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ε-ORTHOGONALITY PRESERVING PAIRS OF MAPPINGS ON HILBERT C^*-MODULES

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ABSTRACT. Let $\mathcal{A}$ be a standard C^*-algebra. In this paper, we will study the continuity of ε-orthogonality preserving mappings between Hilbert $\mathcal{A}$-modules. Moreover, we will show that a local mapping between Hilbert $\mathcal{A}$-modules is $\mathcal{A}$-linear. Furthermore, we will prove that for a pair of nonzero $\mathcal{A}$-linear mappings $T, S : E \longrightarrow F$, between Hilbert $\mathcal{A}$-modules, satisfying ε-orthogonality preserving property, there exists $\gamma \in \mathbb{C}$,

$$\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\| \leq \varepsilon\|T\|\|S\|\|x\|\|y\|, \quad x, y \in E.$$

Our results generalize the known ones in the context of Hilbert spaces.

1. Introduction

Let $\big(H, (.,.)\big)$ be an inner product space, two elements $x, y \in H$ are said to be orthogonal, and is denoted by $x \perp y$, if $(x, y) = 0$. For two inner product spaces H and K, a mapping $T : H \longrightarrow K$ is called orthogonality preserving, OP in short, if it preserves orthogonality, that is if

$$\forall x, y \in H : \ x \perp y \quad \Longrightarrow \quad T(x) \perp T(y)$$

By [4], for a pair of linear mappings $T, S : H \longrightarrow K$ between inner product spaces H and K. The following conditions are equivalent, for

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some $\gamma \in \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$:

$$1.\ \forall x, y \in H:\ x \perp y\ \Rightarrow\ T(x) \perp S(y),$$
$$2.\ \forall x, y \in H:\ \big(T(x), S(y)\big) = \gamma(x, y).$$

A generalization of the orthogonality notion, namely, approximately orthogonality preserving mappings between inner product spaces was considered in [2] and also studied in [11, 14]. Recall that for $\varepsilon \in [0, 1)$ two vectors $x, y \in H$ are approximately orthogonal or ε-orthogonal, denoted by $x \perp^{\varepsilon} y$, if $|(x, y)| \leq \varepsilon \|x\| \|y\|$.

For $(\delta, \varepsilon) \in [0, 1)$, a map $T : H \longrightarrow K$ between inner product spaces H and K is called approximately orthogonality preserving, AOP in short, or (δ, ε)-orthogonality preserving, if

$$\forall x, y \in H:\ x \perp^{\delta} y \quad \Longrightarrow \quad T(x) \perp^{\varepsilon} T(y).$$

In particular, for $\delta = 0$, the mapping $T : H \longrightarrow K$ is said to be ε-orthogonality preserving, ε-OP in short, if

$$\forall x, y \in H:\ x \perp y \quad \Longrightarrow \quad T(x) \perp^{\varepsilon} T(y).$$

For a pair of linear mappings $T, S : H \longrightarrow K$, an analoguse property

$$\forall x, y \in H:\ x \perp y \quad \Longrightarrow \quad T(x) \perp^{\varepsilon} S(y),$$

was characterized by Chmieliński et al in [3].

The notion of an inner product (respectively Hilbert) C^*-module is a generalization of a complex inner product (respectively Hilbert) space in which the inner product takes its values in a C^*-algebra rather than in field of complex numbers. Let A be a C^*-algebra. Let E be a complex linear space which is also algebric left Hilbert A-module with compatible scalar multiplication (i.e., a(λx)=(λa)x=$\lambda(ax)$ for all $x \in E, a \in A,\ \lambda \in \mathbb{C}$) equipped with an "$A$-valued inner product" ${}_A\langle ., . \rangle$ such that the following conditions hold for all $x, y, z \in E$, $a \in A$ and $\alpha, \beta \in \mathbb{C}$:

$$(i) \quad {}_A\langle \alpha x + \beta y, z \rangle = \alpha\ {}_A\langle x, z \rangle + \beta\ {}_A\langle y, z \rangle,$$
$$(ii) \quad {}_A\langle ax, y \rangle = a\ {}_A\langle x, y \rangle,$$
$$(iii) \quad {}_A\langle x, y \rangle^* =\ {}_A\langle y, x \rangle,$$
$$(iv) \quad {}_A\langle x, x \rangle \geq 0,\ and\ {}_A\langle x, x \rangle = 0\ if\ and\ only\ if\ x = 0.$$

If E is complete with respect to the induced norm by the A-valued inner product, $\|x\| = \|{}_A\langle x, x \rangle\|^{\frac{1}{2}}$, $x \in E$, then E is called a left Hilbert C^*-module over A or, simply a left Hilbert A-module (in the sequel, we will omit the subscripts). Similarly, a right Hilbert C^*-module over the C^*-algebra A has been defined. Any C^*-algebra A is a Hilbert C^*-module over itself via $\langle a, b\rangle = ab^*$ $(a, b \in A)$. Hence $|x|^2 = \langle x, x\rangle = xx^*$ for every $x \in E$, for more about Hilbert C^*-modules, see [7].

Two elements x, y in an inner product A-module $(E, \langle ., .\rangle)$ are said to be orthogonal if $\langle x, y\rangle = 0$ and, for a given $\varepsilon \in [0, 1)$, they are approximately orthogonal or ε-orthogonal if $\|\langle x, y\rangle\| \leq \varepsilon\|x\|\|y\|$. A mapping $T : E \longrightarrow F$, where E and F are inner product A-modules, is called ε-orthogonality preserving if $\langle x, y\rangle = 0$ (where $x, y \in E$) implies $\|\langle Tx, Ty\rangle\| \leq \varepsilon\|Tx\|\|Ty\|$.

Throughout, $\mathcal{F}(H)$, $\mathcal{K}(H)$ and $\mathcal{B}(H)$ denote the space of finite rank operators, the C^*-algebras of all compact operators and all bounded operators on a Hilbert spaces H, respectively. We know that $\overline{\mathcal{F}(H)} = K(H)$, is an essential ideal of $\mathcal{B}(H)$, that is, for each $b \in \mathcal{B}(H)$, the equality $\mathcal{K}(H) \cdot b = 0$ implies $b = 0$, see [12].

Recall that $\mathcal{A}$ is a standard C^*-algebra on a Hilbert space H if $\mathcal{K}(H) \subseteq \mathcal{A} \subseteq \mathcal{B}(H)$.

It is natural to explore the approximately orthogonality preserving mappings between inner product C^*-modules. For $\delta, \varepsilon \in [0, 1)$, ε-orthogonality preserving and (δ, ε)-orthogonality preserving property between Hilbert $\mathcal{A}$-modules has been studied for a nonzero $\mathcal{A}$-linear mapping by D. Ilisevic and A. Turnsek [6] and Moslehian and Zamani [10], respectively.

In [5], Frank et al proved that if a pair of nonzero local mappings $T, S : E \longrightarrow F$ between Hilbert $\mathcal{A}$-modules are orthogonal preserving, then there exists $\gamma \in \mathbb{C}$ such that

$$\langle T(x), S(y)\rangle = \gamma\langle x, y\rangle, \quad x, y \in E.$$

It is interesting to ask whether it is possible to consider ε-orthogonality preserving property for two these mappings. In this paper, we study ε-orthogonality preserving property for a pair of nonzero mappings in the setting of Hilbert C^*-modules over standard C^*-algebra $\mathcal{A}$. Then we give the estimate of $\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\|$ for a pair of local ε-orthogonality preserving mappings $T, S : E \longrightarrow F$ when E and F are Hilbert $\mathcal{A}$-modules, where $\gamma \in \mathbb{C}$.

We recall that, for a C^*-algebra A, a complex linear mapping $T : E \longrightarrow F$ between inner product A-modules E and F, is called local if

$$aT(x) = 0 \quad \text{whenever} \quad ax = 0, \quad a \in A; x \in E.$$

2. Preliminaries

Let A be a C^*-algebra. A complex linear mapping $T : E \longrightarrow F$, where E and F are inner product A-modules, is called A-linear if

$T(ax) = aT(x)$ for all $a \in A$ and $x \in E$. As example, linear differential mappings are local mapping, see [13]. Note that every A-linear mapping is local. Conversely, every bounded local mapping is A-linear, see [9, Proposition A.1].

Suppose that E and F are Hilbert A-modules. Let $\mathcal{L}(E, F)$ to be the set of all mappings $T : E \longrightarrow F$ for which there is a mapping $T^* : F \longrightarrow E$ such that for all $x \in E$ and $y \in F$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

By [7], $\mathcal{L}(E, F)$ is called the set of all adjointable mappings from E to F. Every element of $\mathcal{L}(E, F)$ is a bounded A-linear, and in general, a bounded A-linear mapping may fail to possess an adjoint, see [7]. But each bounded $\mathcal{K}(H)$-linear mapping on $\mathcal{K}(H)$-modules is essentially adjointable, see [1].

In the following we give some preliminaries about minimal projections in C^*-algebras and their role in our work.

Let $\xi, \eta \in H$ be elements of a Hilbert space $\big(H, (.,.)\big)$, the rank one operator defined by $[\xi \otimes \eta]\zeta = (\zeta, \eta)\xi$, where $\zeta \in H$.

The operator $\xi \otimes \xi$ is rank one projection if and only if $(\xi, \xi) = 1$. That is, for unit vector ξ, the operator $\xi\otimes\xi$ is the orthogonal projection to the one dimensional subspace spanned by ξ.

Let T be an arbitrary bounded operator on $\big(H, (.,.)\big)$, then

$$[\xi \otimes \xi]T[\xi \otimes \xi] = (T\xi, \xi)\xi \otimes \xi.$$

Recall that a projection (i.e., a self-adjoint idempotent.) e in $\mathcal{A}$ is called minimal if $e\mathcal{A}e = \mathbb{C}e$. Hence, $\xi \otimes \xi$ is a minimal projection.

Let $(E, \langle ., .\rangle)$ be an inner product(respectively Hilbert) $\mathcal{A}$-module, and for a unit vector $\xi \in H$, let $e = \xi \otimes \xi$ be any minimal projection. Then

$$E_e = \{ex : x \in E\},$$

is a complex inner product (respectively Hilbert) space contained in E with respect to the inner product $(x, y) = tr(\langle x, y\rangle)$, $x, y \in E_e$.

Let $x = eu, y = ev$ such that $u, v \in E$,

$$\langle x, y\rangle = e\langle u, v\rangle e = [\xi \otimes \xi]\langle u, v\rangle[\xi \otimes \xi] = \big(\langle u, v\rangle\xi, \xi\big)[\xi \otimes \xi],$$

by $\mathrm{tr}(\langle x, y\rangle) = \big(\langle u, v\rangle\xi, \xi\big)$, thus

$$\langle x, y\rangle = (x, y)e.$$

Authors in [6] showed that:

1) two elements $x, y \in E_e$ are orthogonal in $\big(E_e, (\cdot, \cdot)\big)$ if and only if they are orthogonal in $\big(E, \langle ., .\rangle\big)$,

2) if $x \in E_e$, then $\|x\|_{E_e} = \|x\|_E$, where the norm $\|.\|_{E_e}$ comes from the inner product $(\cdot, \cdot)$,

3) if $T : E \to F$ between Hilbert $\mathcal{A}$-modules E and F is an $\mathcal{A}$-linear OP (respectively ε-OP) mapping, then $T_e = T\mid_{E_e}: E_e \to F_e$ is a linear OP (respectively ε-OP) mapping.

Lemma 2.1. *Let $L \in \mathcal{B}(H)$, then*

$$\|L\| = sup\left\{\|eLf\| \ : \ e, f \text{ are rank one projections}\right\}.$$

3. ε-ORTHOGONALITY-PRESERVING $\mathcal{A}$-LINEAR AND LOCAL PAIR OF MAPPINGS

In this section, we study ε-orthogonality preserving mappings between Hilbert $\mathcal{A}$-modules. As mentioned in previous section, $\mathcal{A}$ is a standard C^*-algebra on a Hilbert space H if $\mathcal{K}(H) \subseteq \mathcal{A} \subseteq \mathcal{B}(H)$.

To achieve our main result, Theorem 3.7, we give some results. First we prove the continuity of ε-orthogonality-preserving nonzero pair of $\mathcal{A}$-linear mappings between Hilbert $\mathcal{A}$-modules.

Theorem 3.1. ***[Chmieliński et al[3]]***
For a given $\varepsilon \in [0,1)$, ε-orthogonality preserving property for two nonzero linear mappings f and g between inner product spaces X and Y, with the same inner product $(.,.)$, is equivalent to

$$\left|(f(x), g(y)) - \frac{(f(y), g(x))}{\|y\|^2}(x, y)\right| \leq \varepsilon \left\| f(x) - \frac{(x, y)}{\|y\|^2} f(y) \right\| \big\| g(y) \big\|,$$

for $x, y \in X, y \neq 0$.

As an immediate generalization, we give the next result in setting of inner product $\mathcal{A}$-modules. Let $\varepsilon \in [0,1)$, and let $T, S : E \longrightarrow F$ be pair of nonzero $\mathcal{A}$-linear ε-orthogonality preserving mappings between inner product $\mathcal{A}$-modules E and F. Then $T_e, S_e : E_e \longrightarrow F_e$ are pair of nonzero linear ε-orthogonality preserving mappings between inner product spaces E_e and F_e. Where e is a minimal projection in $\mathcal{A}$.

Proposition 3.2. *Let $\varepsilon \in [0,1)$, and let $T, S : E \longrightarrow F$ be pair of nonzero $\mathcal{A}$-linear ε-orthogonality preserving mappings between inner product $\mathcal{A}$-modules E and F. Then for every minimal projection $e \in \mathcal{A}$, and for all $x, y \in E_e$,*

$$\|\langle y, y\rangle\langle T(x), S(y)\rangle - \langle x, y\rangle\langle T(y), S(y)\rangle\| \leq \varepsilon\|\langle y, y\rangle T(x) - \langle x, y\rangle T(y)\|\|S(y)\|.$$

Consequently, T_e, S_e are a pair of nonzero linear ε-orthogonality preserving mappings.

Proof. Let e be a minimal projection of $\mathcal{A}$, for all $x, y \in E_e$,

$$\langle y, y\rangle x - \langle x, y\rangle y \perp_E y. \tag{3.1}$$

By $\langle x, y\rangle = (x, y)e$, we have

$$\begin{aligned}\big\langle \langle y, y\rangle x - \langle x, y\rangle y, y\big\rangle &= \big\langle (y, y)ex - (x, y)ey, y\big\rangle \\ &= (y, y)\langle ex, y\rangle - (x, y)\langle ey, y\rangle.\end{aligned}$$

For each $x, y \in E_e$, we have $x = eu, y = ev$ such that $u, v \in E$. Then $ex = e^2u$ and $ey = e^2v$, since e is a projection, then $ex = e^2u = eu = x$ and $ey = e^2v = ev = y$.

Hence

$$\begin{aligned}(y, y)\langle ex, y\rangle - (x, y)\langle ey, y\rangle &= (y, y)\langle x, y\rangle - (x, y)\langle y, y\rangle \\ &= (y, y)(x, y)e - (x, y)(y, y)e = 0.\end{aligned}$$

The last equality holds, because the values of inner product $(.,.)$ are $\mathbb{C}$-valued, then they commute together. Thus (3.1) holds. Now, since T, S are ε-orthogonality preserving mappings, then

$$T(\langle y, y\rangle x - \langle x, y\rangle y) \perp_F^{\varepsilon} S(y).$$

Hence

$$\|\langle y, y\rangle\langle T(x), S(y)\rangle - \langle x, y\rangle\langle T(y), S(y)\rangle\| \leq \varepsilon\|\langle y, y\rangle T(x) - \langle x, y\rangle T(y)\|\|S(y)\|.$$

□

Before considering the continuity of two mappings T, S, we first state the following lemma.

Lemma 3.3. *Let E be a Hilbert $\mathcal{A}$-module and $x \in E$. If $ex = 0$ for all minimal projections in $\mathcal{A}$, then $x = 0$.*

Proof. Let e be an arbitrary minimal projection in $\mathcal{A}$. Let $ex = 0$ for all $x \in E$. Since $0 = \langle ex, x\rangle = e\langle x, x\rangle$. On one side $\langle x, x\rangle$ is a positive element in $\mathcal{A}$, and on the other side, $e = \xi \otimes \xi$, where $\xi \in H$ is unit vector, is a minimal projection, so for any $h \in H$ and by setting $\xi := h$, we get $x = 0$. □

Proposition 3.4. *Let $\varepsilon \in [0, 1)$, and let $\mathcal{A}$ has an approximate unit, which contains finite combinations of minimal projections in $\mathcal{A}$, and let E and F be Hilbert $\mathcal{A}$-modules, and e be an arbitrary minimal projection in $\mathcal{A}$. Suppose that $T, S : E \longrightarrow F$ are a pair of nonzero surjective $\mathcal{A}$-linear ε-orthogonality preserving mappings. Then T and S are continuous.*

Proof. We just prove T is continuous, for another mapping, it proves similarly. Let $\{x_n\}_{n=1}^{\infty}$ be a nonzero sequence in E converges to zero, and $\{T(x_n)\}_{n=1}^{\infty}$ converges to $t \in F$. We must show that $t = 0$. Let $(e_i)_{i\in I}$ be an approximate unit for $\mathcal{A}$ such that $(e_i)_{i\in I}$ contains finite combinations of minimal projections in $\mathcal{A}$. Since $e_i t \to t$, it is enough to prove that $et = 0$ for all minimal projections $e \in \mathcal{A}$. Let there exists a minimal projection $e \in \mathcal{A}$ such that $et \neq 0$. Now, by surjectivity of S, there exists $y \in E$ such that $S(y) = t$. We have $ey \neq 0$.

Now, since a pair of $\mathcal{A}$-linear mappings T, S are ε-orthogonality preserving mappings, then by the Proposition 3.2, we have

$$\begin{aligned}\big\|\langle ey, ey\rangle\langle T(ex_n),&S(ey)\rangle - \langle ex_n, ey\rangle\langle T(ey), S(ey)\rangle\big\| \\ &\leq \varepsilon\big\|\langle ey, ey\rangle T(ex_n) - \langle ex_n, ey\rangle T(y)\big\|\big\|S(ey)\big\|.\end{aligned}$$

Consequently, if $n \to \infty$, we have $x_n \to 0$ and $T(x_n) \to t$, then

$$\|\langle ey, ey\rangle\langle et, eS(y)\rangle\| \leq \varepsilon\|\langle ey, ey\rangle et\|\|eS(y)\|.$$

Since $\langle ey, ey\rangle = e\langle y, y\rangle e = (\langle y, y\rangle\xi, \xi)e$ for all minimal projections $e = \xi \otimes \xi$ in $\mathcal{A}$. Then

$$\begin{aligned}\big\|(\langle y, y\rangle\xi, \xi)e\langle et, eS(y)\rangle\big\| &\leq \varepsilon\big\|(\langle y, y\rangle\xi, \xi)eet\big\|\big\|eS(y)\big\| \\ &= \varepsilon\big\|(\langle y, y\rangle\xi, \xi)et\big\|\big\|eS(y)\big\|.\end{aligned}$$

Therefore

$$\big|(\langle y, y\rangle\xi, \xi)\big|\big\|e\langle et, eS(y)\rangle\big\| \leq \varepsilon\big|(\langle y, y\rangle\xi, \xi)\big|\big\|et\big\|\|eS(y)\|.$$

Hence

$$\big\|\langle e^2 t, eS(y)\rangle\big\| = \big\|e\langle et, eS(y)\rangle\big\| \leq \varepsilon\big\|et\big\|\|eS(y)\|.$$

Since e is a projection, then

$$\big\|\langle et, eS(y)\rangle\big\| \leq \varepsilon\big\|et\big\|\|eS(y)\|.$$

Now, by $S(y) = t$, we have

$$\|\langle et, et\rangle\| \leq \varepsilon\|et\|\|et\|,$$

this implies $et = 0$, because $\varepsilon < 1$. Now, by Lemma 3.3 we have $t = 0$. Therefore, the desired result is obtained. Thus by closed graph theorem, T is continuous. □

In the following, we give a stability result in this context. Note that, in Proposition 3.5 and Theorem 3.7, H is a complex Hilbert space.

Proposition 3.5. *Let $\varepsilon \in [0,1)$. Let $\mathcal{A} = \mathcal{K}(H)$, and let E, F be Hilbert $\mathcal{A}$-modules. Suppose that $T, S : E \longrightarrow F$ are a pair of nonzero surjective $\mathcal{A}$-linear ε-orthogonality preserving mappings. Then there exists $\gamma \in \mathbb{C}$ such that*

$$\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\| \leq \varepsilon\|T\|\|S\|\|x\|\|y\|, \quad x, y \in E.$$

Proof. Let e be a minimal projection in $\mathcal{A}$. Since T_e, S_e are a pair of ε-orthogonality preserving linear mappings from E_e into F_e. Then by [3, Theorem 3.8], there exists $\gamma \in \mathbb{C}$ such that for each $x, y \in E_e$,

$$\|(S_e)^* T_e - \gamma I_e\| \leq \varepsilon\|T_e\|\|S_e\|. \tag{3.2}$$

By [6, Proposition 3.3], $\|T\| = \|T_e\|$ and $\|S\| = \|S_e\|$. Then from (3.2), we have

$$\|S^*T - \gamma I\| \leq \varepsilon\|T\|\|S\|.$$

Now, since each bounded $\mathcal{K}(H)$-linear mapping on $\mathcal{K}(H)$-modules is essentially adjointable, thus for two nonzero bounded $\mathcal{K}(H)$-linear mappings $T, S : E \longrightarrow F$ and for all $x, y \in E$, we have

$$\begin{aligned}\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\| = \|\langle S^*T(x) - \gamma x, y\rangle\| &\leq \|S^*T - \gamma I\|\|x\|\|y\| \\ &\leq \varepsilon\|T\|\|S\|\|x\|\|y\|.\end{aligned}$$

□

As mentioned in previous section, in general, for any C^*-algebra A, a local mapping on Hilbert A-modules is not A-linear.

In the following, for standard C^*-algebra $\mathcal{A}$, we will show that a local mapping between Hilbert $\mathcal{A}$-modules is $\mathcal{A}$-linear.

To achieve this goal, we use from [8, Lemma 3.1]. This lemma states that if A is a C^*-algebra and A_0 is $*$-algebra generated by all the idempotents in A, and if $T : E \longrightarrow F$ on Hilbert A-modules E and F is a local mapping, then T is an A_0-linear mapping.

Since the space generated by projections is subspace of space generated by idempotens, so we have the following proposition.

Proposition 3.6. *Let $T : E \longrightarrow F$ be a local mapping between Hilbert $\mathcal{A}$-modules E and F, then $T : E \longrightarrow F$ is an $\mathcal{A}$-linear mapping.*

Proof. According to the above description, for each projection $p \in \mathcal{A}$ and for all $x \in E$, we have $T(px) = pT(x)$. As $\mathcal{F}(H)$ is linear spanned of its projections, therefore $T(sx) = sT(x)$ for all $s \in \mathcal{F}(H)$ and $x \in E$.

Now, for every $x \in E$, $a \in \mathcal{A}$ and $s \in \mathcal{F}(H)$, we have

$$s(T(ax) - aT(x)) = T(sax) - saT(x) = T(sax) - T(sax) = 0.$$

Hence, if we set $y = T(ax) - aT(x)$, we have $\mathcal{F}(H) \cdot \langle y, y\rangle = 0$, and by $\overline{\mathcal{F}(H)} = \mathcal{K}(H)$, then $\mathcal{K}(H) \cdot \langle y, y\rangle = 0$. By $\mathcal{K}(H)$ is an essential ideal in $\mathcal{B}(H)$, hence $\langle y, y\rangle = 0$, Then we have $y = 0$, i.e., $T(ax) = aT(x)$. Therefore, T is an $\mathcal{A}$-linear mapping. □

In Proposition 3.4, the boundedness of a pair of nonzero $\mathcal{A}$-linear mappings $T, S : E \longrightarrow F$ over Hilbert $\mathcal{A}$-modules is proved. Now, we are in a position to give the main result.

Theorem 3.7. *Let $\varepsilon \in [0, 1)$, and let E and F be Hilbert $\mathcal{A}$-modules. Let $T, S : E \longrightarrow F$ be a pair of nonzero surjective local ε-orthogonality preserving mappings. Then there exists $\gamma \in \mathbb{C}$ such that*

$$\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\| \leq \varepsilon\|T\|\|S\|\|x\|\|y\|, \quad x, y \in E.$$

Proof. Define $\tilde{T}, \tilde{S} : \mathcal{K}(H) \cdot E \longrightarrow \mathcal{K}(H) \cdot F$ (both of $\mathcal{K}(H) \cdot E$ and $\mathcal{K}(H) \cdot F$ being Hilbert $\mathcal{K}(H)$-modules), where $\tilde{T} = T \mid_{\mathcal{K}(H)\cdot E}$ and $\tilde{S} = S \mid_{\mathcal{K}(H)\cdot E}$ by $\tilde{T}(x) = eT(x)$ and $\tilde{S}(y) = fS(y)$ for any rank one projections $e, f \in \mathcal{K}(H)$, respectively.

Now, by previous proposition, $\tilde{T}, \tilde{S}$ are a pair of bounded ε-orthogonality preserving $\mathcal{K}(H)$-linear mappings. Then by Proposition 3.5, there exists $\gamma \in \mathbb{C}$ such that for every $x, y \in \mathcal{K}(H) \cdot E$,

$$\|\langle \tilde{T}(x), \tilde{S}(y)\rangle - \gamma\langle x, y\rangle\| \leq \varepsilon\|\tilde{T}\|\|\tilde{S}\|\|x\|\|y\|.$$

Hence, for any $x, y \in E$ and any rank one projections $e, f \in \mathcal{K}(H)$,

$$\begin{aligned}\|e\langle T(x), S(y)\rangle f - \gamma e\langle x, y\rangle f\| &= \|\langle \tilde{T}(ex), \tilde{S}(fy)\rangle - \gamma\langle ex, fy\rangle\| \\ &\leq \varepsilon\|\tilde{T}\|\|\tilde{S}\|\|ex\|\|fy\| \leq \varepsilon\|T\|\|S\|\|e\|\|x\|\|f\|\|y\|.\end{aligned}$$

Then for all $x, y \in E$ and, all rank one projections $e, f \in \mathcal{K}(H)$,

$$\left\|e\big(\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\big)f\right\| \leq \varepsilon\|T\|\|S\|\|x\|\|y\|,$$

We have $(\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle) \in \mathcal{A}$. On the other hand, by Lemma 2.1, for every $L \in \mathcal{B}(H)$,

$$\|L\| = sup\left\{\|eLf\| \ : \ e, f \text{ are rank one projections}\right\}.$$

Thus, for all $x, y \in E$,

$$\|\langle T(x), S(y)\rangle - \gamma\langle x, y\rangle\| \leq \varepsilon\|T\|\|S\|\|x\|\|y\|.$$

□

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