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CONSTRUCTION OF SYMMETRIC PENTADIAGONAL MATRIX FROM THREE INTERLACING SPECTRUM

K. GHANBARI * AND M. R. MOGHADDAM

Abstract. In this paper, we introduce a new algorithm for constructing a symmetric pentadiagonal matrix by using three interlacing spectrum, say $(\lambda_i)_{i=1}^n$, $(\mu_i)_{i=1}^n$ and $(\nu_i)_{i=1}^n$ such that

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < ... < \lambda_n < \mu_n,$$
$$\mu_1 < \nu_1 < \mu_2 < \nu_2 < ... < \mu_n < \nu_n,$$

where $(\lambda_i)_{i=1}^n$ are the eigenvalues of pentadiagonal matrix A, $(\mu_i)_{i=1}^n$ are the eigenvalues of A^* (the matrix A^* differs from A only in the $(1,1)$ entry) and $(\nu_i)_{i=1}^n$ are the eigenvalues of A^{**} (the matrix A^{**} differs from A^* only in the $(2,2)$ entry). From the interlacing spectrum, we find the first and second columns of eigenvectors. Sufficient conditions for the solvability of the problem are given. Then we construct the pentadiagonal matrix A from these eigenvectors and given eigenvalues by using the block Lanczos algorithm. We also give an example to demonstrate the efficiency of the algorithm.

1. Introduction

An inverse eigenvalue problem (IEP) concerns the reconstruction of a certain matrix from prescribed spectral data. The spectral data may involve a complete or only partial information on eigenvalues or eigenvectors. Such an inverse problem has received considerable attention and arises in many applications and in various areas of science and engineering [5, 7, 11, 15]. Often a physical process is described by a

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*Corresponding author .

mathematical model of which parameters represent important physical quantities. For this problem, an IEP is to determine or estimate the parameters of a certain physical problem from its dynamical behavior. The mathematical model involves matrices whose spectral properties determine the dynamical behavior of the physical system.

For the problem to be more meaningful, either physically or mathematically, it is often necessary to confine the construction to certain special classes of matrices (e.g., tridiagonal, pentadiagonal, Toeplitz, etc.). Thus an IEP is indeed a structured IEP (SIEP). The objective of a SIEP is to construct a matrix that maintains both the specific structure as well the given spectral property.

In this paper we consider the problem of constructing a symmetric pentadiagonal matrix from some spectral information. The symmetric pentadiagonal matrix A has the form

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & & & & \\ b_1 & a_2 & b_2 & c_2 & & & \\ c_1 & b_2 & a_3 & b_3 & c_3 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & c_{n-4} & b_{n-3} & a_{n-2} & b_{n-2} & c_{n-2} \\ & & & c_{n-3} & b_{n-2} & a_{n-1} & b_{n-1} \\ & & & & c_{n-2} & b_{n-1} & a_n \end{pmatrix}.$$

The pentadiagonal case occurs in the inverse problem for a vibrating beam. The inverse problem for constructing a pentadiagonal matrix was first considered by Barcilon [1, 2, 3]. He proved that the reconstruction of such a matrix would require three spectra, corresponding to three different end conditions. The necessary and sufficient conditions for spectra corresponding to a realizable system were introduced by Gladwell [10]. Boley and Golub presented an algorithm for constructing a pentadiagonal matrix from three interlacing spectra corresponding to pentadiagonal matrix and two leading principal submatrix of matrix. In that algorithm, they need to use the Householder transformation to reduce a constructed symmetric matrix to pentadiagonal form, see [4]. Chu et al. proposed a numerical method for constructing a symmetric pentadiagonal Toeplitz matrix from three largest eigenvalues, see [6]. Recently, the inverse problem for constructing a pentadiagonal matrix are studied by many authors, for example see [9, 12, 14]. In these cases the reconstruction procedure requires three real spectra, some partial knowledge about its eigenvectors or some special submatrices. In [13] Mirzaei et al., solved the inverse eigenvalue problem for H-Symmetric pentadiagonal matrices, not necessarily symmetric by using three spectra.

Gladwell [11] presented an algorithm in Chapter 4 to construct a Jacobi matrix from two spectra with interlacing property by using Lanczos algorithm. In this paper we extend the corresponding result to pentadiagonal matrices by using the block Lanczos algorithm.

The set of eigenvalues of matrix A are called the spectrum of A and is denoted by $\sigma(A)$. Let $E_{1,1}$ and $E_{2,2}$ be matrices having exactly one entry of 1 in $(1,1)$ position and $(2,2)$ position, respectively and 0 elsewhere. We consider the following inverse problem.

Given three distinct spectrum $(\lambda_i)_{i=1}^n$, $(\mu_i)_{i=1}^n$ and $(\nu_i)_{i=1}^n$ such that

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < ... < \lambda_n < \mu_n \tag{1.1}$$

$$\mu_1 < \nu_1 < \mu_2 < \nu_2 < ... < \mu_n < \nu_n \tag{1.2}$$

construct a pentadiagonal matrix A such that $\sigma(A) = (\lambda_i)_1^n$, $\sigma(A^*) = (\mu_i)_1^n$, $\sigma(A^{**}) = (\nu_i)_1^n$, where $A^* = A + (a_{11}^* - a_{11})E_{1,1}$, (the matrix A^* differs from A only in the $(1,1)$ entry), $A^{**} = A^* + (a_{22}^* - a_{22})E_{2,2}$, (the matrix A^{**} differs from A^* only in the $(2,2)$ entry).

2. Statement of results

Consider the eigenvalue problem

$$A\mathbf{u}_i = \lambda_i \mathbf{u}_i. \tag{2.1}$$

Use the column vectors $(\mathbf{u_i})_\mathbf{1}^\mathbf{n}$ to construct a square matrix $\mathbf{U}$: $\mathbf{U} = [\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}]$. The orthogonality conditions $\mathbf{u_i^T u_j} = \delta_\mathbf{ij}$ yield

$$\mathbf{U^T U} = \mathbf{I}.$$

This means that $\mathbf{U}^T$ is the inverse of $\mathbf{U}$: $\mathbf{U}$ is an orthogonal matrix. Then $\mathbf{U}\mathbf{U}^T = I$. Now put $\mathbf{U}^T = \mathbf{X}$, then $\mathbf{U}\mathbf{U}^T = \mathbf{X}^T\mathbf{X} = I$. This means that the columns of $\mathbf{X}$, like the columns of $\mathbf{U}$, are orthogonal. Suppose $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$, where

$$\begin{aligned} \mathbf{x}_1 &= [x_{11}, x_{21}, \dots, x_{n1}]^T = [u_{11}, u_{12}, \dots, u_{1n}]^T, \\ \mathbf{x}_2 &= [x_{12}, x_{22}, \dots, x_{n2}]^T = [u_{21}, u_{22}, \dots, u_{2n}]^T, \\ &\vdots \end{aligned} \tag{2.2}$$

The set of equations (2.1) for $i = 1, 2, \dots, n$, may be written as

$$A\mathbf{U} = \mathbf{U}\Lambda.$$

Thus, on transposing we find

$$\mathbf{X}A = \Lambda\mathbf{X},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Suppose $A \in S_n$ is symmetric matrix and that A^* differs from A only in the $(1,1)$ entry, i.e., $A^* = A + (a_1^* - a_1)E_{1,1}$. We will show that $\sigma(A) = (\lambda_i)_1^n$ and $\sigma(A^*) = (\mu_i)_1^n$ determine $\mathbf{x}_1$ in (2.2). Let $A^{**} = A^* + (a_2^* - a_2)E_{2,2}$, A^{**} differs from A^* only in the $(2,2)$ entry. We will show that $\sigma(A^*) = (\mu_i)_1^n$ and $\sigma(A^{**}) = (\nu_i)_1^n$ determine $\mathbf{x}_2$ in (2.2). The eigenvalue equation for A^* is

$$A^*\mathbf{u} = \lambda\mathbf{u} \tag{2.3}$$

which we write as follows

$$A\mathbf{u} + (a_1^* - a_1)u_1 e_1 = \lambda\mathbf{u}. \tag{2.4}$$

Since the eigenvectors $\mathbf{u}_i$ are linearly independent, we may write

$$\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i \tag{2.5}$$

and then

$$A\mathbf{u} = \sum_{i=1}^{n} \alpha_i A\mathbf{u}_i = \sum_{i=1}^{n} \lambda_i \alpha_i \mathbf{u}_i,$$

so that (2.4) becomes

$$\sum_{i=1}^{n} \lambda_i \alpha_i \mathbf{u}_i + (a_1^* - a_1)u_1\mathbf{e}_1 = \lambda \sum_{i=1}^{n} \alpha_i \mathbf{u}_i,$$

and the orthogonality condition $\mathbf{u}_j^T\mathbf{u}_i = \delta_{ij}$ gives

$$\alpha_i(\lambda - \lambda_i) = (a_1^* - a_1)u_1 u_{1i} = (a_1^* - a_1)u_1 x_{i1}, \tag{2.6}$$

where we have used (2.2). Substituting α_i in (2.5) we find

$$\mathbf{u} = (a_1^* - a_1)u_1 \sum_{i=1}^{n} \frac{x_{i1}}{\lambda - \lambda_i}\mathbf{u}_i.$$

Comparing the first components on each side of the above equation, we have

$$1 = (a_1^* - a_1) \sum_{i=1}^{n} \frac{x_{i1}^2}{\lambda - \lambda_i}. \tag{2.7}$$

Note that $u_1 \neq 0$. Because if $u_1 = 0$, then we find from (2.6) $\alpha_i = 0$ or $\lambda = \lambda_i$. The first case leads to $\mathbf{u} = 0$ (see eq. (2.5)) and it contradicts with the fact that $\mathbf{u}$ is an eigenvector. The second case contradicts with the interlacing property.

The zeros of equation (2.7) are $(\mu_i)_i^n$, so that

$$1 - (a_1^* - a_1) \sum_{i=1}^{n} \frac{x_{i1}^2}{\lambda - \lambda_i} = \prod_{i=1}^{n} \Big(\frac{\lambda - \mu_i}{\lambda - \lambda_i}\Big). \tag{2.8}$$

Multiplying both sides of (2.8) by $(\lambda - \lambda_j)$ and then putting $\lambda = \lambda_j$ we find

$$(a_1^* - a_1)x_{i1}^2 = (\mu_i - \lambda_i)\prod_{j=1}^{n}{}'\Big(\frac{\lambda_i - \mu_j}{\lambda_i - \lambda_j}\Big), \quad i = 1, 2, \ldots, n, \tag{2.9}$$

where $'$ indicates that the term $j = i$ has been omitted. The interlacing condition (1.1) ensures that the right hand side of (2.9) is strictly positive for each $i = 1, 2, \ldots, n$. By comparing the traces of A and A^* we see that

$$a_1^* - a_1 = \sum_{j=1}^{n}(\mu_i - \lambda_i) > 0. \tag{2.10}$$

Thus, equation (2.9) yields $(x_{i1})^2$, the first column of matrix X, i.e., $\mathbf{x}_1$ in terms of $\sigma(A)$ and $\sigma(A^*)$.

We repeat previous procedure to matrix A^{**}. The eigenvalue problem for A^{**} is

$$A^{**}\mathbf{v} = \eta\mathbf{v}.$$

The definition of A^{**} yields

$$A^*\mathbf{v} + (a_2^* - a_2)v_2\mathbf{e}_2 = \eta\mathbf{v}. \tag{2.11}$$

Let

$$\mathbf{v} = \sum_{i=1}^{n}\beta_i\mathbf{u}_i \tag{2.12}$$

and then

$$A^*\mathbf{v} = \sum_{i=1}^{n}\beta_i A^*\mathbf{u}_i = \sum_{i=1}^{n}\mu_i\beta_i\mathbf{u}_i,$$

so that (2.11) becomes

$$\sum_{i=1}^{n}\mu_i\beta_i\mathbf{u}_i + (a_2^* - a_2)v_2\mathbf{e}_2 = \eta\sum_{i=1}^{n}\beta_i\mathbf{u}_i,$$

and again the orthogonality condition $\mathbf{u}_j^T\mathbf{u}_i = \delta_{ij}$ gives

$$\beta_i(\eta - \mu_i) = (a_2^* - a_2)v_2u_{2i} = (a_2^* - a_2)v_2x_{i2},$$

where we have used (2.2). Substituting β_i in (2.12) we find

$$\mathbf{v} = (a_2^* - a_2)v_2\sum_{i=1}^{n}\frac{x_{i2}}{\eta - \mu_i}\mathbf{u}_i.$$

Comparing the second components on each side of the equation, we have

$$1 = (a_2^* - a_2)\sum_{i=1}^{n}\frac{x_{i2}^2}{\eta - \mu_i}.$$

Note that $v_2 \neq 0$. The zeros of this equation are $(\nu_i)_1^n$, so that

$$1-(a_2^*-a_2)\sum_{i=1}^{n}\frac{x_{i2}^2}{\eta-\mu_i}=\prod_{i=1}^{n}\Big(\frac{\eta-\nu_i}{\eta-\mu_i}\Big). \tag{2.13}$$

Multiplying both sides of (2.13) by $(\eta-\mu_j)$ and then putting $\eta=\mu_j$ we find

$$(a_2^*-a_2)x_{i2}^2=(\nu_i-\mu_i)\prod_{j=1}^{n}{}'\Big(\frac{\mu_i-\nu_j}{\mu_i-\mu_j}\Big),\quad i=1,2,\ldots,n, \tag{2.14}$$

where $'$ indicates that the term $j=i$ has been omitted. The interlacing condition (1.2) ensures that the right hand side of (2.14) is strictly positive for each $i=1,2,\ldots,n$. By comparing the traces of A^* and A^{**} we see that

$$a_2^*-a_2=\sum_{j=1}^{n}(\nu_i-\mu_i)>0. \tag{2.15}$$

Thus, equation (2.14) yields $(x_{i2})^2$, the second column of matrix $\mathbf{X}$, i.e., $\mathbf{x}_2$ in terms of $\sigma(A^*)$ and $\sigma(A^{**})$.

We showed that if A is an arbitrary symmetric matrix, then $\sigma(A)$ and $\sigma(A^*)$ determine the vector $\mathbf{x}_1$ of first components of the normalized eigenvectors of A and $\sigma(A^*)$ and $\sigma(A^{**})$ determine the vector $\mathbf{x}_2$ of second components of the normalized eigenvectors of A. If we know that A is a pentadiagonal matrix then we can use block Lanczos algorithm to determine the whole matrix A (see the Appendix A). The following algorithm lists the steps for constructing the matrix A:

Construction Algorithm:

Step 1.: Take three eigenvalues $(\lambda_i)_{i=1}^n$, $(\mu_i)_{i=1}^n$ and $(\nu_i)_{i=1}^n$ with interlacing properties (1.1) and (1.2).

Step 2.: Compute the first column of eigenvectors from (2.9).

Step 3.: Compute the second column of eigenvectors from (2.14).

Step 4.: Construct the pentadiagonal matrix A from Lanczos algorithm (see the Appendix A).

3. Numerical example

Here we illustrate the construction procedure by an example. Assume A is a pentadiagonal 8×8 matrix. Let the prescribed three spectra are as follows

$$\begin{aligned}
\sigma(A) &= \{\lambda_1 = 0.0145,\ \lambda_2 = 0.2189,\ \lambda_3 = 1,\ \lambda_4 = 2.7314,\\
&\quad \lambda_5 = 5.5098,\ \lambda_6 = 9,\ \ \lambda_7 = 12.4757,\ \lambda_8 = 15.0496\},\\
\sigma(A^*) &= \{\mu_1 = 0.0422,\ \mu_2 = 0.41,\ \mu_3 = 1.5754,\ \mu_4 = 3.8783,\\
&\quad \mu_5 = 7.2413,\ \ \mu_6 = 11.0764,\ \mu_7 = 14.42,\ \mu_8 = 17.3565\},\\
\sigma(A^{**}) &= \{\nu_1 = 0.0703,\ \nu_2 = 0.6791,\ \nu_3 = 2.5414,\ \nu_4 = 5.9693,\\
&\quad \nu_5 = 10.3808,\ \ \nu_6 = 13.5855,\ \nu_7 = 14.4214,\ \nu_8 = 27.3521\}.
\end{aligned}$$

These eigenvalues satisfy the conditions (1.1) and (1.2). Eqs. (2.9) and (2.14) give the first and second column of eigenvector matrix X as

$$\begin{aligned}
\mathbf{x}_1 &= [0.1614, 0.3030, 0.4082, 0.4642, 0.4642, 0.4082, 0.3030, 0.1612]^T,\\
\mathbf{x}_2 &= [0.1339, 0.3316, 0.4848, 0.5155, 0.4040, 0.1962, 0.0058, 0.4119]^T.
\end{aligned}$$

Here we use the positive square roots of x_{i1} and x_{i2}. Hence different choice for them will lead to different solutions. Now by using block Lanczos algorithm, we find matrix A as follows

$$A = \begin{pmatrix}
4.9997 & -1.8521 & 3.6838 & & & & & \\
-1.8521 & 11.3535 & 3.2857 & 2.3930 & & & & \\
3.6838 & 3.2857 & 10.3087 & -0.4074 & 3.8510 & & & \\
 & 2.3930 & -0.4074 & 4.9082 & -3.0039 & 3.2534 & & \\
 & & 3.8510 & -3.0039 & 3.9887 & -0.9348 & 0.2278 & \\
 & & & 3.2534 & -0.9348 & 5.8839 & 2.4454 & 0.4742\\
 & & & & 0.2278 & 2.4454 & 3.2397 & 1.4942\\
 & & & & & 0.4742 & 1.4942 & 1.3173
\end{pmatrix}.$$

Computing the spectrum of matrix A (the solution matrix) confirms the given eigenvalues of the matrix A.

Note that it is very easy use this algorithm for constructing any higher order pentadiagonal matrices.

4. Conclusion

In this paper, we construct a symmetric pentadiagonal matrix from three given spectra with interlacing properties (1.1) and (1.2). This problem will have multiple solutions because in calculating the values x_{i1} and x_{i2} we have different choice. Also, in constructing process Lanczos algorithm yields different solution. Numerical example shows the accuracy of the method and details of the algorithm confirms the simplicity of the construction algorithm.

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Appendix A. Block Lanczos algorithm [11]

Suppose that $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are given. Let $\mathbf{X}_1 = [\mathbf{x}_1, \mathbf{x}_2]$, this algorithm construct a pentadiagonal matrix A of the block form

$$A = \begin{pmatrix} A_1 & B_1^T & & \\ B_1 & A_2 & B_2^T & \\ & \ddots & \ddots & \\ & & B_{s-1} & A_s \end{pmatrix}$$

and orthogonal matrix $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_s)$ such that

$$\mathbf{X}A = \Lambda\mathbf{X}. \tag{A.1}$$

Here $n = 2s$ for some integer s, A_i, $i = 1, 2, \ldots, s$ are symmetric matrices of order 2×2 and B_i, $i = 1, 2, \ldots, s-1$ are upper triangular matrix of order 2×2. We may write the equation (A.1) as

$$(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_s) \begin{pmatrix} A_1 & B_1^T & & \\ B_1 & A_2 & B_2^T & \\ & \ddots & \ddots & \\ & & B_{s-1} & A_s \end{pmatrix} = \Lambda(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_s). \tag{A.2}$$

Take this equation column by column. The first column is

$$\mathbf{X}_1 A_1 + \mathbf{X}_2 B_1 = \Lambda\mathbf{X}_1. \tag{A.3}$$

Premultiply this by X_1^T, using $X_1^T X_1 = 1$ and $X_1^T X_2 = 0$, we obtain

$$A_1 = \mathbf{X}_1^T \Lambda \mathbf{X}_1.$$

Now rewrite equation(A.3) as

$$\mathbf{X}_2 B_1 = \Lambda\mathbf{X}_1 - \mathbf{X}_1 A_1 = \mathbf{Z}_2. \tag{A.4}$$

The vector $\mathbf{Z}_2$ is known, because $\mathbf{X}_1$, A_1, Λ are all known. Let $\mathbf{X}_2 = [\mathbf{y}_1, \mathbf{y}_2]$, $\mathbf{Z}_2 = [\mathbf{z}_1, \mathbf{z}_2]$ and

$$B_1 = \begin{pmatrix} b_{11} & b_{12} \\ & b_{22} \end{pmatrix}.$$

The first column of equation (A.4) gives

$$b_{11}\mathbf{y}_1 = \mathbf{z}_1.$$

Since the vector $\mathbf{X}_2$ is to be a unit vector, so that

$$b_{11} = \pm||\mathbf{z}_1||, \quad \mathbf{y}_1 = \mathbf{z}_1/b_{11}. \tag{A.5}$$

The second column of (A.4) gives

$$b_{12}\mathbf{y}_1 + b_{22}\mathbf{y}_2 = \mathbf{z}_2$$

which leads to

$$b_{12} = \mathbf{y}_1^T \mathbf{z}_2, \quad b_{22}\mathbf{y}_2 = \mathbf{z}_2 - b_{12}\mathbf{y}_1 = \mathbf{w}_2,$$

so that

$$b_{22} = \pm||\mathbf{w}_2||, \quad \mathbf{y}_2 = \mathbf{w}_2/b_{22}. \tag{A.6}$$

For finding orthonormal vectors $\mathbf{y}_1$, $\mathbf{y}_2$ and the elements of matrix B_1 a Gram-Schmidt process is necessary. Note that different choices for the square roots, as in (A.5) and (A.6) will lead to different matrices A.

Now we can proceed as before. We have found $\mathbf{X}_2$, so that

$$A_2 = \mathbf{X}_2^T A \mathbf{X}_2$$

and

$$\Lambda \mathbf{X}_2 = \mathbf{X}_1 B_1^T + \mathbf{X}_2 A_2 + \mathbf{X}_3 B_2$$

so that

$$\mathbf{X}_3 B_2 = \Lambda \mathbf{X}_2 - \mathbf{X}_1 B_1^T - \mathbf{X}_2 A_2 = \mathbf{Z}_3$$

for which $\mathbf{X}_3$, B_2 may be found, as before by Gram-Schmidt algorithm. By repeating the above process we can find matrices A_i, B_i and therefore A. This algorithm can be applied for $n = 2s - 1$, similarly.

K. Ghanbari
Department of Mathematics, Sahand University of Technology, P.O.Box 1996–51335, Tabriz, Iran.
Email: kghanbari@sut.ac.ir

M. R. Moghaddam
Department of Mathematics, Sahand University of Technology, P.O.Box 1996–51335, Tabriz, Iran.
Email: moghaddam403@gmail.com