Journal of Algebra and Related Topics
Vol. 10, No 22, (2022), pp 43-50

ON THE LASKERIAN PROPERTIES OF EXTENSION FUNCTORS OF LOCAL COHOMOLOGY MODULES

D. HASSANZADEH-LELEKAAMI * AND H. ROSHAN-SHEKALGOURABI

Abstract. Let R be a commutative Noetherian ring with identity and M be an R-module. In this paper, we are going to generalize some results on finiteness of extension functors of local cohomology modules to the category of generalized Laskerian modules. In particular, we study the Laskerianness of $\mathrm{Ext}^1_R(R/\mathfrak{a}, H^t_R(M)))$ and $\mathrm{Ext}^2_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ where t is a non-negative integer.

1. Introduction

Throughout the paper, R is a commutative Noetherian ring with identity and all modules are unitary. Also, $\mathfrak{a}$ is an ideal of R, $V(\mathfrak{a})$ is the set of all prime ideals of R containing $\mathfrak{a}$ and $\mathcal{I}$ denotes an arbitrary set of ideals of R.

An interesting problem in commutative algebra is determining when extension functors of local cohomology modules are finitely generated R-modules. Grothendieck [5] raised the following conjecture: for any finitely generated R-module M, $\mathrm{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated for all i. Hartshorne [6] has given counterexamples to this conjecture. However, this problem is studied by many authors and there are some partial answers to Grothendieck's conjecture. In particular, it is shown in [2, Theorem 2.1] that for a finitely generated R-module M and for a non-negative integer t if $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is finitely generated. Recall that an R-module M is said to be $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ and

MSC(2010): Primary: 13D45; Secondary: 13A15, 13D07
Keywords: Local cohomology modules, Extension functors, Generalized Laskerian module.
Received: 1 August 2021, Accepted: 10 September 2022.
∗Corresponding author .

$\mathrm{Ext}^j_R(R/\mathfrak{a}, M)$ is finitely generated for all $j \geq 0$ (see [6]). Recently, this result generalized by authors to the category of $\mathcal{I}$-Laskerian modules, a category larger than the category of finitely generated modules (for definitions see Section 2). More precisely, if M is an $\mathcal{I}$-Laskerian R-module and t is a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is $\mathcal{I}$-Laskerian for all $i < t$, then for any $\mathcal{I}$-Laskerian submodule N of $H^t_{\mathfrak{a}}(M)$, the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)/N)$ is $\mathcal{I}$-Laskerian (see [11, Theorem 3.3]).

Asadollahi and Schenzel in [1], Dibaei and Yassemi in [3] and Khashyarmanesh in [9] have provided some results on the finiteness of $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ for $j > 0$. As a motivation, it is natural to ask about the $\mathcal{I}$-Laskerianness of $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ for $j > 0$. This paper is devoted to investigating cases $i = 1$ and $i = 2$. In fact, we will generalize the main results of [3] and [1, Theorem 1.2] to $\mathcal{I}$-Laskerian modules.

2. Preliminaries

The notion of $\mathcal{I}$-closure as a semiprime closure operation on submodules of modules was introduced in [7]. Let M be an R-module and P be a proper submodule of M. Then P is said to be *$\mathcal{I}$-prime* if $P = (P :_M I)$ for each $I \in \mathcal{I}$ (see [7, Definition 2.1]). Let N be a submodule of M. Recall that by the *closure of N with respect to $\mathcal{I}$* (or *$\mathcal{I}$-closure* of N), we mean the intersection of all $\mathcal{I}$-prime submodules of M containing N, and designate it by $Cl_{\mathcal{I}}(N)$. Following Kirby [10], a nonempty set $\mathcal{I}$ of ideals of R is said to be *closed* if it has the following properties: (1) If $I \subseteq J$ and $I \in \mathcal{I}$, then $J \in \mathcal{I}$; (2) If $\{a_\lambda\}_{\lambda\in\Lambda}$ generate an ideal I of $\mathcal{I}$ and $\{I_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{I}$, then $\sum_{\lambda\in\Lambda} a_\lambda I_\lambda \in \mathcal{I}$. The intersection of all closed sets of ideals which contain $\mathcal{I}$ is called the *closure of $\mathcal{I}$* and is denoted by $\widetilde{\mathcal{I}}$ (for more details see [10, Definition 4]). It is shown in [7, Theorem 3.8] that if M is an R-module, then for each submodule N of M we have $Cl_{\mathcal{I}}(N) = \{m \in M \mid Jm \subseteq N \text{ for some } J \in \widetilde{\mathcal{I}}\}$.

Recall that R-module M is said to be *Laskerian* if any submodule of M is an intersection of a finite number of primary submodules. Obviously, any Noetherian module is Laskerian. Also, an R-module M is said to be *weakly Laskerian* if the set of associated primes of any quotient module of M is finite (see [4]). Obviously, any Laskerian module is weakly Laskerian. As a generalization of these notions, the concept of the $\mathcal{I}$-Laskerian module was introduced in [8]. An R-module M is said to be *$\mathcal{I}$-Laskerian* if the set of associated primes of $Cl_{\mathcal{I}}(N)/N$ is a finite set for any submodule N of M. If the notation $\Gamma_{\mathcal{I}}(M)$ represents the $\mathcal{I}$-closure of the zero submodule of M, i.e., $\Gamma_{\mathcal{I}}(M) := Cl_{\mathcal{I}}(0_M)$, then according to [7, Lemma 3.10(3)] we have $Cl_{\mathcal{I}}(N)/N = Cl_{\mathcal{I}}(0_{M/N}) = \Gamma_{\mathcal{I}}(M/N)$. Thus, M is *$\mathcal{I}$-Laskerian* if $\mathrm{Ass}_R(\Gamma_{\mathcal{I}}(M/N))$ is finite for any submodule N of M. Obviously, $\mathrm{Ass}(Cl_{\mathcal{I}}(N)/N) \subseteq \mathrm{Ass}(M/N)$ for any submodule N of M. Thus, any weakly Laskerian module is $\mathcal{I}$-Laskerian. In particular, any Noetherian module is $\mathcal{I}$-Laskerian for all $\mathcal{I}$. Moreover, based on [11, Example 2.1], the class of $\mathcal{I}$-Laskerian modules is strictly larger than the class of weakly Laskerian modules. For more results on the $\mathcal{I}$-Laskerian module see [11].

For readers convenience, we list some properties of $\mathcal{I}$-Laskerian modules.

Lemma 2.1. *Let $\mathcal{C}(R)$ be the category of all R-modules and R-homomorphisms.*

(1) $\Gamma_{\mathcal{I}}(-)$ *is a covariant, R-linear and left exact functor from $\mathcal{C}(R)$ to itself.*
(2) *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then M is $\mathcal{I}$-Laskerian if and only if M' and M'' are both $\mathcal{I}$-Laskerian. Thus any subquotient of an $\mathcal{I}$-Laskerian module, as well as any finite direct sum of $\mathcal{I}$-Laskerian modules, is $\mathcal{I}$-Laskerian.*
(3) *Let M and N be two R-modules. If M is $\mathcal{I}$-Laskerian and N is finitely generated, then $\mathrm{Ext}^i_R(N, M)$ and $\mathrm{Tor}^R_i(N, M)$ are $\mathcal{I}$-Laskerian for all $i \geq 0$.*
(4) *Let M be an R-module such that $\mathrm{Supp}_R(M)$ is finite. Then M is $\mathcal{I}$-Laskerian. In particular, any Artinian R-module is $\mathcal{I}$-Laskerian.*

Proof. See [8, Lemma 2.3]. □

We conclude this section by the following useful lemma (see [3, Remark 2.1]).

Lemma 2.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$. Let M be an R-module and E be the injective hull of the R-module $\bar{M} := M/\Gamma_{\mathfrak{a}}(M)$. If $K = E/\bar{M}$, then the following statements hold.*

(1) $\Gamma_{\mathfrak{b}}(E) = 0$;
(2) $\mathrm{Hom}_R(R/\mathfrak{b}, E) = 0$;
(3) $H^i_{\mathfrak{a}}(K) \cong H^{i+1}_{\mathfrak{a}}(M)$ *for all* $i \geq 0$;
(4) $\mathrm{Ext}^i_R(R/\mathfrak{b}, K) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{b}, \bar{M})$ *for all* $i \geq 0$;
(5) $\mathrm{Hom}_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(K)) = \mathrm{Hom}_R(R/\mathfrak{b}, K)$.

3. RESULTS

In the rest of the paper, we use the notation K as the same as we stated in Lemma 2.2. The next result is a generalization of [1, Theorem 1.2] and [3, Theorem B].

Theorem 3.1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$ and M be an R-module. Let t be a non-negative integer such that $\operatorname{Ext}_R^{t+1}(R/\mathfrak{b}, M)$ and $\operatorname{Ext}_R^j(R/\mathfrak{b}, H_\mathfrak{a}^i(M))$ are $\mathcal{I}$-Laskerian for all $i < t$ and all $j \geq 0$. Then $\operatorname{Ext}_R^1(R/\mathfrak{b}, H_\mathfrak{a}^t(M))$ is $\mathcal{I}$-Laskerian.*

Proof. We proceed by induction on t. If $t = 0$ and $\bar{M} = M/\Gamma_\mathfrak{a}(M)$, then the exact sequence

$$0 \to \Gamma_\mathfrak{a}(M) \to M \to \bar{M} \to 0 \tag{3.1}$$

induces the exact sequence

$$0 = \operatorname{Hom}_R(R/\mathfrak{b}, \bar{M}) \to \operatorname{Ext}_R^1(R/\mathfrak{b}, \Gamma_\mathfrak{a}(M)) \to \operatorname{Ext}_R^1(R/\mathfrak{b}, M).$$

So, in view of Lemma 2.1(2), $\operatorname{Ext}_R^1(R/\mathfrak{b}, \Gamma_\mathfrak{a}(M))$ is $\mathcal{I}$-Laskerian.

Now, suppose that $t > 0$ and that the claim has been proved for $t - 1$. By assumption $\operatorname{Ext}_R^j(R/\mathfrak{b}, \Gamma_\mathfrak{a}(M))$ is $\mathcal{I}$-Laskerian for all j. Thus, in the light of the exact sequence (3.1) and Lemma 2.1(2) it follows that $\operatorname{Ext}_R^{t+1}(R/\mathfrak{b}, \bar{M})$ is $\mathcal{I}$-Laskerian. Hence, Lemma 2.2 implies that $\operatorname{Ext}_R^t(R/\mathfrak{b}, K)$ is $\mathcal{I}$-Laskerian and $\operatorname{Ext}_R^j(R/\mathfrak{b}, H_\mathfrak{a}^i(K))$ is $\mathcal{I}$-Laskerian for all j and $i < t - 1$. Thus, by induction hypothesis, $\operatorname{Ext}_R^1(R/\mathfrak{b}, H_\mathfrak{a}^{t-1}(K))$ is $\mathcal{I}$-Laskerian. Therefore, $\operatorname{Ext}_R^1(R/\mathfrak{b}, H_\mathfrak{a}^t(M))$ is $\mathcal{I}$-Laskerian by Lemma 2.2. □

The following theorem is an extension of [3, Theorem A], in a sense.

Theorem 3.2. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$ and M be an R-module such that $\operatorname{Ext}_R^j(R/\mathfrak{b}, M)$ is $\mathcal{I}$-Laskerian for all $j \geq 0$. Let t be a non-negative integer such that $\operatorname{Ext}_R^j(R/\mathfrak{b}, H_\mathfrak{a}^i(M))$ are $\mathcal{I}$-Laskerian for all $j \geq 0$ and $i < t$. Then $\operatorname{Hom}(R/\mathfrak{b}, H_\mathfrak{a}^{t+1}(M))$ is $\mathcal{I}$-Laskerian if and only if $\operatorname{Ext}_R^2(R/\mathfrak{b}, H_\mathfrak{a}^t(M))$ is $\mathcal{I}$-Laskerian.*

Proof. We proceed by induction on t. Let $t = 0$ and $\bar{M} = M/\Gamma_\mathfrak{a}(M)$.

($\Rightarrow$) The short exact sequence (3.1) induces the following exact sequence

$$\operatorname{Ext}_R^1(R/\mathfrak{b}, \bar{M}) \to \operatorname{Ext}_R^2(R/\mathfrak{b}, \Gamma_\mathfrak{a}(M)) \to \operatorname{Ext}_R^2(R/\mathfrak{b}, M).$$

To show $\operatorname{Ext}^2_R(R/\mathfrak{b},\Gamma_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian, by Lemma 2.1 it is enough to show that $\operatorname{Ext}^1_R(R/\mathfrak{b},\bar{M})$ is $\mathcal{I}$-Laskerian. By Lemma 2.2, we have

$$\begin{aligned}\operatorname{Ext}^1_R(R/\mathfrak{b},\bar{M}) &= \operatorname{Hom}_R(R/\mathfrak{b},K)\\ &= \operatorname{Hom}_R(R/\mathfrak{b},\Gamma_{\mathfrak{a}}(K))\\ &= \operatorname{Hom}_R(R/\mathfrak{b},H^1_{\mathfrak{a}}(M)), \qquad (3.2)\end{aligned}$$

as desired.

($\Leftarrow$) The short exact sequence (3.1) induces the following exact sequence

$$\operatorname{Ext}^1_R(R/\mathfrak{b},M)\to \operatorname{Ext}^1_R(R/\mathfrak{b},\bar{M})\to \operatorname{Ext}^2_R(R/\mathfrak{b},\Gamma_{\mathfrak{a}}(M)).$$

It follows from (3.2) and Lemma 2.1, $\operatorname{Hom}_R(R/\mathfrak{b},H^1_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian.

Now suppose, inductively, that $t>0$ and the claim is proved for $t-1$. According to hypothesis and Lemma 2.2 the modules $\operatorname{Ext}^j_R(R/\mathfrak{b},H^i_{\mathfrak{a}}(K))$ are $\mathcal{I}$-Laskerian for all $j\geq 0$ and $i<t-1$. Moreover, by the exact sequence (3.1), $\operatorname{Ext}^j_R(R/\mathfrak{b},\bar{M})\cong \operatorname{Ext}^j_R(R/\mathfrak{b},K)$ are $\mathcal{I}$-Laskerian for all j. Hence, by induction hypothesis $\operatorname{Hom}(R/\mathfrak{b},H^t_{\mathfrak{a}}(K))$ is $\mathcal{I}$-Laskerian if and only if $\operatorname{Ext}^2_R(R/\mathfrak{b},H^{t-1}_{\mathfrak{a}}(K))$ is $\mathcal{I}$-Laskerian. Now, the assertion follows from Lemma 2.2, as desired. □

Corollary 3.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a}\subseteq\mathfrak{b}$ and M be an R-module. Let t be a non-negative integer such that the R-modules $\operatorname{Ext}^j_R(R/\mathfrak{b},H^i_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian for all $j\geq 0$ and $i<t$.*
If $\operatorname{Ext}^t_R(R/\mathfrak{b},M)$ and $\operatorname{Ext}^{t+1}_R(R/\mathfrak{b},M)$ are $\mathcal{I}$-Laskerian, then for any $\mathcal{I}$-Laskerian submodule X of $H^t_{\mathfrak{a}}(M)$, the R-module $\operatorname{Ext}^j_R(R/\mathfrak{b},H^t_{\mathfrak{a}}(M)/X)$ is $\mathcal{I}$-Laskerian for all $j\leq 1$.

Proof. The exact sequence

$$0\to X\to H^t_{\mathfrak{a}}(M)\to H^t_{\mathfrak{a}}(M)/X\to 0$$

induces the following exact sequence

$$\operatorname{Ext}^1_R(R/\mathfrak{b},H^t_{\mathfrak{a}}(M))\to \operatorname{Ext}^1_R(R/\mathfrak{b},H^t_{\mathfrak{a}}(M)/X)\to \operatorname{Ext}^2_R(R/\mathfrak{b},X).$$

By Lemma 2.1(3), $\operatorname{Ext}^2_R(R/\mathfrak{b},X)$ is $\mathcal{I}$-Laskerian. On the other hand, Theorem 3.1 implies that $\operatorname{Ext}^1_R(R/\mathfrak{b},H^t_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian. Thus, it yields from Lemma 2.1(2) that $\operatorname{Ext}^1_R(R/\mathfrak{b},H^t_{\mathfrak{a}}(M)/X)$ is $\mathcal{I}$-Laskerian. □

Proposition 3.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a}\subseteq\mathfrak{b}$ and M be an R-module. If $\operatorname{Ext}^1_R(R/\mathfrak{b},M)$ and $\operatorname{Ext}^2_R(R/\mathfrak{b},\Gamma_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian, then $\operatorname{Hom}_R(R/\mathfrak{b},H^1_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian.*

Proof. If $\bar{M} = M/\Gamma_{\mathfrak{a}}(M)$, then the exact sequence (3.1) induces the exact sequence

$$\mathrm{Ext}^1_R(R/\mathfrak{b}, M) \to \mathrm{Ext}^1_R(R/\mathfrak{b}, \bar{M}) \to \mathrm{Ext}^2_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M)).$$

It follows from Lemma 2.1(2) and assumption that $\mathrm{Ext}^1_R(R/\mathfrak{b}, \bar{M})$ is $\mathcal{I}$-Laskerian. Hence, equation (3.2) completes the proof. □

Corollary 3.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$ and M be an R-module. Let t be a non-negative integer such that the R-modules $\mathrm{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian for all j and $i < t$.*
If both $\mathrm{Ext}^{t+2}_R(R/\mathfrak{b}, M)$ and $\mathrm{Hom}_R(R/\mathfrak{b}, H^{t+1}_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian, then the R-module $\mathrm{Ext}^2_R(R/\mathfrak{b}, H^t_{\mathfrak{a}}(M)/X)$ is $\mathcal{I}$-Laskerian for any $\mathcal{I}$-Laskerian submodule X of $H^t_{\mathfrak{a}}(M)$.

Proof. The exact sequence

$$0 \to X \to H^t_{\mathfrak{a}}(M) \to H^t_{\mathfrak{a}}(M)/X \to 0$$

induces the following exact sequence

$$\mathrm{Ext}^2_R(R/\mathfrak{b}, H^t_{\mathfrak{a}}(M)) \to \mathrm{Ext}^2_R(R/\mathfrak{b}, H^t_{\mathfrak{a}}(M)/X) \to \mathrm{Ext}^3_R(R/\mathfrak{b}, X).$$

By Lemma 2.1(3), $\mathrm{Ext}^3_R(R/\mathfrak{b}, X)$ is $\mathcal{I}$-Laskerian. So, in the light of Lemma 2.1(2), it is enough for us to show that the R-module $\mathrm{Ext}^2_R(R/\mathfrak{b}, H^t_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian. To do this, we use induction on t. Let $t = 0$. Then, by assumption $\mathrm{Ext}^2_R(R/\mathfrak{b}, M)$ and $\mathrm{Hom}_R(R/\mathfrak{b}, H^1_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian. Therefore, by equation (3.2), the R-module $\mathrm{Ext}^1_R(R/\mathfrak{b}, \bar{M})$ is $\mathcal{I}$-Laskerian, where $\bar{M} = M/\Gamma_{\mathfrak{a}}(M)$. Thus, the exact sequence

$$\mathrm{Ext}^1_R(R/\mathfrak{b}, \bar{M}) \to \mathrm{Ext}^2_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}^2_R(R/\mathfrak{b}, M)$$

and Lemma 2.1(2) imply that $\mathrm{Ext}^2_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian. Now suppose, inductively, that $t > 0$ and the result has been proved for $t-1$. By assumption, $\mathrm{Ext}^j_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian for all j. Therefore, the exact sequence

$$\mathrm{Ext}^{t+2}_R(R/\mathfrak{b}, M) \to \mathrm{Ext}^{t+2}_R(R/\mathfrak{b}, \bar{M}) \to \mathrm{Ext}^{t+3}_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M))$$

implies that $\mathrm{Ext}^{t+2}_R(R/\mathfrak{b}, \bar{M})$ is $\mathcal{I}$-Laskerian. Now, by Lemma 2.2 and by using a similar proof as in the proof of Theorem 3.1, we conclude that $\mathrm{Ext}^2_R(R/\mathfrak{b}, H^{t-1}_{\mathfrak{a}}(K))$ is $\mathcal{I}$-Laskerian, and so $\mathrm{Ext}^2_R(R/\mathfrak{b}, H^t_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian by Lemma 2.2, as desired. □

Corollary 3.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$ and M be an R-module. Let t be a non-negative integer such that $\mathrm{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian for all j and $i < t$. If the R-modules $\mathrm{Hom}_R(R/\mathfrak{b}, H^{t+1}_{\mathfrak{a}}(M))$ and $\mathrm{Ext}^j_R(R/\mathfrak{b}, M)$ for $j = t, t+1, t+2$ are $\mathcal{I}$-Laskerian, then for any*

$\mathcal{I}$-Laskerian submodule X of $H^t_{\mathfrak{a}}(M)$ and for any finitely generated R-module L with $\operatorname{Supp}(L) \subseteq V(\mathfrak{b})$, the R-module $\operatorname{Ext}^j_R(L, H^t_{\mathfrak{a}}(M)/X)$ is $\mathcal{I}$-Laskerian for all $j \leq 2$.

Proof. The assertion follows from Corollaries 3.3 and 3.5 and [11, Theorem 2.4]. □

Corollary 3.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R with $\mathfrak{a} \subseteq \mathfrak{b}$ and M be an R-module. Let t be a non-negative integer such that $\operatorname{Ext}^j_R(R/\mathfrak{b}, H^i_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian for all j and $i < t$.*
If $\operatorname{Ext}^{t+1}_R(R/\mathfrak{b}, M)$ and $\operatorname{Ext}^2_R(R/\mathfrak{b}, H^t_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian, then the R-module $\operatorname{Hom}_R(L, H^{t+1}_{\mathfrak{a}}(M)/X)$ is $\mathcal{I}$-Laskerian for any $\mathcal{I}$-Laskerian submodule X of $H^{t+1}_{\mathfrak{a}}(M)$ and for any finitely generated R-module L with $\operatorname{Supp}(L) \subseteq V(\mathfrak{b})$.

Proof. In view of [11, Theorem 2.4] it suffices to show that $\operatorname{Hom}_R(R/\mathfrak{b}, H^{t+1}_{\mathfrak{a}}(M)/X)$ is $\mathcal{I}$-Laskerian. By assumption and Lemma 2.1(3), $\operatorname{Ext}^1_R(R/\mathfrak{b}, X)$ is $\mathcal{I}$-Laskerian. So, according to the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{b}, H^{t+1}_{\mathfrak{a}}(M)) \to \operatorname{Hom}_R(R/\mathfrak{b}, H^{t+1}_{\mathfrak{a}}(M)/X) \to \operatorname{Ext}^1_R(R/\mathfrak{b}, X),$$

it is enough for us to prove that $\operatorname{Hom}_R(R/\mathfrak{b}, H^{t+1}_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian. We proceed by induction on t. Let $t = 0$. Then by assumption $\operatorname{Ext}^1_R(R/\mathfrak{b}, M)$ and $\operatorname{Ext}^2_R(R/\mathfrak{b}, \Gamma_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian. So in view of Proposition 3.4 the R-module $\operatorname{Hom}_R(R/\mathfrak{b}, H^1_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian. Now suppose, inductively, that $t > 0$ and that the claim has been proved for $t-1$. By Lemma 2.2 and using a similar proof as in the proof of Corollary 3.5, we infer that $\operatorname{Hom}_R(R/\mathfrak{b}, H^t_{\mathfrak{a}}(K))$ is $\mathcal{I}$-Laskerian. Therefore, $\operatorname{Hom}_R(R/\mathfrak{b}, H^{t+1}_{\mathfrak{a}}(M))$ is $\mathcal{I}$-Laskerian. □

Corollary 3.8. *Let $\mathfrak{a}$ be an ideal of R and M be an $\mathcal{I}$-Laskerian R-module. Let t be a non-negative integer such that $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ are $\mathcal{I}$-Laskerian for all j and $i < t$. Then the following statements hold.*

1. $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ *is $\mathcal{I}$-Laskerian.*
2. *The following statements are equivalent:*
   - (a) $\operatorname{Hom}(R/\mathfrak{a}, H^{t+1}_{\mathfrak{a}}(M))$ *is $\mathcal{I}$-Laskerian;*
   - (b) $\operatorname{Ext}^2_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ *is $\mathcal{I}$-Laskerian.*

Proof. Since M is $\mathcal{I}$-Laskerian, $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is $\mathcal{I}$-Laskerian for all i, by Lemma 2.1(3). Thus, the assertions follow from Theorems 3.1 and 3.2. □

Acknowledgments

The authors would like to thank the referee for careful reading.

References

1. J. Asadollahi and P. Schenzel, *Some results on associated primes of local cohomology modules*, Japan. J. Math. (N.S.) **29** (2003), 285–296.
2. M. T. Dibaei and S. Yassemi, *Associated primes and cofiniteness of local cohomology modules*, Manusc. Math. **117** (2005), 199–205.
3. M.T. Dibaei and S. Yassemi, *Finiteness of extension functors of local cohomology modules*, Comm. Algebra **34** (2006), 3097–3101.
4. K. Divaani-Aazar and A. Mafi, *Associated primes of local cohomology modules*, Proc. Amer. Math. Soc. (3) **133** (2005), 655–660.
5. A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de lefschetz locaux et globaux (sga 2)*, Adv. Stud. Pure Math. **2** (1968).
6. R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145–164.
7. D. Hassanzadeh-lelekaami, *A closure operation on submodules*, J. Algebra Appl. (12) **16** (2017), 1750229 (22 pages).
8. D. Hassanzadeh-Lelekaami and H. Roshan-Shekalgourabi, *Associated primes of local cohomology modules of generalized Laskerian modules*, Czech. Math. J. **69** (2019), 1101–1109.
9. K. Khashyarmanesh, *On the finiteness properties of extension and torsion functors of local cohomology modules*, Proceedings Of The American Mathematical Society, (5) **135** (2007), 1319–1327.
10. D. Kirby, *Components of ideals in a commutative ring*, Ann. Mat. Pura Appl. (1) **71** (1966), 109–125.
11. Hajar Roshan-Shekalgourabi and Dawood Hassanzadeh-Lelekaami, *On colaskerian module and local cohomology*, Comm. Algebra, (12) **47** (2019), no. 12, 5462–5470.

D. Hassanzadeh-Lelekaami

Department of Basic Sciences, Arak University of Technology, P.O.Box 38135-1177, Arak, Iran.

Email: lelekaami@gmail.com and Dhmath@arakut.ac.ir

H. Roshan-Shekalgourabi

Department of Basic Sciences, Arak University of Technology, P.O.Box 38135-1177, Arak, Iran.

Email: Hrsmath@gmail.com and roshan@arakut.ac.ir