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CLASSICAL AND STRONGLY CLASSICAL n-ABSORBING SECOND SUBMODULES

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Abstract. Let R be a commutative ring with identity and M be an R-module. The main purpose of this paper is to introduce and investigate the notion of classical and strongly classical n-absorbing second submodules as a dual notion of classical n-absorbing submodules. We obtain some basic properties of these classes of modules.

1. Introduction

Throughout this paper, R is a commutative ring with identity. Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [13]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the endomorphism of M given by multiplication by a is is either surjective or zero [19]. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [14].

Let $n \geq 2$ be a positive integer. The concept of 2-absorbing ideals was introduced in [9] and then extended to n-absorbing ideals in [1]. Also, one can see a kind of generalization of 2-absorbing ideals in [15]. A proper ideal I is called an *n-absorbing ideal* of R if whenever $x_1 \dots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$, then there are n of x_i's whose their

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product is in I. A proper submodule N of M is called *n-absorbing submodule* of M if whenever $a_1 \dots a_n m \in N$ for $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \dots a_n \in (N :_R M)$ or there are $n-1$ of a_i's whose their product with m is in N [12]. In [17], the authors introduced the notion of classical n-absorbing submodules as a generalization of n-absorbing submodules and studied some properties of this class of modules. A proper submodule N of M is called *classical n-absorbing submodule* if whenever $a_1, \dots, a_{n+1} \in R$ and $m \in M$ with $a_1 \dots a_{n+1} m \in N$, then there are n of a_i's whose their product with m is in N. The authors in [7], introduced and studied the concept of n-absorbing second and strongly n-absorbing second submodules as dual notion of n-absorbing submodules. A non-zero submodule N of M is said to be an *n-absorbing second submodule* of M if whenever $a_1, \dots, a_n \in R$, L is a completely irreducible submodule of M, and $a_1 \dots a_n N \subseteq L$, then there are $n-1$ of a_i's whose their product with N is a subset of L or $a_1 \dots a_n \in Ann_R(N)$. Also, a non-zero submodule N of M is said to be a *strongly n-absorbing second submodule* of M if whenever $a_1, \dots, a_n \in R$, K is a submodule of M, and $a_1 \dots a_n N \subseteq K$, then there are $n-1$ of a_i's whose their product with N is a subset of K or $a_1 \dots a_n \in Ann_R(N)$. Also, in [2] classical and strongly classical 2-absorbing second submodules was studied. A non-zero submodule N is a *classical 2-absorbing second submodule* of M if whenever $a, b, c \in R$, L is a completely irreducible submodule of M, and $abcN \subseteq L$, then $abN \subseteq L$ or $acN \subseteq L$ or $bcN \subseteq L$. The module M is a classical 2-absorbing second module if M is a classical 2-absorbing second submodule of itself. A non-zero submodule N of M is a *strongly classical 2-absorbing second submodule* of M if whenever $a, b, c \in R$, L_1, L_2, L_3 are completely irreducible submodules of M, and $abcN \subseteq L_1 \cap L_2 \cap L_3$, then $abN \subseteq L_1 \cap L_2 \cap L_3$ or $acN \subseteq L_1 \cap L_2 \cap L_3$ or $bcN \subseteq L_1 \cap L_2 \cap L_3$. Also, M is a strongly classical 2-absorbing second module if M is a strongly classical 2-absorbing second submodule of itself.

The purpose of this paper is to introduce the concepts of classical and strongly classical n-absorbing second submodules of an R-module M as dual notion of classical n-absorbing submodules and provide some information concerning these new classes of modules. Also, classical n-absorbing (resp. strongly classical n-absorbing) second submodules is a generalization of classical 2-absorbing (resp. strongly classical 2-absorbing) second submodules. In this paper, we generalize some results given in [2].

2. Classical n-absorbing second submodules

We begin with the following remark.

Remark 2.1. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$ [6, Theorem 2.1].

The following definition is a generalization of [2, Definition 2.2].

Definition 2.2. Let N be a non-zero submodule of an R-module M. We say that N is a *classical n-absorbing second submodule* of M if whenever $a_1, \dots, a_{n+1} \in R$, L is a completely irreducible submodule of M and $a_1 a_2 \dots a_{n+1} N \subseteq L$, then there are n of a_i's whose product with N is a subset of L. We say M is a classical n-absorbing second module if M is a classical n-absorbing second submodule of itself.

Let t be a positive integer number, $i \in \{1, \dots, t\}$, $a_1, \dots, a_t \in R$ and let $I_1, \dots, I_t$ be ideals of R. In the rest of this paper, we denote by $\widehat{a_{i,t}}$ and $\widehat{I_{i,t}}$ the product of all elements of $\{a_1, \dots, a_t, 1\} \setminus \{a_i\}$ and the product of all elements of $\{I_1, \dots, I_t, R\} \setminus \{I_i\}$, respectively. For abbreviation, we denote $\widehat{a_{i,n}}$ and $\widehat{I_{i,n}}$ by $\widehat{a}_i$ and $\widehat{I}_i$, respectively.

There are interesting results in [2] on classical 2-absorbing second submodules. We extend them for classical n-absorbing second submodules in the next results.

Proposition 2.3. *Let M be an R-module and N be a non-zero submodule of M. Then we have the following:*

(a) *If N is a classical n-absorbing second submodule of M, then N is a classical m-absorbing second submodule of M, for every $m \geq n$.*

(b) *If N is an n-absorbing second submodule of M, then N is a classical n-absorbing second submodule of M.*

(c) *If N is a classical n-absorbing second submodule of M, then rN is a classical n-absorbing second submodule of M, for every $r \in R \setminus Ann_R(N)$.*

Proof. (a) It is clear.

(b) Let $a_1, \dots, a_{n+1} \in R$, L be a completely irreducible submodule of M and let $a_1 \dots a_n a_{n+1} N \subseteq L$. Then $a_1 \dots a_n N \subseteq (L :_M a_{n+1})$. We note that by [8, Lemma 2.1], $(L :_M a_{n+1})$ is a completely irreducible submodule of M. Since N is an n-absorbing second submodule, either $a_1 \dots a_n N = 0$ or $\widehat{a}_i N \subseteq (L :_M a_{n+1})$, for some i, $1 \leq i \leq n$. Hence $\widehat{a_{i,n+1}} N \subseteq L$, for some i, $1 \leq i \leq n+1$ and the proof is complete.

(c) The proof is similar to the proof of previous part. □

Theorem 2.4. *Let M be an R-module. Then N is a classical n-absorbing second submodule of M if and only if $(L :_R N)$ is an n-absorbing ideal of R, for every completely irreducible submodule L of M with $N \not\subseteq L$.*

Proof. First, suppose that N is a classical n-absorbing second submodule of M. Let $a_1 \dots a_{n+1} \in (L :_R N)$, for some $a_1, \dots, a_{n+1} \in R$. Then $a_1 \dots a_{n+1} N \subseteq L$. Since N is a classical n-absorbing second submodule and L is a completely irreducible submodule of M, $\widehat{a_{i,n+1}} N \subseteq L$, for some i, $1 \leq i \leq n+1$. Hence $\widehat{a_{i,n+1}} \in (L :_R N)$ and so $(L :_R N)$ is an n-absorbing ideal of R. The proof of converse is clear. □

We recall that an R-module M is said to be a *cocyclic module* if the sum of all minimal submodules of M is a large and simple submodule of M [20]. A submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R-module [14].

Corollary 2.5. *Let N be a classical n-absorbing second submodule of a cocyclic R-module M. Then $Ann_R(N)$ is an n-absorbing ideal of R.*

Proof. This follows from Theorem 2.4, because (0) is a completely irreducible submodule of M. □

Example 2.6. For every prime integer p, let $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module and $G_t = \langle 1/p^t + \mathbb{Z} \rangle$, for $t \in \mathbb{N}$. We know that $Ann_{\mathbb{Z}} G_t = p^t\mathbb{Z}$. Consider $t \geq n+1$. Let $a_1 = \cdots = a_n = p$ and $a_{n+1} = p^{t-n}$. Then $a_1 \dots a_{n+1} \in p^t\mathbb{Z}$. On the other hand, we have $\widehat{a_{n+1,n+1}} = p^n \notin p^t\mathbb{Z}$ and $\widehat{a_{i,n+1}} = p^{t-1} \notin p^t\mathbb{Z}$, for every i, $1 \leq i \leq n$. This implies that $p^t\mathbb{Z}$ is not n-absorbing ideal of $\mathbb{Z}$, for every $t \geq n+1$. Hence by Corollary 2.5, G_t is not classical n-absorbing second submodule of M, for every $t \geq n+1$. In the next section, we study the case that $t \leq n$.

Proposition 2.7. *Let M be an R-module and let N_i be a classical n_i-absorbing second submodule of M, for $i = 1, \dots, m$. Then $\sum_{i=1}^m N_i$ is a classical n-absorbing second submodule of M, where $n \geq \sum_{i=1}^m n_i$.*

Proof. Let $a_1, \dots, a_{n+1} \in R$ and let L be a completely irreducible submodule of M such that $a_1 \dots a_{n+1} \sum_{i=1}^m N_i \subseteq L$. Since N_i is a classical n_i-absorbing second submodule of M, there exists a subset $\{t_{i_1}, \dots, t_{i_{n_i}}\}$ of $\{1, \dots, n+1\}$ such that $a_{t_{i_1}} \dots a_{t_{i_{n_i}}} N_i \subseteq L$, for every i, $1 \leq i \leq m$. Since $n \geq \sum_{i=1}^m n_i$, $\{1, \dots, n+1\} \setminus \bigcup_{i=1}^m \{t_{i_1}, \dots, t_{i_{n_i}}\} \neq \emptyset$. Let $s \in \{1, \dots, n+1\} \setminus \bigcup_{i=1}^m \{t_{i_1}, \dots, t_{i_{n_i}}\}$. It is not hard to see that $\widehat{a_{s,n+1}} N_i \subseteq L$, for every i, $1 \leq i \leq m$. Hence $\widehat{a_{s,n+1}} \sum_{i=1}^m N_i \subseteq L$. Therefore $\sum_{i=1}^m N_i$ is a classical n-absorbing second submodule of M. □

A commutative ring R is said to be a u-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals. A um-ring is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them [18]. Now, we would like to study the classical n-absorbing second submodules, where R is a u-ring. First, we need the following lemma.

Lemma 2.8. *Let R be a u-ring, M be an R-module, N be a non-zero submodule of M, $n \geq 3$ and let j be a positive integer number with $j \in \{1, \dots, n-2\}$. Then $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$, where (a), (b) and (c) are the following conditions:*

(a) For every $a_1, \dots, a_{n-j} \in R$, every ideals $I_1, \dots, I_j$ of R, and submodule K of M with $a_1 a_2 \dots a_{n-j} I_1 \dots I_j N \nsubseteq K$, $(K :_R a_1 \dots a_{n-j} I_1 \dots I_j N) = (\bigcup_{i=1}^{n-j} (K :_R \widehat{a_{i,n-j}} I_1 \dots I_j N)) \cup (\bigcup_{i=1}^{j} (K :_R a_1 \dots a_{n-j} \widehat{I_{i,j}} N))$.

(b) For every $a_1, \dots, a_{n-j-1} \in R$, every ideals $I_1, \dots, I_{j+1}$ of R, and submodule K of M with $a_1 a_2 \dots a_{n-j-1} I_1 \dots I_{j+1} N \nsubseteq K$, $(K :_R a_1 \dots a_{n-j-1} I_1 \dots I_{j+1} N) = (\bigcup_{i=1}^{n-j-1} (K :_R \widehat{a_{i,n-j-1}} I_1 \dots I_{j+1} N)) \cup (\bigcup_{i=1}^{j+1} (K :_R a_1 \dots a_{n-j-1} \widehat{I_{i,j+1}} N))$.

(c) For every $a_1 \in R$, every ideals $I_1, \dots, I_{n-1}$ of R, and submodule K of M with $a_1 I_1 \dots I_{n-1} N \nsubseteq K$, $(K :_R a_1 I_1 \dots I_{n-1} N) = (K :_R I_1 \dots I_{n-1} N) \cup (\bigcup_{i=1}^{n-1} (K :_R a_1 \widehat{I_{i,n-1}} N))$.

Proof. $(a) \Rightarrow (b)$ For every $a_1, \dots, a_{n-j} \in R$, every ideals $I_1, \dots, I_{j+1}$ of R, and submodule K of M with $a_1 a_2 \dots a_{n-j-1} I_1 \dots I_{j+1} N \nsubseteq K$, suppose that $a_{n-j} \in (K :_R a_1 \dots a_{n-j-1} I_1 \dots I_{j+1} N)$. Then $a_1 \dots a_{n-j} I_1 \dots I_{j+1} N \subseteq K$ and so $I_{j+1} \subseteq (K :_R a_1 \dots a_{n-j} I_1 \dots I_j N)$. By part (a), we find that if $a_1 a_2 \dots a_{n-j} I_1 \dots I_j N \nsubseteq K$, then $I_{j+1} \subseteq (\bigcup_{i=1}^{n-j} (K :_R \widehat{a_{i,n-j}} I_1 \dots I_j N)) \cup (\bigcup_{i=1}^{j} (K :_R a_1 \dots a_{n-j} \widehat{I_{i,j}} N))$. Therefore either $a_1 a_2 \dots a_{n-j} I_1 \dots I_j N \subseteq K$ or $I_{j+1} \subseteq (\bigcup_{i=1}^{n-j} (K :_R \widehat{a_{i,n-j}} I_1 \dots I_j N)) \cup (\bigcup_{i=1}^{j} (K :_R a_1 \dots a_{n-j} \widehat{I_{i,j}} N))$. As R is a u-ring, $I_{j+1} \subseteq (\bigcup_{i=1}^{n-j} (K :_R \widehat{a_{i,n-j}} I_1 \dots I_j N)) \cup (\bigcup_{i=1}^{j} (K :_R a_1 \dots a_{n-j} \widehat{I_{i,j}} N))$ and $a_{n-j} \in (\bigcup_{i=1}^{n-j-1} (K :_R \widehat{a_{i,n-j-1}} I_1 \dots I_{j+1} N)) \cup (\bigcup_{i=1}^{j} (K :_R a_1 \dots a_{n-j-1} \widehat{I_{i,j+1}} N))$ are equivalent, because $a_1 a_2 \dots a_{n-j-1} I_1 \dots I_{j+1} N \nsubseteq K$. Now, we have either $a_1 a_2 \dots a_{n-j} I_1 \dots I_j N \subseteq K$ or $a_{n-j} \in (\bigcup_{i=1}^{n-j-1} (K :_R \widehat{a_{i,n-j-1}} I_1 \dots I_{j+1} N)) \cup (\bigcup_{i=1}^{j} (K :_R a_1 \dots a_{n-j-1} \widehat{I_{i,j+1}} N))$ which implies that $a_{n-j} \in (\bigcup_{i=1}^{n-j-1} (K :_R \widehat{a_{i,n-j-1}} I_1 \dots I_{j+1} N)) \cup (\bigcup_{i=1}^{j+1} (K :_R a_1 \dots a_{n-j-1} \widehat{I_{i,j+1}} N))$. Therefore $(K :_R a_1 \dots a_{n-j-1} I_1 \dots I_{j+1} N) \subseteq (\bigcup_{i=1}^{n-j-1} (K :_R \widehat{a_{i,n-j-1}} I_1 \dots I_{j+1} N)) \cup (\bigcup_{i=1}^{j+1} (K :_R a_1 \dots a_{n-j-1} \widehat{I_{i,j+1}} N))$. Hence part (b) holds because the reverse inclusion is clear.

$(a) \Rightarrow (c)$ This is clear by repeating $(a) \Rightarrow (b)$, $n - j - 1$ times. □

A proper ideal I is a *strongly n-absorbing ideal* of R if whenever $I_1 \dots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R then there are n of the I_i's whose their product is in I [1]. Clearly a strongly n-absorbing ideal of

R is also an n-absorbing ideal of R. Anderson and Badawi conjectured that these two concepts are equivalent, e.g., they proved that an ideal I of a Prüfer domain R is strongly n-absorbing if and only if I is an n-absorbing ideal of R [1, Corollary 6.9]. Now, we are in a position to prove one of the main results.

Theorem 2.9. *Let R be a u-ring, M be an R-module and let N be a non-zero submodule of M. Then the following statements are equivalent:*

(a) *N is a classical n-absorbing second submodule of M;*

(b) *For every $a_1, \ldots, a_n \in R$ and completely irreducible submodule L of M with $a_1a_2 \ldots a_nN \nsubseteq L$, $(L :_R a_1a_2 \ldots a_nN) = \bigcup_{i=1}^{n}(L :_R \widehat{a_i}N)$;*

(c) *For every $a_1, \ldots, a_n \in R$ and completely irreducible submodule L of M with $a_1a_2 \ldots a_nN \nsubseteq L$, $(L :_R a_1a_2 \ldots a_nN) = (L :_R \widehat{a_i}N)$, for some i, $1 \leq i \leq n$;*

(d) *For every $a_1, \ldots, a_n \in R$, every ideal I of R, and completely irreducible submodule L of M with $a_1a_2 \ldots a_nIN \subseteq L$, either $a_1a_2 \ldots a_nN \subseteq L$ or $\widehat{a_i}IN \subseteq L$, for some i, $1 \leq i \leq n$;*

(e) *For every $a_1, \ldots, a_{n-1} \in R$ and for ideal I of R and completely irreducible submodule L of M with $a_1 \ldots a_{n-1}IN \nsubseteq L$, either $(L :_R a_1 \ldots a_{n-1}IN) = (L :_R a_1 \ldots a_{n-1}N)$ or $(L :_R a_1 \ldots a_{n-1}IN) = (L :_R \widehat{a_{i,n-1}}IN)$, for some i, $1 \leq i \leq n-1$;*

(f) *For every $a_1, \ldots, a_{n-1} \in R$ and for ideals I_1, I_2 of R and completely irreducible submodule L of M with $a_1 \ldots a_{n-1}I_1I_2N \subseteq L$, either $a_1 \ldots a_{n-1}I_1N \subseteq L$ or $a_1 \ldots a_{n-1}I_2N \subseteq L$ or $\widehat{a_{i,n-1}}I_1I_2N \subseteq L$, for some i, $1 \leq i \leq n-1$;*

(g) *For ideals $I_1, \ldots, I_n$ of R and completely irreducible submodule L of M with $I_1I_2 \ldots I_nN \nsubseteq L$, $(L :_R I_1I_2 \ldots I_nN) = (L :_R \widehat{I_i}N)$, for some i, $1 \leq i \leq n$;*

(h) *For ideals $I_1, \ldots, I_{n+1}$ of R and completely irreducible submodule L of M with $I_1I_2 \ldots I_{n+1}N \subseteq L$, $\widehat{I_{i,n+1}}N \subseteq L$, for some i, $1 \leq i \leq n+1$.*

(i) *For each completely irreducible submodule L of M with $N \nsubseteq L$, $(L :_R N)$ is a strongly n-absorbing ideal of R.*

Proof. (a) $\Rightarrow$ (b) Let $a \in (L :_R a_1a_2 \ldots a_nN)$. Then $aa_1a_2 \ldots a_nN \subseteq L$. Since $a_1a_2 \ldots a_nN \nsubseteq L$, and N is a classical n-absorbing second submodule of M, we conclude that $a \in (L :_R \widehat{a_i}N)$, for some i, $1 \leq i \leq n$. Therefore $(L :_R a_1a_2 \ldots a_nN) \subseteq (L :_R \widehat{a_1}N) \cup (L :_R \widehat{a_2}N) \cup \ldots \cup (L :_R \widehat{a_n}N)$. This completes the proof because the reverse inclusion is clear.

(b) $\Rightarrow$ (c) This follows from the fact that R is a u-ring.

(c) $\Rightarrow$ (d) Suppose that for some $a_1, \ldots, a_n \in R$, an ideal I of R, and completely irreducible submodule L of M, $a_1a_2 \ldots a_nIN \subseteq L$ and $a_1a_2 \ldots a_nN \nsubseteq L$. This yields that $I \subseteq (L :_R a_1 \ldots a_nN)$. Now, by part (c), $(L :_R a_1a_2 \dots a_nN) = (L :_R \widehat{a_i}N)$, for some i, $1 \le i \le n$. So, $I \subseteq (L :_R \widehat{a_i}N)$, for some i, $1 \le i \le n$, as desired.

$(d) \Rightarrow (e) \Rightarrow (f)$ The proofs are similar to that of the previous implications.

$(f) \Rightarrow (g)$ If $n = 2$, then we are done. So we may assume that $n \ge 3$. Let $I_1, \dots, I_n$ be ideals of R, L be a completely irreducible submodule M with $I_1I_2 \dots I_nN \not\subseteq L$ and let $a_1 \in (L :_R I_1I_2 \dots I_nN)$. By part (f) we find that part (a) of Lemma 2.8 is true for $j = 1$. Therefore part (c) of Lemma 2.8 holds. So, for every $a_1 \in R$ every ideals $I_1, \dots, I_{n-1}$ we have that $(L :_R a_1I_1 \dots I_{n-1}N) = (L :_R I_1 \dots I_{n-1}N) \cup (\bigcup_{i=1}^{n-1}((L :_R a_1\widehat{I_{i,n-1}}N)))$. In particular, suppose that $a_1I_1 \dots I_nN \subseteq L$ and $I_1 \dots I_nN \not\subseteq L$. This shows that $I_n \subseteq (L :_R a_1I_1 \dots I_{n-1}N)$. Hence $I_n \subseteq (L :_R I_1 \dots I_{n-1}N) \cup (\bigcup_{i=1}^{n-1}((L :_R a_1\widehat{I_{i,n-1}}N)))$. Since R is a u-ring, either $I_n \subseteq (L :_R I_1 \dots I_{n-1}N)$ or $I_n \subseteq (L :_R a_1\widehat{I_{i,n-1}}N)$, for some i, $1 \le i \le n-1$. Therefore either $I_1 \dots I_nN \subseteq L$ or $a_1\widehat{I_{i,n-1}}I_nN \subseteq L$, for some i, $1 \le i \le n-1$. As $I_1 \dots I_nN \not\subseteq L$, $(L :_R I_1I_2 \dots I_nN) = (L :_R \widehat{I_i}N)$, for some i, $1 \le i \le n$. Hence part (g) holds.

$(g) \Rightarrow (h) \Rightarrow (i)$ The proofs are clear.

$(i) \Rightarrow (a)$ It is clear by Theorem 2.4 and the fact that every strongly n-absorbing ideal is an n-absorbing ideal. □

Proposition 2.10. *Let N be a classical n-absorbing second submodule of an R- module M. Then we have the following statements:*

(a) If $a \in R$, then $a^iN = a^{i+1}N$, for all $i \ge n$.

(b) If L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $\sqrt{(L :_R N)}$ is an n-absorbing ideal of R.

Proof. (a) It is enough to show that $a^nN = a^{n+1}N$. Clearly, $a^{n+1}N \subseteq a^nN$. Let L be a completely irreducible submodule of M such that $a^{n+1}N \subseteq L$. Since N is a classical n-absorbing second submodule, $a^nN \subseteq L$. Hence by Remark 2.1, $a^nN \subseteq a^{n+1}N$.

(b) Assume that $a_1 \dots a_{n+1} \in \sqrt{(L :_R N)}$. Then there is a positive integer t such that $a_1^t \dots a_{n+1}^tN \subseteq L$. Since N is a classical n-absorbing second submodule of M, $\widehat{a_{i,n+1}}^tN \subseteq L$, for some i, $1 \le i \le n+1$. This implies that $\widehat{a_{i,n+1}} \in \sqrt{(L :_R N)}$, for some i, $1 \le i \le n+1$ and the proof is complete. □

Theorem 2.11. *Let N be a submodule of an R-module M. Then we have the following statements:*

(a) If R is a u-ring and N is a classical n-absorbing second submodule of M, then IN is a classical n-absorbing second submodule of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.

(b) *If R is a um-ring and N is a classical n-absorbing submodule of M, then $(N :_M I)$ is a classical n-absorbing submodule of M for all ideals I of R with $I \nsubseteq (N :_R M)$.*

(c) *Let $f : M \to M'$ be a monomorphism of R-modules. If N' is a classical n-absorbing second submodule of $f(M)$, then $f^{-1}(N')$ is a classical n-absorbing second submodule of M.*

Proof. (a) Let I be an ideal of R with $I \nsubseteq Ann_R(N)$. Since $I \nsubseteq Ann_R(N)$, IN is a non-zero submodule of M. Let $a_1, \ldots, a_{n+1} \in R$, L be a completely irreducible submodule of M, and $a_1 \ldots a_{n+1}IN \subseteq L$. Then by Theorem 2.9 ($a$) $\Rightarrow$ (d), we find that $a_1a_2 \ldots a_nN \subseteq L$ or $\widehat{a_{i,n+1}}IN \subseteq L$, for some i, $1 \leq i \leq n$. If $\widehat{a_{i,n+1}}IN \subseteq L$, for some i, then we are done. Also, if $a_1a_2 \ldots a_nN \subseteq L$, then $\widehat{a_{n+1,n+1}}IN \subseteq a_1a_2 \ldots a_nN \subseteq L$. This completes the proof.

(b) The proof is clear with [17, Theorem 2.6].

(c) The proof is similar to that of [2, Theorem 2.9]. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [10]. An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [5].

Corollary 2.12. *Let M be an R-module. Then we have the following statements:*

(a) *If R is a u-ring and M is a multiplication classical n-absorbing second R-module, then every non-zero submodule of M is a classical n-absorbing second submodule of M.*

(b) *If R is a um-ring, M is a comultiplication module and the zero submodule of M is a classical n-absorbing submodule, then every proper submodule of M is a classical n-absorbing submodule of M.*

Proof. This follows from Theorem 2.11 parts (a) and (b). □

Proposition 2.13. *Let M be an R-module and $\{K_i\}_{i \in I}$ be a chain of classical n-absorbing second submodules of M. Then $\sum_{i \in I} K_i$ is a classical n-absorbing second submodule of M.*

Proof. We use the technique of the proof of [2, Proposition 2.11]. Let $a_1, \ldots, a_{n+1} \in R$, L be a completely irreducible submodule of M, and let $a_1 \ldots a_{n+1} \sum_{i \in I} K_i \subseteq L$. Assume that $\widehat{a_{i,n+1}} \sum_{i \in I} K_i \nsubseteq L$, for every $i = 1, \ldots, n$. We prove that $\widehat{a_{n+1,n+1}} \sum_{i \in I} K_i \subseteq L$. There are $t_1, \ldots, t_n \in I$, such that $\widehat{a_{i,n+1}}K_{t_i} \nsubseteq L$, for every $i = 1, \ldots, n$. Let $K_{t_i} \subseteq K_h$, for every $i = 1, \ldots, n$. Clearly, $\widehat{a_{i,n+1}}K_h \nsubseteq L$, for every

$i = 1, \ldots, n$. Since K_h is a classical n-absorbing second submodules of M, $\widehat{a_{n+1,n+1}} K_h \subseteq L$. As $\{K_i\}_{i\in I}$ is a chain, $\sum_{i\in I} K_i = \sum_{K_i \subseteq K_h} K_i + \sum_{K_h \subset K_i} K_i = K_h + \sum_{K_h \subset K_i} K_i$. Let $K_{h'} \in \{K_i\}_{i\in I}$ and $K_h \subset K_{h'}$. As we saw before, $\widehat{a_{n+1,n+1}} K_{h'} \subseteq L$ and so $\widehat{a_{n+1,n+1}} \sum_{i\in I} K_i \subseteq L$, as needed. □

Definition 2.14. We say that a classical n-absorbing second submodule N of an R-module M is a *maximal classical n-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a classical n-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 2.15. *Let M be an R-module. Then every classical n-absorbing second submodule of M is contained in a maximal classical n-absorbing second submodule of M.*

Proof. This is proved easily by Zorn's Lemma and Proposition 2.13. □

Theorem 2.16. *Let M be an Artinian R-module. Then every non-zero submodule of M has only a finite number of maximal classical n-absorbing second submodules.*

Proof. We use the technique of the proof of [2, Theorem 2.14]. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal classical n-absorbing second submodules. Let S be a submodule of M chosen minimal such that S has an infinite number of maximal classical n-absorbing second submodules because M is an Artinian R-module. Then S is not a classical n-absorbing second submodule. Thus there exist $a_1, \ldots, a_{n+1} \in R$ and a completely irreducible submodule L of M such that $a_1 \ldots a_{n+1} S \subseteq L$ and $\widehat{a_{i,n+1}} S \nsubseteq L$, for every i, $1 \leq i \leq n+1$. Let V be a maximal classical n-absorbing second submodule of M contained in S. Then $\widehat{a_{i,n+1}} V \subseteq L$, for some i, $1 \leq i \leq n+1$. Therefore $V \subseteq (L :_M \widehat{a_{i,n+1}})$, for some i, $1 \leq i \leq n+1$. Hence $V \subseteq (L :_S \widehat{a_{i,n+1}})$, for some i, $1 \leq i \leq n+1$. By choice of S, the module $(L :_S \widehat{a_{i,n+1}})$ has only finitely many maximal classical n-absorbing second submodules, for every $i = 1, \ldots, n+1$. This implies that there is only a finite number of possibilities for the module S, a contradiction. □

3. Strongly classical n-absorbing second submodules

In this section, the notion of strongly classical n-absorbing submodules is introduced and some of their basic properties are given. Most of the results below are the same as ones in [2] when $n = 2$.

Definition 3.1. Let N be a non-zero submodule of an R-module M. We say that N is a *strongly classical n-absorbing second submodule* of M if whenever $a_1, \dots, a_{n+1} \in R$, $L_1, \dots, L_{n+1}$ are completely irreducible submodules of M, and $a_1 \dots a_{n+1} N \subseteq \bigcap_{i=1}^{n+1} L_i$, then $\widehat{a_{i,n+1}} N \subseteq \bigcap_{i=1}^{n+1} L_i$, for some i, $1 \leq i \leq n+1$. We say M is a strongly classical n-absorbing second module if M is a strongly classical n-absorbing second submodule of itself.

Theorem 3.2. *Let N be a submodule of an R-module M. Then N is a strongly classical n-absorbing submodule of M if and only if $(K :_R N)$ is an n-absorbing ideal of R, for each submodule K of M with $N \nsubseteq K$.*

Proof. First, suppose that N is a strongly classical n-absorbing submodule of M. Let $a_1, \dots, a_{n+1} \in R$ and let K be a submodule of M with $a_1 \dots a_{n+1} N \subseteq K$. By contradiction, suppose that $\widehat{a_{i,n+1}} N \nsubseteq K$, for every i, $1 \leq i \leq n+1$. There exist completely irreducible submodules $L_1, \dots, L_{n+1}$ of M such that K is a submodule of them and $\widehat{a_{i,n+1}} N \nsubseteq L_i$, for every i, $1 \leq i \leq n+1$. Clearly, $a_1 \dots a_{n+1} N \subseteq \bigcap_{i=1}^{n+1} L_i$. Since N is a strongly classical n-absorbing submodule of M, $\widehat{a_{j,n+1}} N \subseteq \bigcap_{i=1}^{n+1} L_i$, for some j, $1 \leq j \leq n+1$. This shows that $\widehat{a_{j,n+1}} N \subseteq L_j$, for some j, $1 \leq j \leq n+1$, which is a contradiction. Conversely, assume that $(K :_R N)$ is an n-absorbing ideal of R, for each submodule K of M with $N \nsubseteq K$. Let $a_1, \dots, a_{n+1} \in R$, $L_1, \dots, L_{n+1}$ be completely irreducible submodule of M and let $a_1 \dots a_{n+1} N \subseteq \bigcap_{i=1}^{n+1} L_i$. Then $a_1 \dots a_{n+1} \in (K :_R N)$, where $K = \bigcap_{i=1}^{n+1} L_i$. Since $(K :_R N)$ is an n-absorbing ideal of R, $\widehat{a_{i,n+1}} \in (K :_R N)$, for some i, $1 \leq i \leq n+1$ and the proof is complete. □

Remark 3.3. Let N be a non-zero submodule of an R-module M. By the above theorem, N is a strongly classical n-absorbing second submodule of M if whenever $a_1, \dots, a_{n+1} \in R$, K is a submodules of M, and $a_1 \dots a_{n+1} N \subseteq K$, then $\widehat{a_{i,n+1}} N \subseteq K$, for some i, $1 \leq i \leq n+1$.

Remark 3.4. Let N be a strongly classical n-absorbing second submodule of an R-module M and let $a_1, \dots, a_{n+1} \in R$. Then $a_1 \dots a_{n+1} N \subseteq a_1 \dots a_{n+1} N$ implies that $\widehat{a_{i,n+1}} N \subseteq a_1 \dots a_{n+1} N$, for some i, $1 \leq i \leq n+1$ by Theorem 3.2. Therefore $\widehat{a_{i,n+1}} N = a_1 \dots a_{n+1} N$, for some i, $1 \leq i \leq n+1$. Hence we conclude that N is a strongly classical n-absorbing second submodule of M if and only if for every $a_1, \dots, a_{n+1} \in R$, $a_1 \dots a_{n+1} N = \widehat{a_{i,n+1}} N$, for some i, $1 \leq i \leq n+1$. This yields that N is a strongly classical n-absorbing second submodule of M if and only if N is a strongly classical n-absorbing second module.

Example 3.5. By the above remark, we find that the $\mathbb{Z}$-module $\mathbb{Z}$ has no strongly classical n-absorbing second submodule.

Corollary 3.6. *Let N be a non-zero submodule of an R-module M. Then we have the following:*

(a) If N is a strongly classical n-absorbing second submodule, then N is a classical n-absorbing second submodule.

(b) If N is a strongly classical n-absorbing second submodule, then N is a strongly classical m-absorbing second submodule, for every $m \geq n$.

(c) If N is a strongly n-absorbing second submodule, then N is a strongly classical n-absorbing second submodule.

(d) If N is a strongly classical n-absorbing second submodule of M, then rN is a strongly classical n-absorbing second submodule of M, for every $r \in R \setminus Ann_R(N)$.

Proof. Parts (a) and (b) are clear.

(c) Let $a_1, \dots, a_{n+1} \in R$, K be a submodule of M and let $a_1 \dots a_n a_{n+1} N \subseteq K$. Then $a_1 \dots a_n N \subseteq (K :_M a_{n+1})$. Since N is a strongly n-absorbing second submodule, either $a_1 \dots a_n N = 0$ or $\widehat{a_t} N \subseteq (K :_M a_{n+1})$, for some t, $1 \leq t \leq n$. Hence $\widehat{a_{t,n+1}} N \subseteq K$, for some t, $1 \leq t \leq n+1$ and the proof is complete.

(d) The proof is similar to the part (c). □

A non-zero submodule N of an R-module M is said to be a *weakly second submodule* of M if $a_1 a_2 N \subseteq K$, where $a_1, a_2 \in R$ and K is a submodule of M, implies either $a_1 N \subseteq K$ or $a_2 N \subseteq K$ [6].

Proposition 3.7. *Let M be an R-module and let N_i be a strongly classical n_i-absorbing second submodule of M, for $i = 1, \dots, m$. Then $\sum_{i=1}^{m} N_i$ is a strongly classical n-absorbing second submodule of M, where $n \geq \sum_{i=1}^{m} n_i$. In particular, if $N_1, \dots, N_n$ are weakly second submodules of M, then $\sum_{i=1}^{n} N_i$ is a strongly classical n-absorbing second submodule of M.*

Proof. Use Remark 3.3 and apply the proof of Proposition 2.7. □

In the next theorem, we argue about u-ring. Compare parts (a) and (k) of the following result with Theorem 3.2.

Theorem 3.8. *Let R be a u-ring, M be an R-module and let N be a non-zero submodule of M. Then the following statements are equivalent:*

(a) N is strongly classical n-absorbing second submodule of M;

(b) For every $a_1, \dots, a_{n+1} \in R$, K a submodule of M with $a_1 \dots a_n N \subseteq K$, then $\widehat{a_{i,n+1}} N \subseteq K$, for some i, $1 \leq i \leq n+1$;

(c) *For every* $a_1,\dots,a_{n+1}\in R$, $a_1\dots a_{n+1}N=\widehat{a_{i,n+1}}N$, *for some* i, $1\le i\le n+1$;

(d) *For every* $a_1,\dots,a_n\in R$, K *a submodule of* M *with* $a_1a_2\dots a_nN\nsubseteq K$, $(K:_R a_1a_2\dots a_nN)=\bigcup_{i=1}^{n}(K:_R\widehat{a_i}N)$;

(e) *For every* $a_1,\dots,a_n\in R$, K *a submodule of* M *with* $a_1a_2\dots a_nN\nsubseteq K$, $(K:_R a_1a_2\dots a_nN)=(K:_R\widehat{a_i}N)$, *for some* i, $1\le i\le n$;

(f) *For every* $a_1,\dots,a_n\in R$, *every ideal* I *of* R, *and submodule* K *of* M *with* $a_1a_2\dots a_nIN\subseteq K$, *either* $a_1a_2\dots a_nN\subseteq K$ *or* $\widehat{a_i}IN\subseteq K$, *for some* i, $1\le i\le n$;

(g) *For every* $a_1,\dots,a_{n-1}\in R$, *every ideal* I *of* R, *and submodule* K *of* M *with* $a_1a_2\dots a_{n-1}IN\nsubseteq K$, *either* $(K:_R a_1\dots a_{n-1}IN)=(K:_R a_1\dots a_{n-1}N)$ *or* $(K:_R a_1\dots a_{n-1}IN)=(K:_R\widehat{a_{i,n-1}}IN)$, *for some* i, $1\le i\le n-1$;

(h) *For every* $a\in R$, *ideals* $I_1,\dots,I_n$ *of* R, *and submodule* K *of* M *with* $aI_1\dots I_nN\subseteq K$, *either* $I_1\dots I_nN\subseteq K$ *or* $a\widehat{I_i}N\subseteq K$, *for some* i, $1\le i\le n$;

(i) *For ideals* $I_1,\dots,I_n$ *of* R, *and submodule* K *of* M *with* $I_1\dots I_nN\nsubseteq K$, $(K:_R I_1\dots I_nN)=(K:_R\widehat{I_i}N)$, *for some* i, $1\le i\le n$;

(j) *For ideals* $I_1,\dots,I_{n+1}$ *of* R, *and submodule* K *of* M *with* $I_1\dots I_{n+1}N\subseteq K$, $\widehat{I_{i,n+1}}N\subseteq K$, *for some* i, $1\le i\le n+1$.

(k) *For each submodule* K *of* M *with* $N\nsubseteq K$, $(K:_R N)$ *is a strongly* n*-absorbing ideal of* R.

Proof. $(a)\Rightarrow(b)$ The proof is clear by Theorem 3.2.

$(b)\Rightarrow(c)$ It is clear by Remark 3.4.

$(c)\Rightarrow(d)$ Suppose that $a\in(K:_R a_1\dots a_nN)$. Then $aa_1\dots a_nN\subseteq K$. Since $a_1\dots a_nN\nsubseteq K$, $a\widehat{a_i}N\subseteq K$, for some i, $1\le i\le n$. Hence $a\in(K:_R\widehat{a_i}N)$, as needed.

$(d)\Rightarrow(e)$ This follows from the fact that R is a u-ring.

$(e)\Rightarrow(f)$ Let for some $a_1,\dots a_n\in R$, an ideal I of R and submodule K of M, $a_1\dots a_nIN\subseteq K$. Then $I\subseteq(K:_R a_1\dots a_nN)$. If $a_1\dots a_nN\subseteq K$, then we are done. Otherwise, by part (e), we find that $I\subseteq(K:_R\widehat{a_i}N)$, for some i, $1\le i\le n$, as desired.

$(f)\Rightarrow(g)$ Trivial.

$(g)\Rightarrow(h)$ By part (g) and Lemma 2.8, we have that for every $a\in R$, every ideals $I_1,\dots,I_{n-1}$ of R, and submodule K of M, either $aI_1\dots I_{n-1}N\subseteq K$ or $(K:_R aI_1\dots I_{n-1}N)=(K:_R I_1\dots I_{n-1}N)\cup(\bigcup_{i=1}^{n-1}(K:_R a\widehat{I_i}N))$. Now, let I_n be an ideal of R and $aI_1\dots I_nN\subseteq K$. Then either $aI_1\dots I_{n-1}N\subseteq K$ or $I_n\subseteq(K:_R I_1\dots I_{n-1}N)\cup(\bigcup_{i=1}^{n-1}(K:_R a\widehat{I_{i,n-1}}N))$. Thus part (h) holds.

$(h) \Rightarrow (i) \Rightarrow (j)$ The proofs are similar to that of the previous implications.

$(j) \Rightarrow (k)$ Trivial.

$(k) \Rightarrow (a)$ It is clear by Theorem 3.2 and the fact that every strongly n-absorbing ideal is an n-absorbing ideal. □

Proposition 3.9. *Let N be a strongly classical n-absorbing second submodules of an R-module M. Then we have the following statements:*

(a) If I is an ideal of R, then $I^iN = I^{i+1}N$, for all $i \geq n$.

(b) If K is a submodule of M such that $N \not\subseteq K$, then $\sqrt{(K :_R N)}$ is an n-absorbing ideal of R.

Proof. (a) It is enough to show that $I^nN = I^{n+1}N$. But it is clear by Remark 3.4.

(b) The proof is similar to the proof of Proposition 2.10 part (b). □

Theorem 3.10. *Let N be a submodule of an R-module M and let $f : M \to M'$ be a monomorphism. Then we have the following:*

(a) If N is a strongly classical n-absorbing second submodule of M, then $f(N)$ is a strongly n-absorbing second submodule of M'.

(b) If N' is a strongly classical n-absorbing second submodule of M', then $f^{-1}(N')$ is a strongly classical n-absorbing second submodule of M.

Proof. (a) Since $N \neq 0$ and f is a monomorphism, $f(N) \neq 0$. Let $a_1 \dots a_{n+1} \in R$. By Remark 3.4, we have $a_1 \dots a_{n+1}N = \widehat{a_{i,n+1}}N$, for some i, $1 \leq i \leq n+1$. Thus $a_1 \dots a_{n+1}f(N) = f(a_1 \dots a_{n+1}N) = f(\widehat{a_{i,n+1}}N) = \widehat{a_{i,n+1}}f(N)$. This implies that $f(N)$ is a strongly classical n-absorbing submodule of M' by Remark 3.4.

(b) If $f^{-1}(N') = 0$, then $f(M) \cap N' = ff^{-1}(N') = f(0) = 0$. Thus $N' = 0$, a contradiction. Therefore $f^{-1}(N') \neq 0$. Let $a_1, \dots, a_{n+1} \in R$, K a submodule of M and $a_1 \dots a_{n+1}f^{-1}(N') \subseteq K$. Then $a_1 \dots a_{n+1}N' = a_1 \dots a_{n+1}(f(M) \cap N') = a_1 \dots a_{n+1}ff^{-1}(N') \subseteq f(K)$. Since N' is a strongly classical n-absorbing second submodule of M', $\widehat{a_{i,n+1}}N' \subseteq f(K)$, for some i, $1 \leq i \leq n+1$. Therefore $\widehat{a_{i,n+1}}f^{-1}(N') \subseteq f^{-1}f(K) = K$. □

The following examples show that the two concepts of classical n-absorbing submodules and strongly classical n-absorbing second submodules are different in general.

Example 3.11. For every prime integer p, let $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module and $G_t = \langle 1/p^t + \mathbb{Z} \rangle$, for $t \in \mathbb{N}$. Consider $t \leq n$. We prove that G_t is a strongly classical n-absorbing second submodule of $\mathbb{Z}_{p^\infty}$. By Remark 3.4, it is enough to show that for every $a_1, \dots, a_{n+1} \in \mathbb{Z}$, $a_1 \dots a_{n+1}G_t =$

$\widehat{a_{i,n+1}}G_t$, for some i, $1 \leq i \leq n+1$. We know that if $(a_i, p) = 1$, for some i, $1 \leq i \leq n+1$, then $a_iG_t = G_t$. Hence $a_1 \dots a_{n+1}G_t = \widehat{a_{i,n+1}}G_t$. Therefore we may assume that $(a_i, p) \neq 1$, for every i, $1 \leq i \leq n+1$. Then we have $a_1G_t \subseteq G_{t-1}$, $a_1a_2G_t \subseteq G_{t-2}$, $\dots$, $a_1 \dots a_tG_t = 0$ and so $a_1 \dots a_{n+1}G_t = 0$. Since $t \leq n$, $\widehat{a_{n+1,n+1}}G_t = 0 = a_1 \dots a_{n+1}G_t$. This completes the proof.

We note that G_t is not a classical n-absorbing submodule of $\mathbb{Z}_{p^\infty}$. Because $p^{n+1}(1/p^{t+n+1} + \mathbb{Z}) = 1/p^t + \mathbb{Z} \in G_t$ and $p^n(1/p^{t+n+1} + \mathbb{Z}) = 1/p^{t+1} + \mathbb{Z} \notin G_t$.

Example 3.12. Let p be a prime integer and let $t \in \{1, \dots, n\}$. The submodule $p^t\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is classical n-absorbing submodule which is not strongly classical n-absorbing second module.

Proposition 3.13. *Let M be an R-module. Then we have the following:*

(a) *Let R be a u-ring. If M is a comultiplication R-module and N is a strongly classical n-absorbing second submodule of M, then N is a strongly n-absorbing second submodule of M.*

(b) *If N is a strongly classical n-absorbing second submodule of M, then IN is a strongly classical n-absorbing second submodule of M for all ideals I of R with $I \not\subseteq Ann_R(N)$.*

(c) *If M is a multiplication strongly classical n-absorbing second R-module, then every non-zero submodule of M is a classical n-absorbing second submodule of M.*

(d) *If M is a strongly classical n-absorbing second R-module, then every non-zero homomorphic image of M is a classical n-absorbing second R-module.*

Proof. (a) By Theorem 3.8 part (k), $Ann_R(N)$ is a strongly n-absorbing ideal of R. Now, the result follows from [7, Theorem 2.12].

(b) It is clear with Remark 3.4.

(c) This follows from part (b).

(d) It is clear with Remark 3.4. □

For a submodule N of an R-module M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [11]).

Theorem 3.14. *Let R be a Prüfer domain and let M be a finitely generated comultiplication R-module. If N is a strongly classical n-absorbing second submodule of M, then $sec(N)$ is a strongly n-absorbing second submodule of M.*

Proof. Let N be a strongly classical n-absorbing second submodule of M. By Theorem 3.2, $Ann_R(N)$ is a n-absorbing ideal of R. Thus by [1, Theorem 2.1], $\sqrt{Ann_R(N)}$ is an n-absorbing ideal of R. By [3, Theorem 2.12], $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$. Therefore, $Ann_R(sec(N))$ is an n-absorbing ideal of R. Since R is a Prüfer domain, $Ann_R(sec(N))$ is a strongly n-absorbing ideal by [1, Corollary 6.9]. Now, the result follows from [7, Theorem 2.12]. □

If N is a strongly classical n-absorbing second submodule of M for some positive integer n, then $w_M(N) = min\{n| \ N$ is strongly classical n-absorbing second submodule of M $\}$; otherwise, set $w_M(N) = \infty$ (we will just write $w(N)$ when the context is clear). Moreover, we define $w_M(0) = 0$. Therefore, for any submodule N of M, we have $w_M(N) \in \mathbb{N} \cup \{0, \infty\}$, with $w_M(N) = 1$ if and only if N is a weakly second submodule of M and $w_M(N) = 0$ if and only if $N = 0$. Then $w_M(N)$ measures, in some sense, how far N is from being a weakly second submodule of M.

Let M_i be an R_i-module for each $i = 1, 2, \ldots, m$ and $m \in \mathbb{N}$. Assume that $M = M_1 \times \cdots \times M_m$ and $R = R_1 \times \cdots \times R_m$. Then M is clearly an R-module with componentwise addition and multiplication. Also, each submodule of M is of the form $N = N_1 \times \cdots \times N_m$ where N_i is a submodule of M_i. We are now ready for one of the main result of this section.

Theorem 3.15. *Let $R = R_1 \times \cdots \times R_m$ $(2 \leq m < \infty)$ be a decomposable ring and $M = M_1 \times \cdots \times M_m$ be an R-module where for every $1 \leq i \leq m$, M_i is an R_i-module, respectively. Suppose that $N = N_1 \times \cdots \times N_m$ is a non-zero submodule of M. Then N is a strongly classical n-absorbing second submodule of M if and only if one of the following conditions holds:*

(a) $w_{M_t}(N_t) \leq n$, for some $t \in \{1, \ldots, m\}$ and $N_i = 0$ for every $i \in \{1, \ldots, m\} \setminus \{t\}$;

(b) $w_{M_i}(N_i) \leq n-1$, for every $i \in \{1, \ldots, m\}$. Moreover, $\sum_{i=1}^{m} w_{M_i}(N_i) \leq n$.

Proof. First, assume that N is a strongly classical n-absorbing second submodule of M. Let $A = \{i | 1 \leq i \leq m, N_i \neq 0\}$ and let $|A| = t$. With no loss of generality, we may assume that $N_1, \ldots, N_t \neq 0$. Consider two following cases:

Case 1. $|A| = 1$. Then $N = N_1 \times 0 \times \cdots \times 0$. Set $M' = M_1 \times 0 \times \cdots \times 0$. One can see that N is a strongly classical n-absorbing second submodule of M'. Also, it is clear that $M' \cong M_1$ and $N \cong N_1$. Therefore part (a) holds.

Case 2. $|A| \geq 2$. We prove that N_1 is a strongly classical $n-1$-absorbing second submodule of M_1. Since $N_2 \neq 0$, there exists a completely irreducible submodules L_2 of M_2 such that $N_2 \nsubseteq L_2$. Let $a_1 \dots a_n N_1 \subseteq K_1$, for some $a_1, \dots, a_n \in R_1$ and submodule K_1 of M_1, $\alpha_i = (a_i, 1, 0, \dots, 0)$, for $i = 1, \dots, n$ and let $\alpha_{n+1} = (1, 0, \dots, 0)$. Then $\alpha_1 \dots \alpha_{n+1}(N_1 \times \cdots \times N_m) \subseteq K$, where

$$K = \begin{cases} K_1 \times L_2, & \text{if } m = 2; \\ K_1 \times L_2 \times M_3 \times \cdots \times M_m, & \text{otherwise.} \end{cases}$$

Therefore $\widehat{\alpha_{s,n+1}} N \subseteq K$, for some s, $1 \leq s \leq n+1$. If $s = n+1$, then we conclude that $N_2 \subseteq L_2$, a contradiction. Hence $\widehat{\alpha_{s,n+1}} N \subseteq K$, for some s, $1 \leq s \leq n$ which shows that $\widehat{a_s} N_1 \subseteq K_1$. Thus N_1 is a strongly classical $n-1$-absorbing second submodule of M_1. Similarly, we can show that N_i is a strongly classical $n-1$-absorbing second submodule of M_i, for every i, $2 \leq i \leq t$. Therefore $w_{M_i}(N_i) \leq n-1$, for every $i \in \{1, \dots, m\}$.
Now, we prove that $\sum_{i=1}^m w_{M_i}(N_i) \leq n$. Let $w_{M_i}(N_i) = n_i$, for $i = 1, \dots, m$. Since $w_{M_i}(N_i) = n_i > 0$ for $i = 1, \dots, t$, there exist submodules K_i of M_i, distinct elements $a_1, \dots, a_{n_1} \in R_1$, $a_{n_1+1}, \dots, a_{n_1+n_2} \in R_2$, $\dots$, $a_{(\sum_{i=1}^{t-1} n_i)+1}, \dots, a_{\sum_{i=1}^t n_i} \in R_t$ such that the following t conditions hold:

(1) $a_1 \dots a_{n_1} N_1 \subseteq K_1$ and $\overline{a_s^{(1)}} N_1 \nsubseteq K_1$, for every s, $1 \leq s \leq n_1$ (Here $\overline{a_s^{(1)}}$ is the product of all elements of $\{a_1, \dots, a_{n_1}, 1\} \setminus \{a_s\}$);

(2) $a_{n_1+1} \dots a_{n_1+n_2} N_2 \subseteq K_2$ and $\overline{a_s^{(2)}} N_2 \nsubseteq K_2$, for every s, $n_1+1 \leq s \leq n_1+n_2$ (Here $\overline{a_s^{(2)}}$ is the product of all elements of $\{a_{n_1+1}, \dots, a_{n_1+n_2}, 1\} \setminus \{a_s\}$);

$\vdots$

(t) $a_{(\sum_{i=1}^{t-1} n_i)+1} \dots a_{\sum_{i=1}^t n_i} N_t \subseteq K_t$ and $\overline{a_s^{(t)}} N_t \nsubseteq K_t$, for every s,(Here $\overline{a_s^{(t)}}$ is the product of all elements of $\{a_{\sum_{i=1}^{t-1} n_i+1}, \dots, a_{\sum_{i=1}^t n_i}, 1\} \setminus \{a_s\}$).

Let e_j be a $1 \times m$ vector whose the j'th component is 1_{R_j} and other components are 0 and let

$$\beta_i = \begin{cases} a_i e_1 + \sum_{j \neq 1} e_j, & \text{if } 1 \leq i \leq n_1; \\ a_i e_2 + \sum_{j \neq 2} e_j, & \text{if } n_1+1 \leq i \leq n_1+n_2; \\ \vdots & \vdots \\ a_i e_t + \sum_{j \neq t} e_j, & \text{if } (\sum_{i=1}^{t-1} n_i)+1 \leq i \leq \sum_{i=1}^t n_i. \end{cases}$$

It is not hard to see that $\beta_1 \dots \beta_{\sum_{i=1}^{t} n_i}(N_1 \times \cdots \times N_m) \subseteq K$, where

$$K = \begin{cases} K_1 \times K_2 \times \cdots \times K_t, & \text{if } t = m; \\ K_1 \times K_2 \times \cdots \times K_t \times M_{t+1} \times \cdots \times M_m, & \text{if } t < m. \end{cases}$$

On the other hand, $\widehat{\beta_{s,\sum_{i=1}^{t} n_i}}(N_1 \times \cdots \times N_m) \nsubseteq K$ for all $1 \leq s \leq \sum_{i=1}^{t} n_i$. This yields that $n \geq \sum_{i=1}^{t} n_i$. Therefore part (b) holds.
Conversely, assume that one of the conditions (a) and (b) holds. Let $N_1' = N_1 \times 0 \times \cdots \times 0$, $M_1' = M_1 \times 0 \times \cdots \times 0$, $N_2' = 0 \times N_2 \times 0 \times \cdots \times 0$, $M_2' = 0 \times M_2 \times 0 \times \cdots \times 0, \dots,$ $N_m' = 0 \times \cdots \times 0 \times N_m$ and let $M_m' = 0 \times \cdots \times 0 \times M_m$. Clearly, $N_i' \cong N_i$, $M_i' \cong M_i$, $M \cong \sum_{i=1}^{m} M_i'$ and $N \cong \sum_{i=1}^{m} N_i'$. Now, the result follows from Proposition 3.7 and Theorem 3.10. □

Proposition 3.16. *Let M be a non-zero R-module. Then we have the following:*

(a) *If M is a finitely generated strongly classical n-absorbing second R-module, then the zero submodule of M is a classical n-absorbing submodule.*

(b) *If M is a multiplication strongly classical n-absorbing second R-module, then the zero submodule of M is a classical n-absorbing submodule.*

(c) *Let R be a um-ring. If M is a Artinian R-module and the zero submodule of M is a classical n-absorbing submodule, then M is a strongly classical n-absorbing second R-module.*

(d) *Let R be a um-ring. If M is a comultiplication R-module and the zero submodule of M is a classical n-absorbing submodule, then M is a strongly classical n-absorbing second R-module.*

Proof. (a) Let $a_1, \dots, a_{n+1} \in R$, $m \in M$, and $a_1 \dots a_{n+1} m = 0$. By Remark 3.4, we can assume that $a_1 \dots a_{n+1} M = a_1 \dots a_n M$. Since M is finitely generated, by using [16, Theorem 76], $Ann_R(a_1 \dots a_n M) + Ra_{n+1} = R$. It follows that $(0 :_M a_1 \dots a_{n+1}) = (0 :_M a_1 \dots a_n)$. This implies that $a_1 \dots a_n m = 0$, as needed.

(b) Use Remark 3.4 and the technique of [2, Theorem 3.11, part (b)].

(c) Let $a_1, \dots, a_{n+1} \in R$. Then by [17, Theorem 2.6], we can assume that $(0 :_M a_1 \dots a_{n+1}) = (0 :_M a_1 \dots a_n)$. Now, apply the proof of [2, Theorem 3.11, part (c)] and Remark 3.4.

(d) Let $a_1, \dots, a_{n+1} \in R$. Then by [17, Theorem 2.6], we can assume that $(0 :_M a_1 \dots a_{n+1}) = (0 :_M a_1 \dots a_n)$. Now, the proof is similar to [2, Theorem 3.11, part (d)]. □

Proposition 3.17. *Let M be an R-module and $\{K_i\}_{i \in I}$ be a chain of strongly classical n-absorbing second submodules of M. Then $\sum_{i \in I} K_i$ is a strongly classical n-absorbing second submodule of M.*

Proof. The proof is similar to the proof of Proposition 2.13. □

Definition 3.18. We say that a strongly classical n-absorbing second submodule N of an R-module M is a *maximal strongly classical n-absorbing second submodule* of a submodule K of M, if $N \subseteq K$ and there does not exist a strongly classical n-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 3.19. *Let M be an R-module. Then every strongly classical n-absorbing second submodule of M is contained in a maximal strongly classical n-absorbing second submodule of M.*

Proof. This is proved easily by Zorn's Lemma and Proposition 3.17. □

Theorem 3.20. *Let M be an Artinian R-module. Then every non-zero submodule of M has only a finite number of maximal strongly classical n-absorbing second submodules.*

Proof. Use the technique of Theorem 2.16 and apply the above lemma. □

Theorem 3.21. *Let R be a um-ring and M be an R-module. If E is an injective R-module and N is a classical n-absorbing submodule of M such that $Hom_R(M/N, E) \neq 0$, then $Hom_R(M/N, E)$ is a strongly classical n-absorbing second R-module.*

Proof. Use [17, Theorem 2.6] and apply the proof of [2, Theorem 3.21]. □

Theorem 3.22. *Let M be a strongly classical n-absorbing second R-module and F be a right exact linear covariant functor over the category of R-modules. Then $F(M)$ is a strongly classical n-absorbing second R-module if $F(M) \neq 0$.*

Proof. It is clear by [6, Lemma 3.14] and Remark 3.4. □

By previous theorem, we deduce the following result.

Corollary 3.23. *Let M be an R-module, S be a multiplicative subset of R and N be a strongly classical n-absorbing second submodule of M. Then $S^{-1}N$ is a strongly classical n-absorbing second submodule of $S^{-1}M$ if $S^{-1}N \neq 0$.*

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