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n-SUPER FINITELY COPRESENTED AND n-WEAK PROJECTIVE MODULES

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Abstract. Let R be a ring and n an non-negative integer. In this paper, we first introduce the concept of n-super finitely copresented R-modules and via these modules, we give a concept of n-weak projective modules and investigate some characterizations of these modules over any arbitrary ring. For example, we obtain that $(\mathcal{WP}^n(R), \mathcal{WP}^n(R)^{\perp})$ is a perfect hereditary cotorsion theory and for any $N \in \mathcal{WP}^n(R)^{\perp}$, there exists an n-weak projective cover with the unique mapping property if and only if every R-module is n-weak projective.

1. Introduction and Preliminaries

In 1994, Costa in [5] via n-presented modules introduced the notion of n-coherent rings for a nonnegative integer n. A left R-module M is said to be *n-presented* if it has a finite n-presentation, that is, there exists an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i finitely generated free R-module, and a ring R is called left *n-coherent* if every n-presented left R-module is $(n+1)$-presented, for more details see [4, 6]. In 2014, Gao and Wang in [8], introduced the notion of super finitely presented modules. A left R-module U is called *super finitely presented* if there exists an exact sequence $\cdots \to F_2 \to F_1 \to F_0 \to U \to 0$, where each F_i is finitely generated and projective for any $i \geq 0$. Then in 2021, Amini, Amzil and Bennis

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in [1], introduced the notion of n-super finitely presented modules as a generalization of the notion of super finitely presented modules by using finitely generated and projective modules. Let n be a non-negative integer. Then, a left R-module U is called *n-super finitely presented* if there exists an exact sequence

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \to F_0 \to U \to 0$$

of projective R-modules F_i for any $i \geq 0$, where each F_i finitely generated for any $i \geq n$. They also in [1], introduce n-weak injective and n-weak flat modules as a generalization of the notion weak injective and weak flat modules, where first weak injective and weak flat modules was introduced by Gao and Wang in [9]. A left R-module M is called *n-weak injective* if $\mathrm{Ext}_R^{n+1}(U, M) = 0$ for every n-super finitely presented left R-module U. A right R-module N is called *n-weak flat* if $\mathrm{Tor}_{n+1}^R(N, U) = 0$ for every n-super finitely presented left R-module U.

As we know, cogenerated modules and cocoherent rings as a dual notions of generated modules and coherent rings have been characterized in various ways, and many nice properties were obtained for such rings in [12, 15, 17, 20]. A right R-module M is said to be *finitely cogenerated* if for every family $\{M_i\}_{i\in I}$ of submodules of M with $\cap_{i\in I} M_i = 0$, there is a finite subset $J \subset I$ such that $\cap_{i\in J} M_i = 0$. Also, in 1999, Weimin Xue in [16] via finitely cogenerated modules introduced n-copresented modules and n-cocoherent rings as a dual notion of n-presented modules and n-coherent rings, respectively. A right R-module M is said to be *n-copresented* if there is an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^n$ of right R-modules, where each E^i is finitely cogenerated injective. A ring R is called right *n-cocoherent* if every n-copresented R-module is $(n+1)$-copresented. n-cocoherent rings have been studied by several authors (see, for example [2, 3, 20]).

In this paper, first via finitely cogenerate modules, we introduce the concept of n-super finitely copresented right modules as a dual notion of n-super finitely presented left modules. Then, we introduce the concept of n-weak projective right modules by using n-super finitely copresented right modules. Every n-super finitely copresented module and every n-weak projective module are m-super finitely copresented and m-weak projective, respectively for $m \geq n$. But, m-super finitely copresented and m-weak projective modules are not n-super finitely copresented and n-weak projective for any $m > n$, see Examples 2.4 and 2.15.

Moreover, we study the relative homological theory of these modules and also, the properties of special super finitely copresented modules, defined via finitely cogenerated injective resolutions of n-super finitely copresented modules, play a crucial role. For example, over any arbitrary ring, we obtain some equivalent characterizations in terms of n-weak projective right modules on the spesial super short exact sequences, see Proposition 2.7.

Also, on every arbitrary ring, we prove that, (1) there exist some equivalent characterizations of right modules of n-weak projective dimension at most k, (2) If $\mathcal{WP}^n(R)$ denotes the class of n-weak projective right modules, then $(\mathcal{WP}^n(R), \mathcal{WP}^n(R)^{\perp})$ is hereditary cotorsion theory, (3) if $\mathcal{WI}^n(R)$ is a class of n-weak injective left modules, then $(\mathcal{WP}^n(R))^* \subseteq \mathcal{WI}^n(R)$, (4) every right R-module is n-weak projective if and only if $(\mathcal{WP}^n(R), \mathcal{WP}^n(R)^{\perp})$ is a perfect hereditary cotorsion theory and N has an n-weak projective cover with the unique mapping property for any $N \in \mathcal{WP}^n(R)^{\perp}$ if and only if $id(U) \leq n-1$ for any n-super finitely copresented right R-module U if and only if N is injective for any $N \in \mathcal{WP}^n(R)^{\perp}$.

2. n-Super finitely copresented and n-weak projective modules

In this section, we first introduce the special super finitely copresented and special super finitely cogenerated modules via n-super finitely copresented right modules. Then by using of these modules, some properties of n-weak projective modules are discussed. We start with the following definition.

Definition 2.1. Let n be a non-negative integer. A right R-module U is said to be n-super finitely copresented if there exists an exact sequence

$$0 \to U \to E^0 \to E^1 \to \cdots \to E^n \to E^{n+1} \to \cdots$$

of injective R-modules E_i, where each E^i is finitely cogenerated and injective for any $i \geq n$.

If $K^{n-1} = \mathrm{Coker}(E^{n-2} \to E^{n-1})$ and $K^n = \mathrm{Coker}(E^{n-1} \to E^n)$, then we call the module K^{n-1} special super finitely copresented right R-module and K^n special super finitely cogenerated right R-module. Also, we shall say the sequence $0 \longrightarrow K^{n-1} \longrightarrow E^n \longrightarrow K^n \longrightarrow 0$ of right R-modules is a special super short exact sequence. Moreover, if $\mathrm{Hom}_R(-, K^{n-1})$ is exact with respect to a short exact sequence $0 \to X \to Y \to Z \to 0$ of right R-modules, then we say that this sequence is special super copure and X is said to be super copure in Y. See the concepts of pure submodules in [7].

Remark 2.2. Let n, m, k be non-negative integers. Then:

(1) Every 0-super finitely copresented (or, super finitely copresented) right R-module is n-super finitely copresented.
(2) Every n-super finitely copresented right R-module is m-super finitely copresented for any $m \geq n$, but not conversely (see Example 2.4). If we denote by $\mathrm{Copres}_{\infty}^{n}$ the class of all n-super finitely copresented right R-modules, then:

$$\mathrm{Copres}_{\infty}^{n} \subseteq \mathrm{Copres}_{\infty}^{n+1} \subseteq \mathrm{Copres}_{\infty}^{n+2} \subseteq \cdots$$

If $n = 0$, then $\mathrm{Copres}_{\infty}^{0}$ is simply the class of all super finitely copresented right R-modules. We denote this class simply by Copres_{∞}.

The finitely presented dimension of an R-module A is defined as $\mathrm{f.p.dim}_R(A) = \inf\{n \mid$ there exists an exact sequence $\mathrm{F}_{n+1} \to \mathrm{F}_n \to \cdots \to \mathrm{F}_1 \to \mathrm{F}_0 \to \mathrm{A} \to 0$ of R-modules, where each F_i is projective, and F_n and F_{n+1} are finitely generated$\}$. We also define the finitely presented dimension of R (denoted by $\mathrm{f.p.dim}(R)$) as $\sup\{\mathrm{f.p.dim}_R(A) \mid A$ is a finitely generated R-module$\}$.

Also, R is called an (a, b, c)-ring if $\mathrm{w.gl.dim}(R) = a$, $\mathrm{gl.dim}(R) = b$ and $\mathrm{f.p.dim}(R) = c$ (see [13]).

Example 2.3. Let $S = k[x_1, \cdots, x_3] \oplus S'$, where $k[x_1, \cdots, x_3]$ is a ring of polynomials in 3 indeterminates over a field k, and S' is a valuation ring with global dimension 3. Then by [13, Proposition 3.10]) S' is a $(3, 3, 4)$-ring.

Example 2.4. Let $R = k[x_1, \cdots, x_5] \oplus S_2$, where $k[x_1, \cdots, x_5]$ is a ring of polynomials in 5 indeterminates over a field k, and S_2 is a $(3, 3, 4)$-ring (see, Example 2.3 and [13, Proposition 3.10]). Then by [13, Proposition 3.8], R is a coherent $(4, 4, 4)$-ring. Hence, $\mathrm{f.p.dim}(R) = 4$ and so there exists a finitely generated R-module U with $\mathrm{f.p.dim}_R(U) = 4$. Hence, there exists an exact sequence $F_5 \to F_4 \to F_3 \to F_2 \to F_1 \to F_0 \to U \to 0$, where F_4 and F_5 are finitely generated and projective R-modules. Also, $K_3 := \mathrm{Im}(F_4 \to F_3)$ is a special super finitely presented module, since R is coherent. So, we have

$$0 \to U^* \to F_0^* \to \cdots \to F_4^* \to F_5^* \to \cdots ,$$

where by Lemma 2.17, U^* is 4-super finitely copresented. If every 4-super finitely copresented module is 3-super finitely copresented, then, U^* is 3-super finitely copresented. Also, F_i^* is finitely cogenerated and injective if and only if F_i is finitely generated and projective for any $i \geq 3$, since R is coherent. So U is 3-super finitely presented otherwise $\mathrm{f.p.dim}(R) = 3$, a contradiction.

Definition 2.5. Let n be a non-negative integer. A right R-module M is called n-weak projective if $\mathrm{Ext}_R^{n+1}(M,U)=0$ for every n-super finitely copresented right R-module U.

Remark 2.6. Let n,m,k be non-negative integers. Then:

(1) $\mathrm{Ext}_R^{n+1}(-,U)\cong \mathrm{Ext}_R^{1}(-,K^{n-1})$, where U is an n-super finitely copresented right R-module with a special super finitely copresented module K^{n-1}. If $n=0$, then n-weak projective right R-modules, n-super finitely pcoresented right R-modules are simply weak projective right R-modules and super finitely copresented right R-module, respectively.

(2) Every n-weak projective right R-module is m-weak projective for any $n\leq m$, but not conversely (see Example 2.15). If U is an $(n+1)$-super finitely copresented right R-module, then there exists an exact sequence

$$0\to U\to E^0\to E^1\to\cdots$$

where K^n is a special super finitely copresented right module. Also, we have the short exact sequence $0\to U\to E^0\to K^0\to 0$, where K^0 is an n-super finitely copresented right module. So, if M is an n-weak projective right R-module, then $\mathrm{Ext}_R^{n+1}(M,K^0)=0$. On the other hand, $\mathrm{Ext}_R^{n+2}(M,U)\cong \mathrm{Ext}_R^{n+1}(M,K_0)=0$, and hence M is $(n+1)$-weak projective.

(5) If $\mathcal{P}$, $\mathcal{WP}(R)$, $\mathcal{WP}^n(R)$ are the classes of projective, weak projective and n-weak projective right R-modules, respectively, then

$$\mathcal{P}\subseteq\mathcal{WP}(R)\subseteq\mathcal{WP}^n(R)\subseteq\mathcal{WP}^{n+1}(R)\subseteq\mathcal{WP}^{n+2}(R)\subseteq\cdots.$$

Proposition 2.7. *Let M be a right R-module. Then, the following assertions are equivalent:*

(1) *M is n-weak projective.*

(2) *Every short exact sequence $0\to A\to B\to M\to 0$ of right R-modules is special super copure.*

(3) *M is n-weak projective with respect to all special super short exact sequences in Mod-R.*

(4) *There exists a special super copure short exact sequence $0\to K\to P\to M\to 0$ of right R-modules, where P is projective.*

(5) *There exists a special super copure short exact sequence $0\to K\to P\to M\to 0$ of right R-modules, where P is n-weak projective.*

Proof. (1) $\Longrightarrow$ (3) Let $0 \to K^{n-1} \to E^n \to K^n \to 0$ be a special super short exact sequence with respect to any n-super finitely copresented right R-module U. Then by hipothesis and Remark 2.6, $\mathrm{Ext}_R^1(M, K^{n-1}) \cong \mathrm{Ext}_R^{n+1}(M, U) = 0$.

(3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are clear.

(1) $\Leftrightarrow$ (4) Let M be a right R-module. Then, there is exact sequence $0 \to K \to P \to M \to 0$ of right R-modules with P is projective. Hence Remark 2.6, $\mathrm{Ext}_R^1(M, K^{n-1}) \cong \mathrm{Ext}_R^{n+1}(M, U)$ for any special super finitely copresented K^{n-1} and any n-super finitely copresented right module U.

(4) $\Longrightarrow$ (5), (5) $\Longrightarrow$ (1) are clear. □

Definition 2.8. (1) The n-weak projective dimension of a right module M is defined by
$n\text{-wpd}_R(M) = \inf\{k : \mathrm{Ext}_R^{k+1}(M, K^{n-1}) = 0\}$ for every special super finitely copresented K^{n-1}.

(2) The right n-super finitely copresented dimension $\mathrm{l.n.scop.gldim}(R)$ of R is defined as: $\mathrm{l.n.scop.gldim}(R) := \sup\{\mathrm{id}_R(K^{n-1})\}$ for any special super finitely copresented right R-module K^{n-1}.

Lemma 2.9. *Let M be a right R-module. Then the following statements are equivalent:*

1. $n\text{-wpd}_R(M) \leq k$.
2. $\mathrm{Ext}_R^{k+1}(M, K^{n-1}) = 0$ *for any special super finitely copresented module* K^{n-1}.

Proof. (2) $\Longrightarrow$ (1) is trivial by Definition 2.8.

(1) $\Longrightarrow$ (2) Use induction on k. Clear if $n\text{-wpd}_R(M) = k$. Let $n\text{-wpd}_R(M) \leq k-1$. If $0 \to K^{n-1} \to E^n \to K^n \to 0$ is a special super short exact sequence of right R-module with respect to any n-super finitely copresented right R-module U, then we deduce that K^n is special super finitely copresented, too. Also, we have $\mathrm{Ext}_R^k(M, K^n) \cong \mathrm{Ext}_R^{k+1}(M, K^{n-1})$. So by induction hypothesis, $\mathrm{Ext}_R^k(M, K^n) = 0$ and consequently $\mathrm{Ext}_R^{k+1}(M, K^{n-1}) = 0$ which completes the proof. □

Theorem 2.10. *Let M be a right R-module and k a non-negative integer. Then the following statements are equivalent:*

1. $n\text{-wpd}_R(M) \leq k$.
2. $\mathrm{Ext}_R^{k+l}(M, K^{n-1}) = 0$ *for any special super finitely copresented* K^{n-1} *and all positive integers* l.
3. $\mathrm{Ext}_R^{k+1}(M, K^{n-1}) = 0$ *for any special super finitely copresented* K^{n-1}.

(4) *There exists an exact sequence*

$$0 \longrightarrow P_k \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} \cdots P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

of right R-modules with $P_0, P_1, \cdots, P_k$ are n-weak projective.

Proof. (1) $\Longrightarrow$ (2) If $n\text{-wpd}_R(M) \leq k$, then $n\text{-wpd}_R(M) \leq k+l-1$. So by Lemma 2.9, $\text{Ext}_R^{k+l}(M, K^{n-1}) = 0$.

(4) $\Longrightarrow$ (1) By Lemma 2.9, $\text{Ext}_R^{j}(P_i, K^{n-1}) = 0$ for any special super finitely copresented module K^{n-1}, all positive integers j and any $0 \leq i \leq k$. So by (4), we have:

$$\text{Ext}_R^{k+1}(M, K^{n-1}) \cong \text{Ext}_R^{k}(ker(f_0), K^{n-1}) \cong \cdots \cong \text{Ext}_R^{1}(P_k, K^{n-1}).$$

Hence by Lemma 2.9, $n\text{-wpd}_R(M) \leq k$.

(2) $\Longrightarrow$ (3) obvious.

(3) $\Longrightarrow$ (4) For every right R-module M, there exists an exact sequence

$$0 \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \cdots P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of right R-modules with $P_0, P_1, \cdots, P_{k-1}$ are projective. Therefore for any positive integers l, we have $\text{Ext}_R^{l}(P_i, K^{n-1}) = 0$ for all special super finitely copresented modules K^{n-1} and any $0 \leq i \leq k-1$. Let $K_i = \ker(P_i \to P_{i-1})$. Then,

$$\text{Ext}_R^{k+1}(M, K^{n-1}) \cong \text{Ext}_R^{k}(K_0, K^{n-1}) \cong \text{Ext}_R^{k-1}(K_1, K^{n-1}) \cong \text{Ext}_R^{1}(P_k, K^{n-1}).$$

By (3), $\text{Ext}_R^{k+1}(M, K^{n-1}) = 0$, and so $\text{Ext}_R^{1}(P_k, K^{n-1}) = 0$, which means that P_k is n-weak projective. □

Corollary 2.11. *Let R be a ring .Then the following statements are equivalent:*

(1) *If $\phi : N \to M$ is an n-weak projective preenvelope, then N has an epic n-weak projective preenvelope.*
(2) *The cokernel of any n-weak projective preenvelope of a right R-module is n-weak projective.*
Moreover, if every submodule of an n-weak projective right R-module has an n-weak projective preenvelope, the above are equivalent to
(3) $\text{r.n.scop.gldim}(R) \leq 1$.

Proof. (1) $\Longrightarrow$ (2) Let $\phi : N \to M$ be an n-weak projective preenvelope. Then $f : \text{Im}(\phi) \to M$ is an n-weak projective preenvelope. By (1), there is an epic n-weak projective preenvelope $g : \text{Im}(\phi) \to C$.

Consider the following commutative diagram, where D is a pushout of two maps f and g:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}(\phi) & \xrightarrow{f} & M & \longrightarrow & \mathrm{Coker}(\phi) & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow \beta & & \downarrow id & & \\ & & C & \xrightarrow{\alpha} & D & \longrightarrow & \mathrm{Coker}(\phi) & \longrightarrow & 0 \end{array}$$

By [14, Exercise 5.10], α is injective and β is surjective. On the other hand, $D = \alpha(C) + \beta(M)$. Since β is surjective, $D = \alpha(C) + D$ and so, $\alpha(C) \subseteq D$. Also, by using of preenvelopes f and g, there is a morphism $h : D \to C$ such that $h\alpha = 1_C$. Hence, $D \cong C \oplus \mathrm{Coker}(\phi)$. Similarly $D \cong M$. Therefore from n-weak projectivity M, we deduce that $\mathrm{Coker}(\phi)$ is n-weak projective.

(2) $\Longrightarrow$ (1) Let $\phi : N \to M$ be an n-weak projective preenvelope. It is enough to show that $\mathrm{Im}(\phi)$ is n-weak projective. Consider the exact sequence $0 \to \mathrm{Im}(\phi) \to M \to \mathrm{Coker}(\phi) \to 0$. By hypothesis, $\mathrm{Coker}(\phi)$ is n-weak projective, and so for every special super finitely copresented K^{n-1}, we have:

$$0 = \mathrm{Ext}^1_R(M, K^{n-1}) \longrightarrow \mathrm{Ext}^1_R(\mathrm{Im}(\phi), K^{n-1}) \longrightarrow \mathrm{Ext}^2_R(\mathrm{Coker}(\phi), K^{n-1}).$$

Hence by Theorem 2.10 and (2), $\mathrm{Ext}^2_R(\mathrm{Coker}(\phi), K^{n-1}) = 0$. Hence,

$$0 = \mathrm{Ext}^1_R(\mathrm{Im}(\phi), K^{n-1}) \cong \mathrm{Ext}^{n+1}_R(\mathrm{Im}(\phi), U)$$

for any n-super finitely copresented right module U, and then it follows that $\mathrm{Im}(\phi)$ is n-weak projective.

(2) $\Longrightarrow$ (3) Let M be a right R-module. Then, there exists an exact sequence $0 \to K \to P \to M \to 0$ of right R-modules , where P is projective. If K is n-weak projective, then $\text{r.n.scop.gldim}(R) \leq 1$. So, we show that K is n-weak projective. If $\phi : K \to N$ is an n-weak projective preenvelope, then ϕ is injective. So similar to proof of (2) $\Longrightarrow$ (1), we get that K is n-weak projective.

(3) $\Longrightarrow$ (1) Let $\phi : N \to M$ be an n-weak projective preenvelope. Then by Theorem 2.10, the exact sequence $0 \to \mathrm{Im}(\phi) \to M \to \mathrm{Coker}(\phi) \to 0$ implies that $\mathrm{Im}(\phi)$ is n-weak projective, and so $N \to \mathrm{Im}(\phi)$ is an epic n-weak projective preenvelope of N. □

Corollary 2.12. *For every exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, A is n-weak projective if B and C are n-weak projective.*

Proof. Let U be an n-super finitely copresented right R-module. Then by Theorem 2.10(2), we have: $0 = \mathrm{Ext}^{n+1}_R(B, U) \to \mathrm{Ext}^{n+1}_R(A, U) \to$

$\text{Ext}_R^{n+2}(C,U)=0$, since B and C are n-weak projective. So $\text{Ext}_R^{n+1}(A,U)$ $=0$, and hence A is n-weak projective. □

Corollary 2.13. *Let M be a right R-module. Then, M is n-weak projective if and only if every super copure epimorphic image and super copure submodule of M is n-weak projective.*

Proof. ($\Longrightarrow$) Let N be a super copure submodule of n-weak projective right R-module M. Then, the exact sequence $0\to N\to M\to \frac{M}{N}\to 0$ is special super copure. So by Proposition 2.7, $\frac{M}{N}$ is n-weak projective and hence by Corollary 2.12, N is n-weak projective.

($\Longleftarrow$) is clear. □

The $(n,0)$-projective dimension of a right module M is defined by $(n,0).\text{pd}_R(M)$=inf$\{k:\text{Ext}_R^{k+1}(M,U)=0\ \textit{for every}\ \text{n-}\textit{copresented}\ U\}$. Also, the $(n,0)$-projective global dimension of a ring R is defined by $r.(n,0).\text{dim(R)}$=sup$\{(n,0).\text{pd}_R(M)\ \mid\ M\ \textit{is a right R-module}\}$, see [20, Definition 2.8].

Corollary 2.14. *If R is a n-cocoherent ring, then* $\text{l.n.scop.gldim}(R)=\text{r.(n},0).\text{dim}(R)$.

Proof. is clear. □

A ring R is called a right V-ring if every simple right R-module is injective. A ring R is called right hereditary if every submodule of a projective right R-module is projective. A right R-module N is called $(1,0)$-projective if $\text{Ext}_R^1(N,U)=0$ for every finitely copresented right R-module U. A ring R is called right cosemihereditary if every submodule of a projective right R-module is $(1,0)$-projective, see [20, Definitions 2.2 and 3.6]. It is clear that hereditary rings are cosemihereditary.

Example 2.15. Let R be a cosemihereditary ring but not V-ring, for example hereditary ring $R=k[X]$ where k is field. Then by [20, Theorem 3.9], there exists an R-module which is not $(1,0)$-projective. But, R is cocoherent by [20, Theorem 3.7]. So, every finitely cogenerated is finitely copresented and hence every R-module is $(n,0)$-projective if and only if n-weak projective. Therefore, there exists an R-module which is not 1-weak projective. Also by [20, Theorem 3.7], $r.(1,0).\text{dim}R\leq 1$. So, $(1,0).\text{pd}_R(M)\leq 1$ for any right R-module M. Hence by Corollary 2.14, $\text{l.1.scop.gldim}(R)\leq 1$. So by Theorem 2.10, there is an exact sequence $0\to P_1\to P_0\to M\to 0$ of right modules, where P_0 and P_1 are 1-weak projective. If U is a 2-super finitely copresented right module, then $\text{Ext}_R^2(P_1,U)=0$, since every 2-super finitely copresented right module is 1-super finitely copresented. Therefore from

$\mathrm{Ext}^2_R(P_1, U) \cong \mathrm{Ext}^3_R(M, U)$ we get that every right R-module M is 2-weak projective.

Before the next results, we first introduce the following symbols and definitions given in [10, 19].

For every class $\mathcal{X}$ of right R-modules, denote the classes

$$\mathcal{X}^{\perp} = \{Y \in Mod\text{-}R \ : \mathrm{Ext}^1_R(X, Y) = 0 \ for \ all \ X \in \mathcal{X}\}$$

and

$$^{\perp}\mathcal{X} = \{Y \in Mod\text{-}R \ : \mathrm{Ext}^1_R(Y, X) = 0 \ for \ all \ X \in \mathcal{X}\}.$$

Given two classes of right R-modules $\mathcal{F}$ and $\mathcal{C}$, then we say that $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory in Mod-R if $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called hereditary if whenever $0 \to F^{'} \to F \to F^{''} \to 0$ is exact in Mod-R with $F, F^{''} \in \mathcal{F}$ then $F^{'}$ is also in $\mathcal{F}$.

A duality pair over a ring R is a pair $(\mathcal{F}, \mathcal{C})$, where $\mathcal{F}$ is a class of right (resp. left) R-modules and $\mathcal{C}$ is a class of left (resp. right) R-modules, subject to the following conditions: (1) For any module F, one has $F \in \mathcal{F}$ if and only if $F^* \in \mathcal{C}$. (2) $\mathcal{C}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{F}, \mathcal{C})$ is called (co)product-closed if the class of $\mathcal{F}$ is closed under direct (co)products, and a duality pair $(\mathcal{F}, \mathcal{C})$ is called perfect if it is coproduct-closed, $\mathcal{F}$ is closed under extensions and R belongs to $\mathcal{F}$.

Theorem 2.16. *The pair ($\mathcal{WP}^n(R)$, $\mathcal{WP}^n(R)^{\perp}$) is hereditary cotorsion theory.*

Proof. Note that we have to show that ${}^{\perp}(\mathcal{WP}^n(R)^{\perp}) = \mathcal{WP}^n(R)$. Let K^{n-1} be a special super finitely copresented with respect to any n-super finitely copresented right R-module U and $M \in {}^{\perp}(\mathcal{WP}^n(R)^{\perp})$. Then, $K^{n-1} \in \mathcal{WP}^n(R)^{\perp}$. Therefore, $\mathrm{Ext}^1_R(M, K^{n-1}) = 0$ and consequently by Theorem 2.10, M is n-weak projective, and hence $M \in \mathcal{WP}^n(R)$. Let $0 \to F^{'} \to F \to F^{''} \to 0$ be a exact sequence of modules right R-modules, where F and $F^{''}$ are n-weak projective. Then by Corollary 2.12, $F^{'}$ is n-weak projective. So, it follows that ($\mathcal{WP}^n(R)$, $\mathcal{WP}^n(R)^{\perp}$) is a hereditary cotorsion theory. □

We denote $(\mathcal{WP}^n(R))^* = \{M^* \mid \ M \in \mathcal{WP}^n(R)\}$. The following lemma shows the connection between n-weak projective and n-weak injective modules.

Lemma 2.17. *Let R be a ring.*

(1) *If U is an n-super finitely presented left R-module, then U^* is an n-super finitely copresented right R-module.*
(2) $(\mathcal{WP}^n(R))^* \subseteq \mathcal{WI}^n(R)$.

Proof. (1) Let U be an n-super finitely presented left R-module. Then, there exists an exact sequence

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow U \longrightarrow 0$$

of projective left R-modules with each F_i is finitely generated and projective for any $i \geq n$. So by [14, Lemma 3.53], there is an exact sequence

$$0 \longrightarrow U^* \longrightarrow F_0^* \longrightarrow \cdots \longrightarrow F_{n-1}^* \longrightarrow F_n^* \longrightarrow \cdots$$

of right R-modules. It suffices to show that every F_i^* is injective for any $i \geq 0$ and also, every F_i^* is finitely cogenerated for any $i \geq n$. It is clear that any F_i^* is finitely cogenerated for any $i \geq n$. So, we prove that every F_i^* is injective, too. Consider the short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules. Then, there exists the following commutative diagram with the upper row exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(C, F_i^*) & \longrightarrow & \mathrm{Hom}_R(B, F_i^*) & \longrightarrow & \mathrm{Hom}_R(A, F_i^*) & & \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Hom}_R(F_i, C^*) & \longrightarrow & \mathrm{Hom}_R(F_i, B^*) & \longrightarrow & \mathrm{Hom}_R(F_i, A^*) & \longrightarrow & 0
\end{array}$$

So, $\mathrm{Ext}_R^1(C, F_i^*) = 0$ and hence any F_i^* is injective.

(2) By [1, Definition 2.1], let $0 \to K_{n-1} \to P_{n-1} \to K_n \to 0$ be a special super short exact sequence of left R-modules with respect to any n-super finitely presented left R-module U. Then by (1), K_{n-1}^* is special super finitely copresented right R-module. So if M is n-weak projective right R-module, then similar to the proof (1), $0 = \mathrm{Ext}_R^1(M, K_{n-1}^*) \cong \mathrm{Ext}_R^1(K_{n-1}, M^*)$. On the other hand by [1, Remark 2.3], $\mathrm{Ext}_R^{n+1}(U, M^*) \cong \mathrm{Ext}_R^1(K_{n-1}, M^*)$. Therefore by [1, Definition 2.2], M^* is an n-weak injective left R-module, and then we conclude that $(\mathcal{WP}^n(R))^* \subseteq \mathcal{WI}^n(R)$. □

Also, as for the classical projective notion, the class $\mathcal{WP}^n(R)$ is closed under direct limits.

Proposition 2.18. *Let R be a graded ring. Then, the class $\mathcal{WP}^n(R)$ is closed under direct limits.*

Proof. Let U be an n-super finitely copresented right module and let $\{M_i\}_{i \in I}$ be a family of n-weak projective right modules . Then,

$$\mathrm{Ext}_R^{n+1}(\varinjlim M_i, U) \cong \mathrm{Ext}_R^1(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{Ext}_R^1(M_i, K^{n-1}) \cong$$

$$\varprojlim \mathrm{Ext}_R^{n+1}(M_i, U),$$

where K^{n-1} is special super finitely copresented. □

In the following theorem, by using the previous results, we present some equivalent characterizations to that each right R-module is n-weak projective.

Theorem 2.19. *Let R be a ring . Then, the following statements are equivalent:*

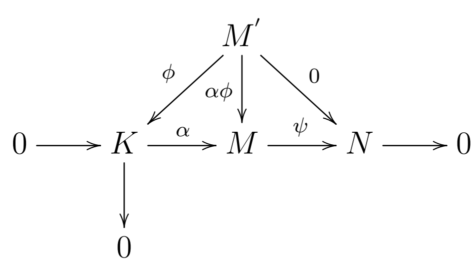
(1) *Every right R-module is n-weak projective.*
(2) *$id(U) \leq n-1$ for any n-super finitely copresented right R-module U.*
(3) *Every special super finitely copresented right R-module is injective.*
(4) *$(\mathcal{WP}^n(R), \mathcal{WP}^n(R)^\perp)$ is perfect hereditary cotorsion and N has an n-weak projective cover with the unique mapping property for any $N \in \mathcal{WP}^n(R)^\perp$.*
(5) *N is injective for any $N \in \mathcal{WP}^n(R)^\perp$.*
(6) *Every right R-module has an n-weak projective cover with the unique mapping property.*
(7) *R-module N is n-weak projective for any $N \in \mathcal{WP}^n(R)^\perp$.*

Proof. (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are clear by Proposition 2.7.

(1) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (3) are obvious.

(1) $\Longrightarrow$ (6) Frist, we show that the class $\mathcal{WP}^n(R)$ is covering. If $M \in \mathcal{WP}^n(R)$, then by Lemma 2.17, $M^* \in \mathcal{WI}^n(R)$. Contrary, if $M^* \in \mathcal{WI}^n(R)$, then [1, Proposition 2.6] implies that M is an n-weak flat right R-module, and hence by (1), $M \in \mathcal{WP}^n(R)$. On the other hands, the class $\mathcal{WI}^n(R)$ is closed under direct summands and direct sums by [1, Proposition 2.10]. So, we obtain that $(\mathcal{WP}^n(R), \mathcal{WI}^n(R))$ is a duality pair. Also By (1), it follows that the class $\mathcal{WP}^n(R)$ is closed under copure submodules, copure quotients and copure extensions. Therefore by Proposition 2.18 and [11, Theorem 3.1], the class $\mathcal{WP}^n(R)$ is covering and hence by hypothesis, (6) follows.

(6) $\Longrightarrow$ (1) Let N be a right R-module. Then there is a commutative diagram with exact rows:



where,ψ and ϕ are n-weak projective cover with the unique mapping property. Since $\psi\alpha\phi = 0 = \psi$, we have $\alpha\phi = 0$ by (7). Therefore, $K = \mathrm{im}(\phi) \subseteq \ker(\alpha) = 0$ and so $K = 0$. Thus $N \cong M$ and hence every right module N is n-weak projective.

(1) $\Longrightarrow$ (7) By hypothesis, N is n-weak projective for any $N \in \mathcal{WP}^n(R)^\perp$.

(7) $\Longrightarrow$ (3) Cosider the special super short exact sequence $0 \to K^{n-1} \to E^{n-1} \to K^n \to 0$ of right R-modules with respect to any n-super finitely copresented right module U, where K^{n-1} is super finitely copresented and K^n is super finitely cogenerated. But K_n is super finitely copresented, too. Thus $K^n \in \mathcal{WP}^n(R)^\perp$ and consequently $0 = \mathrm{Ext}^{n+1}_R(K^n, U) \cong \mathrm{Ext}^1_R(K^n, K^{n-1})$, since K_n is n-weak projective by (7). Therefore, the special super short exact sequence $0 \to K^{n-1} \to E^{n-1} \to K^n \to 0$ is split and we deduce that K^{n-1} is injective.

(4) $\Longrightarrow$ (7) Let $N \in \mathcal{WP}^n(R)^\perp$. If $\phi : M \to N$ is an n-weak projective cover with the unique mapping property, then $\ker\phi \in \mathcal{WP}^n(R)^\perp$. Thus, similar to the proof of (6) $\Longrightarrow$ (1), we get that N is n-weak projective.

(7) $\Longrightarrow$ (4) By Theorem 2.16, $(\mathcal{WP}^n(R), \mathcal{WP}^n(R)^\perp)$ is hereditary cotorsion theory. Also $R \in \mathcal{WP}^n(R)$ and by Corollary 2.18 and (7) $\Longrightarrow$ (3) $\Longrightarrow$ (1), $\mathcal{WP}^n(R)$ is closed under direct sum and extensions. Therefore, we deduce that $(\mathcal{WP}^n(R), \mathcal{WP}^n(R)^\perp)$ is a perfect hereditary cotorsion theory. If N is n-weak projective for any $N \in \mathcal{WP}^n(R)^\perp$, then it is clear that N has an n-weak projective cover with the unique mapping property. □

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