
$(3,3)*$	6	$(7,4)$	26	$(9,6)$	36
$(4,2)$	11	$(7,5)$	25	$(9,7)$	35
$(4,3)*$	9	$(7,6)*$	21	$(9,8)$	30
$(4,4)*$	9	$(7,7)*$	21	$(9,9)$	29
$(5,2)*$	16	$(8,2)$	51	$(10,2)$	71
$(5,3)*$	16	$(8,3)$	31	$(10,3)$	58
$(5,4)*$	13	$(8,4)$	41	$(10,4)$	69
$(5,5)*$	13	$(8,5)*$	30	$(10,5)$	43
$(6,2)$	27	$(8,6)*$	30	$(10,6)$	52
$(6,3)$	20	$(8,7)$	26	$(10,7)$	42
$(6,4)$	19	$(8,8)$	24	$(10,8)$	40
$(6,5)$	17	$(9,2)$	52	$(10,9)$	35
$(6,6)$	16	$(9,3)$	60	$(10,10)$	33

3. SEMIDIRECT PRODUCT AND WREATH PRODUCT

The *direct product* of two semigroups, denoted by $S \times T$ is the set $S \times T$ with the binary operation $(s_1,t_1)(s_2,t_2) = (s_1s_2, t_1t_2)$. Evidently, $S \times T = T \times S$ dose not hold in general and the semigroup $S \times T$ is commutative if and only if both of the semigroups are commutative.

We follow [3, 5] and adopt the definitions of semidirect product and the wreath product for a non-group semigroup $S = C_{n_1,r_1} = \langle a \rangle$ by a non-group monogenic semigroup $T = C_{n_2,r_2} = \langle b \rangle$, where $2 \leq r_i \leq n_i$ $(i = 1, 2)$.

Define the semigroup homomorphism $\varphi : T \to End(S)$ by $\varphi_{b^j}(a^i) = a^{ij\alpha}$, where $\alpha = n_1 - r_1 + 1$. Then, the multiplication in the set $S \times T$ will be defined by $(a^i, b^j)(a^{i'}, b^{j'}) = (a^{i+ij\alpha}, b^{j+j'})$, for every $a^i \in S$ and $b^j \in T$. The non-commutative semigroup defined by this multiplication is called the *semidirect product* of S by T. Since a^α is an idempotent of S, the multiplication may be simplified as $(a^i, b^j)(a^{i'}, b^{j'}) = (a^{i+\alpha}, b^{j+j'})$. The *standard wreath product* (or simply, the wreath product) of S by T, denoted by $S \wr T$ is defined to be the semi-direct product $S_1 \rtimes T$, where $S_1 = \underbrace{S \times S \times \cdots \times S}_{|T|-copies}$. The wreath product is also a non-commutative semigroup and the multiplication in $S \wr T$ may be defined by

$$(s, b^j)(s', b^{j'}) = (a^{i_1 + ji'_1\alpha}, a^{i_2 + ji'_2\alpha}, \ldots, a^{i_{n_2} + ji'_{n_2}\alpha}, b^{j+j'}),$$

where $s = (a^{i_1}, a^{i_2}, \ldots, a^{i_{n_2}})$ and $s' = (a^{i'_1}, a^{i'_2}, \ldots, a^{i'_{n_2}})$ belong to S_1 and $b^j, b^{j'} \in T$. The simplified and applicable form of this multiplication is indeed,

$$(s, b^j)(s', b^{j'}) = (a^{i_1+\alpha}, a^{i_2+\alpha}, \ldots, a^{i_{n_2}+\alpha}, b^{j+j'}).$$

The following theorem is our first result in this section concerning these products.

Theorem 3.1. *Let* $S = C_{n_1,r_1} = \langle a\rangle$ *and* $T = C_{n_2,r_2} = \langle b\rangle$ *where* $2 \leq r_i \leq n_i$. *Then,*

(i). $\psi(S \rtimes T) = n_1 \times (1 + \psi(T))$.
(ii). $\psi(S \wr T) = n_1^{n_2} \times (1 + \psi(T))$.

Proof. To prove (i) let $x = (a^i, b^j) \in S \rtimes T$. By an induction method we may prove that $x^k = (a^{i+\alpha}, b^{kj})$, for every positive integer k. Let $\ell = o(x)$ which should be the least positive integer satisfying $x^{\ell+1} = x^s$, for some positive integer $s \leq n_2$. This yields in turn the relator $b^{(\ell+1)j} = b^{sj}$. There are two possible cases $j = n_2 - r_2 + 1$ and $j \neq n_2 - r_2 + 1$ to consider. In the first case, $x^3 = x^2$ holds for every $x = (a^i, b^{n_2-r_2+1})$ and then $o(x) = 2$. However, in the second case suppose that $o(b^j = \ell$. Then, ℓ is the least positive integer such that for some positive integer $s \leq \ell$ the relator $b^{\ell+1} = b^s$ holds. Consequently, $x^{\ell+1} = (a^{i+\alpha}, b^{(\ell+1)j}) = (a^{i+\alpha}, b^s) = x^s$. So, $o(x) = o(b^j)$, for every $i \in \{1, 2, \ldots, n_1\}$ and every $j \neq n_2 - r_2 + 1$. So,

$$\begin{aligned}\psi(C_{n_1,r_1} \rtimes C_{n_2,r_2}) &= \sum_{i=1}^{n_1}\sum_{i=2}^{n_2} o(a^i, b^j) \\ &= \sum_{i=1}^{n_1}\left(\sum_{j=1, j\neq n_2-r_2+1}^{n_2} o(b^j)\right) + \sum_{i=1}^{n_1} 2 \\ &= \sum_{i=1}^{n_1} (\psi(C_{n_2,r_2}) - 1) + 2n_1 \\ &= n_1.(1 + \psi(C_{n_2,r_2})).\end{aligned}$$

To prove (ii) let $x = (t, b^j) \in T \wr S$, where $t = (a^{i_1}, a^{i_2}, \ldots, a^{i_{n_2}})$ and $1 \leq i_1, i_2, \ldots, i_{n_2} \leq n_1$. Let $T_1 = T \times T \times \cdots \times T$ (direct product of n_2-copies). Then, for every positive integer k the relator,

$$x^k = (a^{i_1+\alpha}, a^{i_2+\alpha}, \ldots, a^{i_{n_2}+\alpha}, b^{kj}),$$

may be proved by an induction method on k. As it is the case in (i), we conclude $x^3 = x^2$, for every $x = (t, b^{n_2-r_2+1})$. Hence, $o(x) = 2$ and then,

$$\begin{aligned}\psi(C_{n_1,r_1} \wr C_{n_2,r_2}) &= \sum_{t\in T_1, 1\leq j\leq n_2} o(t, b^j) \\ &= \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_1}\cdots\sum_{i_{n_2}=1}^{n_1}\left(\sum_{j=1, j\neq n_2-r_2+1}^{n_2} o(b^j)+2\right) \\ &= \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_1}\cdots\sum_{i_{n_2}=1}^{n_1}(-1+\psi(C_{n_2,r_2}))+2\times n_1^{n_2} \\ &= n_1^{n_2}\times(1+\psi(C_{n_2,r_2})).\end{aligned}$$

□

The following theorem examines the sum of element orders of a direct product of non-group monogenic semigroups.

Theorem 3.2. *Let p be an odd prime p and $1 < r \leq p-1$. Then, $\psi(C_{p,r}\times C_{p,r}) = p^3 - p + 1$.*

Proof. Let $S = C_{p,r}\times C_{p,r}$. For every $r > 1$ we have to give a minimal generating set for S to calculate the orders of elements. Let $X = \{x, A_i, B_i \mid 2 \leq i \leq p\}$, where $x = (a, b)$, $A_i = (a^i, b)$ and $B_i = (a, b^i)$. We may easily verify that any element $y = (a^i, b^j) \in S$ may be presented as

$$y = \begin{cases} x^i, & i = j, \\ x^{i-1}B_{j-1}, & i < j, \\ x^{i-1}A_{j-1}, & j < i. \end{cases}$$

Obviously, x^p is an idempotent element and then $o(y) = 1$, if $i = j = p$. However, by the relators $a^{p+1} = a$ and $b^{p+1} = b$, we conclude that $y^{p+1} = y^p$, for all other values of i and j. This in turn proves that $o(y) = p$ and $\psi(S) = 1 + p(p^2 - 1) = p^3 - p + 1$, as required. □

4. Presentation involvement

A finite semigroup presentation $\pi = \langle X \mid R\rangle$ can also be considered as a group presentation. It is a well-known fact that if either $Sg(\pi)$ or $gp(\pi)$ is finite then the group $gp(\pi)$ is a homomorphic image of $Sg(\pi)$ under the natural homomorphism $Sg(\pi) \to gp(\pi)$, where X and R are non-empty. This homomorphism is used to identify the structures of almost all of the semigroups studied in [3, 4, 13] showing that $Sg(\pi)$ contains a certain number of copies of the group $gp(\pi)$ and then $\psi(Sg(\pi))$ is a multiple of $\psi(gp(\pi))$. However, this is not the case in general. In this section we study two infinite classes of semigroup, where $Sg(\pi)$ is not a multiple of $gp(\pi)$. Indeed, we give explicit values for the sum of element orders to show that $\psi(Sg(\pi))$ and $\psi(gp(\pi))$ are approximately equal.

Theorem 3.2 stands for $r_1, r_2 \geq 2$ and in not the case when $r_1 = r_2 = 1$. Our studied non-commutative semigroups are here $T_1 = C_{p^2,1} \rtimes C_{p,1}$ and $T_2 = C_{p,1} \wr C_{p,1}$, for a given odd prime p. The semigroup $C_{p^2,1} = \langle a \rangle$ accepts the automorphism $\theta_1 : a \to a^{p+1}$ of order p. Also, $C_{p,1} \times C_{p,1} = \langle a_1 \rangle \times \langle a_2 \rangle$ accepts the automorphism $\theta_2 : (a_1, a_2) \to (a_1^2, a_2^2)$. So, we get the following presentations by using the definitions of the semi-direct product and the wreath product of semigroups.

$$\begin{aligned}
&T_1 = Sg(\pi_1), \pi_1 = \langle a, b \mid a^{p^2+1} = a, b^{p+1} = b, ab = ba^{p+1} \rangle, \mathit{and}\\
&T_2 = Sg(\pi_2),\\
&\pi_2 = \langle a, b, c \mid a^{p+1} = a, b^{p+1} = b, c^{p+1} = c, ab = bac, ac = ca, bc = cb \rangle.
\end{aligned}$$

Suppose that $G_i = gp(\pi_i)$, $(i = 1, 2)$. The group G_i is indeed a homomorphic image of T_i under the natural homomorphism $(i = 1, 2)$. Both of the groups G_1 and G_2 are extra special p-groups of order p^3, and are of exponents p^2 and p, respectively. In the following remarks we aim to compare $\psi(T_i)$ and $\psi(G_i)$ to show that $\lim_{p\to\infty} \frac{\psi(T_i)}{\psi(G_i)} = 1$.

Remark 4.1. For every odd prime p, $|T_1| = p^3 + p^2 + p$ and $\psi(T_1) = p^5 + 2p^2 - 3p + 3$. Moreover, for sufficiently large values of p, $\psi(T_1) = \psi(G_1)$.

Proof. By using the relator $ab = ba^{p+1}$ of T_1, this semigroup will be partitioned into the union of the left Green's classes

$$L_a \cup L_b \cup L_{ba^{p^2}},$$

where $|L_a| = p^2$, $|L_b| = p$ and $|L_{ba^{p^2}}| = p^3$. Hence, $|T_1| = p^3 + p^2 + p$. Again by using the relators of T_1, we conclude that the elements a^p and b^p are central elements and T_1 contains exactly three idempotent elements a^{p^2}, b^p and $a^{p^2}b^p$. Moreover,

$$\begin{aligned}
&(1). \qquad a^i b^j = b^j a^{i(1+jp)},\\
&(2). \qquad (b^j a^i)^k = b^{kj} a^{i(k+jp\frac{k(k-1)}{2})},
\end{aligned}$$

for every i and j where $1 \leq i \leq p^2 - 1$ and $1 \leq j \leq p$. These may be proved by an induction method. These relators help us to prove,

$$o(x) = \begin{cases} 1, & x \text{ is an idempotent},\\ p^2, & x \in \{a^i, b^j a^i \mid p \nmid i\},\\ p, & \text{otherwise}, \end{cases}$$

for every element $x \in T_1$. Indeed,

For every $x = b^j a^i$ or $x = a^i$, where $1 \leq j \leq p-1$, $1 \leq i \leq p^2-1$ and $p \nmid i$ the integers i and p^2 are co-prime to each other. Then,

$$\begin{array}{ll} (b^p a^i)^{p^2+1} = b^p a^{i(p^2+1)} = b^p a^i & \Rightarrow o(b^p a^i) = p^2, \\ (b^j a^i)^{p^2+1} = b^{jp^2+j} a^{i(p^2+1)+jp\frac{p^2(p^2+1)}{2})} = b^j a^i & \Rightarrow o(b^j a^i) = p^2, \\ (a^i)^{p^2+1} = a^i & \Rightarrow o(a^i) = p^2. \end{array}$$

Hence, there are $(p^2-p)+(p-1)(p^2-p)+(p^2-p) = p^3-p$ elements of order p^2. Also, there are three elements of order 1 and other elements of T_1 which may be classified as:

$$\begin{array}{l} \{b^j \mid 1 \leq j \leq p-1\}, \\ \{a^{kp} \mid 1 \leq k \leq p-1\}, \\ \{b^j a^{kp} \mid 1 \leq k \leq p-1, 1 \leq j \leq p\}, \\ \{b^j a^{p^2} \mid 1 \leq j \leq p-1\}, \end{array}$$

which are of order p (proving $x^{p+1} = x$ is easy by using the relators (1) and (2).) So, T_1 contains exactly p^2+2p-3 elements of order p. Consequently, $\psi(T_1) = 3+(p^3-p)p^2+(p^2+2p-3)p = p^5+2p^2-3p+3$, as required. To complete the proof we classify the elements of the group G_1 and easily we get $\psi(G_1) = p^5-p^4+p^3-p+1$. So, $\lim_{p\to\infty} \frac{\psi(T_1)}{\psi(G_1)} = 1$ □

Remark 4.2. For every odd prime p, $|T_2| = p^3+3p^2+3p$ and $\psi(T_2) = p^4+3p^3+4p^2-7p+7$. Moreover, for sufficiently large values of p, $\psi(T_2) = \psi(G_2)$.

Proof. As well as in the Remark 4.1, T_1 may be partitioned as

$$L_a \cup L_b \cup L_c \cup L_{bac} \cup L_{a^p c} \cup L_{b^p c} \cup_{i=1}^{p} L_{ba^i}$$

for T_2. Evidently, $|L_a| = |L_b| = |L_c| = p$, $|L_{bac}| = p^3$ and the other classes are of the cardinality p^2. So, $|T_2| = p^3+3p^2+3p$.

For the second semigroup consider the partition $T_2 = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7$ such that

$$\begin{array}{ll} A_1 = \{a^p, b^p, c^p, b^p a^p c^p, a^p c^p, b^p c^p\}, & A_2 = \{b^p a^p\}, \\ A_3 = \{b^i a^j \mid 1 \leq i, j \leq p\} \setminus \{b^p a^p\}, & A_4 = \{a^i, b^i, c^i | 1 \leq i \leq p-1\}, \\ A_5 = \{b^i a^j c^k \mid 1 \leq i, j, k \leq p\} \setminus \{b^p a^p c^p\}, & A_6 = \{a^i c^j \mid 1 \leq i, j \leq p\} \setminus \{a^p c^p\}, \\ A_7 = \{b^i c^j \mid 1 \leq i, j \leq p\} \setminus \{b^p c^p\}. & \end{array}$$

Now, we may prove:

$$o(x) = \begin{cases} 1, & x \in A_1, \\ 2, & x \in A_2, \\ p+1, & x \in A_3, \\ p, & x \in A_4 \cup A_5 \cup A_6 \cup A_7. \end{cases}$$

Since, each element of A_1 is an idempotent, then by using $ab = bac$ we get $a^k b^k = b^k a^k c^{k^2}$, for every positive integer k. This yields in turn

$$a^p b^p = b^p a^p c^{p^2} = b^p a^p c^{p^2-1+1} = b^p a^p c^{(p+1)(p-1)+1} = b^p a^p c^{(p-1)+1} = b^p a^p c^p$$

and $(b^p a^p)^3 = (b^p a^p)^2$, showing that $o(b^p a^p) = 2$. Again by using $a^k b^k = b^k a^k c^{k^2}$ we may verify the relators:

(3). $a^i b^j = b^j a^i c^{ij}$,
(4). $(b^j a^i)^m = b^{mj} a^{mi} c^{mij}$,
(5). $(b^j a^i c^k)^m = b^{mj} a^{mi} c^{m(ij+k)}$,

for every positive integers i, j, k and m. Using the relators (3), (4) and (5) yield, $x^{p+2} = x^2$ (or $x^{p+1} = x$) if $x \in A_3$ (or $x \in A_4 \cup A_5 \cup A_6 \cup A_7$). Hence,

$$\psi(T_2) = 6+2+(p^2-1)(p+1)+[3(p-1)+(p^3-1)+2(p^2-1)]p = p^4+3p^3+4p^2-7p+7.$$

We classify the elements of G_2 and get $\psi(T_2) = p^4 - p + 1$, as it is the case in the previous example. So,

$$\lim_{p\to\infty} \frac{\psi(T_2)}{\psi(G_2)} = 1.$$

□

5. Conclusion

Our final computational result in this paper is the following remark as an application of the Corollary 2.2. Looking for the cyclic groups C_m satisfying $\psi(C_m) = m^2 - m + 1$ is of interest specially when m is a non-prime integer. The following remark gives us a large class of such groups.

Remark 5.1. For every odd prime p and any odd integer $t \geq 1$ consider the non-prime integer $m = 2^{pt} - 1$. Then, $\psi(C_m) = m^2 - m + 1$.

Proof. Let $n = 2^{pt}$. As a quick result of the Corollary 2.2 we get

$$\psi(C_m) = \psi(C_{n-1}) = n^2-3n+3 = (m+1)^2-3(m+1)+3 = m^2-m+1,$$

where, $m = n - 1$. □

Certain examples of this remark are gathered in the following tables. In the first table we let $t = 1$ and consider all of the primes less that 50 to get non-prime values of m. In the second table we let $p = 3$ and $t = 3, 5, 7$ and get evident examples.

TABLE 2.

p	$m = 2^p - 1$	$\psi(C_m)$
11	23×89	$4188163 = m^2 - m + 1$
23	47×178481	$70368719011843 = m^2 - m + 1$
29	$233 \times 1103 \times 2089$	$288230374541099011 = m^2 - m + 1$
37	223×616318177	$18889465931066263994371 = m^2 - m + 1$
41	13367×164511353	$4835703278451919629058051 = m^2 - m + 1$
43	$431 \times 9719 \times 2099863$	$77371252455309878902128643 = m^2 - m + 1$
47	$2351 \times 4513 \times 13264529$	$19807040628565662185920921603 = m^2 - m + 1$

TABLE 3.

t	$m = 2^{3t} - 1$	$\psi(C_m)$
3	7×73	$260611 = m^2 - m + 1$
5	$7 \times 31 \times 151$	$1073643523 = m^2 - m + 1$
7	$7^2 \times 127 \times 337$	$4398040219651 = m^2 - m + 1$

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