Journal of Algebra and Related Topics
Vol. 10, No 2, (2022), pp 61-67

THREE BOUNDS FOR IDENTIFYING CODE NUMBER

E. VATANDOOST * AND K. MIRASHEH

ABSTRACT. Let $G = (V, E)$ be a simple graph. A set C of vertices G is an identifying set of G if for every two vertices x and y belong to V the sets $N_G[x] \cap C$ and $N_G[y] \cap C$ are non-empty and different. Given a graph G, the smallest size of an identifying set of G is called the identifying code number of G and is denoted by $\gamma^{ID}(G)$. Two vertices x and y are twins when $N_G[x] = N_G[y]$. Graphs with at least two twin vertices are not identifiable graphs. In this paper, we present three bounds for identifying code number.

1. Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $G = (V, E)$ to denote the graph with non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. The order of a graph is its number of vertices and the size of a graph is its number of edges. An edge of G with end vertices u and v is denoted by $u - v$. For every vertex $x \in V(G)$, the *open neighborhood* of vertex x is denoted by $N_G(x)$ and is defined as $N_G(x) = \{y \in V(G) \ : \ x - y\}$. Also, the *closed neighborhood* of vertex $x \in V(G)$, $N_G[x]$, is $N_G[x] = N_G(x) \cup \{x\}$. The *complement* of graph G is denoted by $\overline{G}$ is a graph with vertex set $V(G)$ which $e \in E(\overline{G})$ if and only if $e \notin E(G)$. A graph G is said to be *self- complementary* if $G \cong \overline{G}$. A graph G is called *empty graph*, if $\overline{G}$ is isomorphic to the complete graph K_n. Given a graph G, *Laplacian matrix* $L_a(G)$ is defined as $L_a(G) = D - A$, where D is the degree matrix and A is the adjacency matrix of the graph

MSC(2010): Primary: 05C69; Secondary: 05C75
Keywords: Code, Strongly, Rayleigh quotient.
Received: 21 September 2021, Accepted: 13 September 2022.
∗Corresponding author .

G. A set of vertices G such as C is a *dominating set* of graph G if for every vertex x belong to $V(G)$, is either in C or is adjacent to a vertex in C. Also, a set C is called a *separating set* of G if for each pair u, v of vertices of G, $N_G[u] \cap C \neq N_G[v] \cap C$, equivalently, $(N_G[u] \bigtriangleup N_G[v]) \cap C = ((N_G[u] \setminus N_G[v]) \cup (N_G[v] \setminus N_G[u])) \cap C \neq \emptyset$. If a dominating set C in graph G is a separating set of G, then we say that C is an identifying set of graph G and if G has an identifying set, then we say that G is an *identifiable graph*. Given a graph G, the smallest size of an identifying set of G is called the *identifying code number* of G and denoted by $\gamma^{ID}(G)$. If for two distinct vertices x and y, $N_G[x] = N_G[y]$, then they are called *twins*. It is noteworthy that a graph G is an identifiable graph if and only if G is twin free.
In recent years much attention drawn to the domination theory which is very interesting branch in graph theory. The concept of domination expanded to other parameters of domination such as 2-rainbow domination, signed domination, Roman domination, total Roman domination number, and identifying code number. For more details, we refer the reader to [2, 12, 13, 14].The identifying code concept was introduced by Karpovsky et al [11] in 1998. Later, several various families of graphs have been studied; cycles and paths [4, 8], trees [1], triangular and square grids [10] and triangle-free graphs [6]. Karpovsky et al [11] shown that for every identifiable graph G of order n, $\gamma^{ID}(G) \geq \lceil log_2(n+1) \rceil$. Also they proved that $\gamma^{ID}(G) \geq \dfrac{2n}{\Delta(G)+2}$. For every identifiable graph G of order n with at least one edge there exists a famous bound as $\gamma^{ID}(G) \leq n-1$ (see [5]). In 2012 Foucaud et al [6], had a conjecture that for every connected identifiable graph G, there exist a constant c such that $\gamma^{ID}(G) \geq n - \dfrac{n}{\Delta(G)} + c$. It is noteworthy that in 2006 Gravier et al [8] investigated the identifying code number of cycles. According to their theorems, this conjecture holds for graphs of maximum degree 2.

This paper deals with the study of identifying code numbers of graphs. Also we present three bounds for identifying code number.

2. Main results

In this section, we give three bounds for identifying code number of graphs.

Lemma 2.1. *Let G be an identifiable graph. Then $\overline{G}$ is not an identifiable graph if and only if there exist two distinct vertices u and v belong $V(G)$ such that $N_G(u) = N_G(v)$.*

Proof. Let $\overline{G}$ is not an identifiable graph. Then there exist two distinct vertices u and v belong to $V(G)$ such that $N_{\overline{G}}[u] = N_{\overline{G}}[v]$. So u is not adjacent to v in G. Suppose that $x \in N_G(u)$. Then $x \notin N_{\overline{G}}[u]$ and so $x \notin N_{\overline{G}}[v]$. Hence x is adjacent to v in G. Thus $x \in N_G(v)$. This shows that $N_G(u) \subseteq N_G(v)$. Similarly we have $N_G(v) \subseteq N_G(u)$. Therefore $N_G(u) = N_G(v)$.
Conversely let there exist two distinct vertices u and v belong to $V(G)$ and $N_G(u) = N_G(v)$. Then u is not adjacent to v in G. So u is adjacent to v in $\overline{G}$.
If $x \in N_{\overline{G}}[u]$, then x is not adjacent to u in G. So $x \notin N_G(u)$. Since $N_G(u) = N_G(v)$, $x \notin N_G(v)$. Hence x is adjacent to v in $\overline{G}$. Thus $N_{\overline{G}}[u] \subseteq N_{\overline{G}}[v]$. Similarly, we have $N_{\overline{G}}[v] \subseteq N_{\overline{G}}[u]$. Therefore $N_{\overline{G}}[u] = N_{\overline{G}}[v]$. This shows that $\overline{G}$ is not an identifiable graph. □

Definition 1. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and C be an identifying set of G. We define
$E_0(C) = \{u - v \in E(G) : |\{u, v\} \cap C| = 0\}$ and $m_0(C) = |E_0(C)|$,
$E_1(C) = \{u - v \in E(G) : |\{u, v\} \cap C| = 1\}$ and $m_1(C) = |E_1(C)|$,
$E_2(C) = \{u - v \in E(G) : |\{u, v\} \cap C| = 2\}$ and $m_2(C) = |E_2(C)|$.
Also for $x_i \in \{0, 1\}$, we define $X_C^t = [x_1, x_2, \ldots, x_n]$ such that $x_i = 1$ if and only if $v_i \in C$.

Theorem 2.2. *Let G be an identifiable graph such that for every two distinct vertices u and v belong $V(G)$, $N_G(u) \neq N_G(v)$. If $\lambda_{max}(G)$ and $\lambda_{max}(\overline{G})$ are the largest eigenvalues of G and $\overline{G}$, respectively, then $\gamma^{ID}(G) + \gamma^{ID}(\overline{G}) \leq 2(\lambda_{max}(G) + \lambda_{max}(\overline{G}) + 1)$.*

Proof. Let A and $\widehat{A}$ be adjacency matrices of G and $\overline{G}$, respectively. Also let J_n be an square matrix of order n with $(J_n)_{ij} = 1$ for every $1 \leq i, j \leq n$. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and C be an identifying set of G with $\gamma^{ID}(G) = |C|$. By Rayleigh quotient, $\frac{X_C^t A X_C}{X_C^t X_C} \leq \lambda_{max}(G)$ and $\frac{X_C^t \widehat{A} X_C}{X_C^t X_C} \leq \lambda_{max}(\overline{G})$. So $\frac{X_C^t A X_C + X_C^t \widehat{A} X_C}{X_C^t X_C} \leq \lambda_{max}(G) + \lambda_{max}(\overline{G})$. Since $A + \widehat{A} = J_n - I_n$ and $X_C^t X_C = \gamma^{ID}(G)$, so

$$X_C^t A X_C + X_C^t \widehat{A} X_C = X_C^t (J_n - I_n) X_C = \gamma^{ID}(G)(\gamma^{ID}(G) - 1).$$

Hence $\gamma^{ID}(G) - 1 \leq \lambda_{max}(G) + \lambda_{max}(\overline{G})$. By Lemma 2.1, $\overline{G}$ is an identifiable graph and so we have

$$\gamma^{ID}(\overline{G}) - 1 \leq \lambda_{max}(G) + \lambda_{max}(\overline{G}).$$

Therefore $\gamma^{ID}(G) + \gamma^{ID}(\overline{G}) \leq 2(\lambda_{max}(G) + \lambda_{max}(\overline{G}) + 1)$. □

A graph G is called self- complementary , if $G \cong \overline{G}$.

Corollary 2.3. *Let G be an self- complementary, which is an identifiable graph . If $\lambda_{max}(G)$ is the largest eigenvalue of G, then $\gamma^{ID}(G) \leq 2\lambda_{max}(G) + 1$.*

Proof. By Theorem 2.2, the proof is straightforward. □

Let G be a regular graph that is neither complete not empty. Then G is said to be strongly regular graph with parameters (n, k, a, c) if it is k-regular, every pair of adjacent vertices has a common neighbors, and every pair of distinct nonadjacent vertices has c common neighbors.

Lemma 2.4. [9] *Let G be an (n, k, a, c) strongly regular graph. Then the following are equivalent:*

i) *G is not connected,*
ii) *$c = 0$,*
iii) *$a = k - 1$,*
iv) *G is isomorphic to mK_{k+1} for some $m > 1$.*

Theorem 2.5. *Let G be a strongly regular graph with parameters (n, k, a, c). If G is connected graph, then G is an identifiable graph.*

Proof. If G is not an identifiable graph, then there exist two distinct vertices u and v belong to $V(G)$ such that $N_G[u] = N_G[v]$. Suppose that $N_G[u] = \{u, v, x_1, x_2, \ldots, x_{k-1}\}$. Then $a = k - 1$. By Lemma 2.4, G is not connected graph, which is a contradiction. Therefore G is an identifiable graph. □

Theorem 2.6. [3] *The number of walks of length ℓ in G, from v_i to v_j, is the entry in position (i, j) of the matrix A^ℓ.*

Theorem 2.7. [3] *Let G be a regular graph of degree k. Then:*

i) *k is an eigenvalue of G;*
ii) *if G is connected, then the multiplicity of k is 1;*
iii) *for any eigenvalue λ of G, we have $|\lambda| \leq k$.*

Theorem 2.8. *Let G be a connected strongly regular graph with parameters (n, k, a, c). If $a \geq 1$, then $\gamma^{ID}(G) \leq \frac{k^2-k+\alpha}{\alpha}$, where $\alpha = min\{a, c\}$.*

Proof. By Theorem 2.5, G is an identifiable graph.
Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and C be an identifying set of G with minimum cardinality. By Theorem 2.7, the largest eigenvalue of G, $\lambda_{max}(G) = k$ and so k^2 is the largest eigenvalue of A^2, where A is the adjacent matrix of G. By Rayleigh quotient, $\frac{X_C^t A^2 X_C}{X_C^t X_C} \leq k^2$.
Since $X_C^t X_C = \gamma^{ID}(G)$, $X_C^t A^2 X_C \leq k^2 \gamma^{ID}(G)$. By computing not so

hard, we have

$$X_C^t A^2 X_C = \sum_{i=1}^{n}(A^2)_{i1}x_ix_1 + \cdots + \sum_{i=1}^{n}(A^2)_{in}x_ix_n.$$

.

By Theorem 2.6,

$$(A^2)_{ij} = \begin{cases} k & i = j, \\ a \text{ or } c & i \neq j. \end{cases}$$

So $X_C^t A^2 X_C = k\sum_{i=1}^{n} x_i^2 + \sum_{i=2}^{n}(A^2)_{i1}x_ix_1 + \sum_{i=1,i\neq 2}^{n}(A^2)_{i2}x_ix_2 + \sum_{i=1,i\neq 3}^{n}(A^2)_{i3}x_ix_3 + \cdots + \sum_{i=1}^{n-1}(A^2)_{in}x_ix_n$.

Since $x_ix_j = x_jx_i$, $(A^2)_{ij} = (A^2)_{ji}$ and $\sum_{i=1}^{n} x_i^2 = \gamma^{ID}(G)$, we have:

$$X_C^t A^2 X_C = k\gamma^{ID}(G) + 2(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(A^2)_{ij}x_ix_j).$$

If $\alpha = min\{a, b\}$, then:

$$X_C^t A^2 X_C \geq k\gamma^{ID}(G) + 2\alpha(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_ix_j).$$

Since $\sum_{i=1}^{n} x_i = \gamma^{ID}(G)$, so:

$$\gamma^{ID}(G)k^2 \geq X_C^t A^2 X_C \geq k\gamma^{ID}(G) + \alpha\gamma^{ID}(G)(\gamma^{ID}(G) - 1).$$

By Lemma 2.4, $\alpha \geq 1$. Therefore $\gamma^{ID}(G) \leq \frac{k^2-k+\alpha}{\alpha}$. □

Theorem 2.9. *Let G be an identifiable graph of order n. If μ_{max} is the largest eigenvalue of Laplacian matrix $L_a(G)$, then* $\lceil \frac{n}{1+\mu_{max}} \rceil \leq \gamma^{ID}(G)$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and C be an identifying set of G with minimum cardinality. It is easy to see that:

$$X_C^t L_a(G) X_C = \sum_{i=1}^{n} x_i^2 \; deg \; v_i - \sum_{i=1}^{n} x_i|N_G(v_i) \cap C|.$$

Since $x_i \in \{0, 1\}$ and $x_i = 1$ if and only if $v_i \in C$, so:

$$X_C^t L_a(G) X_C = \sum_{v_i \in C}(deg \; v_i - |N_G(v_i) \cap C|) = m_1(C).$$

Since $m_1(C) \geq |V(G) \setminus C| = n - \gamma^{ID}(G)$, so:

$$X_C^t L_a(G) X_C \geq n - \gamma^{ID}(G).$$

By Rayleigh quotient, we have $\frac{X_C^t L_a(G) X_C}{X_C^t X_C} \leq \mu_{max}$. Hence

$$n - \gamma^{ID}(G) \leq X_C^t L_a(G) X_C \leq \gamma^{ID}(G)\mu_{max}.$$

This completes the proof. □

Theorem 2.10. *Let G be an identifiable graph of order n and size m. If λ_{max} and μ_{max} are the largest eigenvalues of A and $L_a(G)$, respectively and $\frac{2m-2}{n-2} > \lambda_{max} + 2\mu_{max}$, then $\gamma^{ID}(G) \neq n-2$.*

Proof. Let $\gamma^{ID}(G) = n-2$ and C be an identifying set with $|C| = n-2$. Then we have $m_0(C) \leq 1$, $X_C^t A X_C = 2m_2(C)$ and $2X_C^t L_a(G) X_C = 2m_1(C)$. By Rayleigh quotient, we have:

$$X_C^t A X_C + 2X_C^t L_a(G) X_C \leq \gamma^{ID}(G)(\lambda_{max} + 2\mu_{max}).$$

Since $m_0(C) + m_1(C) + m_2(C) = m$ and $m_0(C) \leq 1$, so:

$$\frac{2m-2}{\lambda_{max} + 2\mu_{max}} \leq \gamma^{ID}(G).$$

Therefore $\frac{2m-2}{n-2} \leq \lambda_{max} + 2\mu_{max}$, which is a contradiction. □

Acknowledgments

The authors are very grateful to the referee for his/her useful comments.

References

1. D. Auger, *Minimal identifying codes in trees and planar graphs with large girth*, European J. Combin. (5) **31** (2010), 1372–84.
2. J. Amjadi , S. Nazari-Moghaddam , SM. Sheikholeslami, L. Volkmann, *Total roman domination number of trees*, Australas. J. Combin. (2) **69** (2017), 271–85.
3. N. Biggs, *Algebraic graph theory*. Cambridge university press, Cambridge, 1993.
4. C. Chen, C. Lu , Z. Miao, *Identifying codes and locatingdominating sets on paths and cycles*, Discrete Appl. Math, (15) **159** (2011), 1540-7.
5. F. Foucaud, E. Guerrini , M. Kovse, R. Naserasr, A. Parreau, and P. Valicov, *Extremal graphs for the identifying code problem*. European J. Combin. (4) **32** (2011), 628–638.
6. F. Foucaud , R. Klasing, A. Kosowski, A. Raspaud, *On the size of identifying codes in triangle-free graphs*, Discrete Appl. Math, (10-11) **160** (2012), 1532–46.
7. A. Frieze , R. Martin, J. Moncel, M. Ruszinké, C. Smyth, *Codes identifying sets of vertices in random networks*, Discrete Math. (9-10) **307** (2007), 1094–107.
8. S. Gravier, J. Monce, A. Semri, *Identifying codes of cycles*, European J. Combin. (5) **27** (2006), 767–76.

9. C. Godsil, GF. Royle, *Algebraic graph theory*, Springer Science & Business Media; 2001 Apr 20.
10. I. Honkala, T. Laihonen, *On identifying codes in the triangular and square grids*, SIAM J. Comput. (2) **33** (2004), 304–12.
11. MG. Karpovsky, K. Chakrabarty, LB. Levitin, *On a new class of codes for identifying vertices in graphs*, IEEE Trans. Inf. Theory, (2) **44** (1998),599–611.
12. Z. Mansouri, DA. Mojdeh, *Outer independent rainbow dominating functions in graphs*, Opuscula Math. (5) **40** (2020), 599–615.
13. F. Ramezani , ED. Rodriguez-Bazan, JA. Rodriguez-Velazquez, *On the roman domination number of generalized Sierpinski graphs*, Filomat, (20) **31** (2017), 6515–6528.
14. A. Shaminezhad , E. Vatandoost, *On 2-rainbow domination number of functigraph and its complement*, Opuscula Math. (5) **40** (2020), 617-627.

E. Vatandoost
Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, P. O. Box 3414896818, Qazvin, Iran.
Email: vatandoost@sci.ikiu.ac.ir

K. Mirasheh
Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, P. O. Box 3414896818, Qazvin, Iran.
Email: K.mir.1185@gmail.com