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Analysis of $GI/M/1/N$ and $GI/Geo/1/N$ queues with balking and vacation interruptions

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Abstract. This paper addresses renewal input continuous and discrete time queues with balking and vacation interruptions. An arriving client may join the system or balk with some state-dependent probability. Whenever the server finds an empty system, he leaves for a working vacation. During working vacations, if there are clients to be served at a service completion instant, the server interrupts the working vacation and switches to regular service period. The embedded Markov chain technique has been adopted for evaluating pre-arrival epoch probabilities and supplementary variable approach is employed to evaluate arbitrary instant probabilities. Few performance characteristics and sojourn time distribution have also been demonstrated. Finally, numerical investigations have been figured out to depict the impact of the model variables on the performance indices.

Keywords: Renewal input, balking, vacation interruption, embedded Markov chain, sojourn time.
AMS Subject Classification 2010: 60K25, 68M20, 90B22.

1 Introduction

Queues with server vacations have been considered widely due to their various implementations in transmission systems, manufacturing industries, food industries, etc. Incorporation of various vacation policies in queueing systems provide more mobility for optimal planning and operation control of the system. In many real life scenarios, the server can be utilized in different forms with a different service rate during the vacation period. Such queueing systems are termed as queues with multiple working vacations

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(MWV) and were proposed by Servi and Finn [18]. The study of a bulk arrival exponential queue with MWV has been carried out by Baba [1]. A finite capacity state-dependent general input queue with MWV was researched by Goswami et al. [6] employing supplementary variable and recursive methods. A review on queueing systems with working vacations has been conducted by Chandrasekaran et al. [3]. On the other hand, due to their wide range of applications in numerous fields, queueing models with working vacations (WV) in discrete time environment have received extensive study in recent years. Goswami and Mund [4] used supplementary variable and embedded Markov chain approaches to analyse a $GI/Geo/1/N$ queue with MWV. The discrete time bulk arrival queue with WV has been considered by Li et al. [10] employing embedded Markov chain method. The discrete time bulk arrival queueing system with unreliable server and WV was analysed by Jain et al. [8] using maximum entropy approach.

According to WV rules, the server returns to regular service after a WV completion if there are still waiting clients in the system. In the real-world scenario, these circumstances seem to be rare. To get over this restriction, Li and Tian [12] developed the concept of vacation interruption (VI) in an exponential queue. Li and Tian [13] have used the matrix-geometric approach to analyse the discrete time general input queue with VI. The generalization of the study of Li and Tian [12] to a renewal input queue with VI was considered by Li et al. [14]. Zhang and Hou [27] studied a general service queue with VI adopting the supplementary variable method. The discrete time bulk arrival queue with VI has been analysed by Shan et al. [21]. Li et al. [11] has explored a Bernoulli schedule vacation queue with VI under discrete time environment using matrix analytic method. The retrial queue with VI in discrete time environment has been considered by Shan and Jinting [20] using supplementary variable method. An exponential queue with VI under Bernoulli schedule has been dealt by Manoharan and Ashok [15]. Rajdurai et al. [16] explored an unreliable queue with WV and VI under Bernoulli schedule using supplementary variable technique.

Queueing models wherein clients balk arise in many distinct application domains. Numerous researchers have put their efforts to investigate the influence of client intolerance of waiting in queueing systems. Further, queueing models with impatient arrivals and server vacations are widely in use due to their appropriateness in various fields such as telephone switchboard, packet transmission and other notable applications. Employing embedded Markov chain together with supplementary variable approach, Vijaya Laxmi and Jyothsna [23] investigated a finite capacity general input queue with balking and WV. Suganya [22] analyzed an $M^X/G(a,b)/1$ queue with VI, optional re-service and balking using supplementary variable method. Vijaya Laxmi and Jyothsna [24] computed the stationary probabilities at different epochs for a discrete time queue with balking and MWV. The impatient actions of arrivals in an N-policy queue with unreliable server and VI has been studied by Richa [17]. Shakir and Manoharan [19] considered an impatient client queue with WV using generating functions. Ivo et al. [7] considered a continuous time queue with two types of impatient clients. Goswami and Mund [5] applied the displacement operator approach to analyse the general input bulk service queue with client impatience and MWV. Vijaya Laxmi and Kassahun [26] considered the transient study of a multi-server queue with synchronized MWV and impatient clients. A feedback queue with variant of multiple vacations and impatience has been researched by Amina et al. [2].

In this article, the analytical analysis of a finite capacity renewal input queue with balking and VI in continuous and discrete time environments has been done. Such queueing systems have many significant applications. For example, in the transmission system of emails, Simple Mail Transfer Protocol (SMTP) is a protocol for the transfer of emails on a TCP/IP network. A picture depicting the SMTP is shown in Figure 1. SMTP provides the mean of transmission and collection of email messages. When an email is sent by a mail client to a mail server using SMTP, a TCP connection is formed. Once the connection is in place, the program follows SMTP wherein a sender is identified, a recipient is specified and the email is transferred. In some situations, the messages may be ended up by the sender before reaching the mail server. When the messages approach the mail server, the first message is selected for transfer and the rest will enter the buffer. For proper functioning of the mail server, some servicing activities such as virus scan, spam filtering, etc., are performed. These activities are done during the idle time of the server on a regular basis. In this context, the buffer, the receiver, ending up of the messages before arriving at the mail server and the servicing activities correlate to queue, server, balking and vacation policy in queueing nomenclature, respectively.

To the best of our knowledge, general arrival queues with balking and VI in both continuous and discrete time environments have not been studied so far. Inspired by the lack of investigations in this domain and the various practical applications as discussed above, this paper aims at the study of both $GI/M/1$ and $GI/Geo/1$ queues with balking and VI. The finite buffer queue with impatient clients and Bernoulli-schedule VI can be found in Vijaya Laxmi et al. [25] wherein they have utilized supplementary variable approach together with recursive technique. However, in the present paper we have analyzed both $GI/M/1/N/WV$ and $GI/Geo/1/N/WV$ queues with balking and VI using embedded Markov chain and supplementary variable approaches. Another highlight of the present paper is that the actual sojourn time analysis has been carried out and is analytically matched with the sojourn time attained from Little's rule in both the models. The regular service durations, WV service durations and vacation durations are presumed to be exponentially and geometrically distributed in the continuous and discrete time models, respectively. The pre-arrival instant probabilities at steady-state are obtained using embedded Markov chain technique. Adopting the supplementary variable method the arbitrary epoch distributions at steady-state are obtained. Different performance characteristics like the mean system size, blocking probability, mean balking rate, sojourn time analysis, etc., are presented. The parameter impact on the performance indices is exhibited through few numerical investigations.

The remaining of the article is organized as follows. An overview of the considered continuous and discrete time queues is depicted in Section 2. In Sections 3 and 4, employing embedded Markov chain and supplementary variable approaches, respectively, the probability distributions at various instants are obtained. Few performance metrics and sojourn time analysis for both the models is accomplished in Section 5. Numerical investigations are displayed in Section 6 followed by conclusions in Section 7.

2 Model description

We examine a general arrival queue with balking and vacation interruptions in both continuous and discrete time environments. The buffer capacity is taken to be finite (N). In the continuous (discrete) time model, we presume that the inter-arrival times (i.a.t.) of consecutive arrivals are identically distributed and independent random variables with pdf (pmf) $\hat{a}(x), x \geq 0$ ($\hat{a}_i,\ i \geq 1$), Laplace-Stieltjes transform (LST) (probability generating function) $A^*(s)(A^*(z))$ and mean i.a.t. $1/\lambda = -A^{*(1)}(0)(A^{*(1)}(1))$, where $h^{(1)}(0)(h^{(1)}(1))$ indicates the derivative of $h(x)$ computed at $x = 0$ $(x = 1)$. Assume that an arrival on finding n clients in the system is not compelled to join the queue and may balk with some state-dependent probability. When n clients exist in the system, let β_n denote the probability with which a client joins the system and $\bar{\beta}_n$ be the probability of balking, where $\bar{x} = 1 - x$. Furthermore, it is also assumed that $\beta_0 = 1,\ 0 < \beta_{n+1} \leq \beta_n \leq 1, 1 \leq n \leq N-1$ and $\beta_N = 0$.

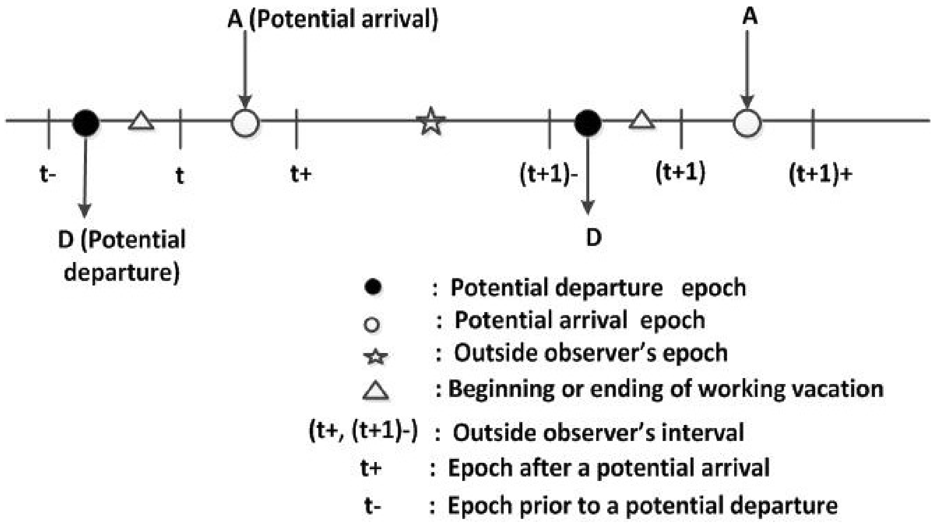

Figure 1: Early arrival system

As and when the service facility finds no clients in the system, he exits for a WV. When the service is completed in WV, if there are clients in the queue to be served, VI occurs; otherwise, the vacation continues. On the other hand, if there are no clients to be served at a vacation completion epoch, another WV starts; otherwise, the server resumes regular service. The regular service durations, service durations in working vacations and vacation durations are presumed to be exponentially (geometrically) distributed with rates μ_b, μ_v and φ in the continuous (discrete) time queue, respectively.

In the discrete time queue, the axis of time be plotted as $0,1,2,\dots,t,\dots$. The discrete time model is addressed for an early arrival system (EAS). In this model, a potential arrival may occur in time $(t,t+)$ and a potential departure in time $(t-,t)$. Figure 1 illustrates the different time instants at which events take place in an EAS.

The following random variables explain the system state at time t. $\hat{N}(t)$ denotes the total clients in the system, the remaining i.a.t. is denoted by $\hat{X}(t)$, $\varsigma(t)=0$, when the server is in WV and $\varsigma(t)=1$, when the server is busy with regular service.

3 Steady-state probability distributions at pre-arrival instants

For both continuous and discrete time queues under discussion, we compute the pre-arrival epoch probabilities using the embedded Markov chain technique under this section. Let the clients arrive at successive time epochs $t_i (i=1,2,\dots)$ and the i.a.t. $t_{i+1}-t_i$ be identically distributed and mutually independent random variables with common distribution function $A(x)$. The system state at t_i^- is denoted as $\{\hat{N}(t_i^-),\varsigma(t_i^-)\}$. This results an embedded Markov chain whose state space is $\{(k,l),\ l=0,1;\ l\le k\le N\}$. At a pre-arrival epoch, let $\omega_{n,j}^-$ represent the probability of the server in state j with n clients in the system and is defined as $\omega_{n,j}^-=\lim_{t\to\infty}P\{\hat{N}(t_i^-)=n,\varsigma(t_i^-)=j\},\ j=0,1;\ j\le n\le N$.

3.1 Embedded Markov chain analysis

During an i.a.t., let $\hat{f}_k$ $(k \geq 0)$ express the probability of k service completions in the regular service period and $\hat{g}_k$ $(k \geq 0)$ be the probability that the service time during WV is shorter than the vacation duration, then a client is served with rate μ_v and the rest $(k-1)$ clients are served with rate μ_b due to VI. Let $\hat{h}_k$ $(k \geq 0)$ denote the probability that the service time during WV is not shorter than the vacation duration, then the vacation terminates before a service completion and the k clients are served during regular service period with rate μ_b. Hence, in the continuous time model, for $k \geq 0$, the probabilities $\hat{f}_k, \hat{g}_k$ and $\hat{h}_k$ are given as

$$\hat{f}_k = \int_0^{\infty} \frac{(\mu_b x)^k}{k!} e^{-\mu_b x} dA(x), k \geq 0, \tag{1}$$

$$\hat{g}_0 = \int_0^{\infty} e^{-(\varphi+\mu_v)x} dA(x), \tag{2}$$

$$\hat{g}_k = \int_0^{\infty} \int_0^{x} \mu_v \frac{(\mu_b(x-t))^{k-1}}{(k-1)!} e^{-((\varphi+\mu_v-\mu_b)t+\mu_b x)} dt \; dA(x), k \geq 1, \tag{3}$$

$$\hat{h}_k = \int_0^{\infty} \int_0^{x} \varphi \frac{(\mu_b(x-t))^{k}}{k!} e^{-((\varphi+\mu_v-\mu_b)t+\mu_b x)} dt \; dA(x), k \geq 0. \tag{4}$$

For the discrete time queue, the probabilities $\hat{f}_k, \hat{g}_k$ and $\hat{h}_k$, for $k \geq 0$, are defined as

$$\hat{f}_k = \sum_{i=k}^{\infty} \hat{a}_i \begin{pmatrix} i \\ k \end{pmatrix} \mu_b^k \bar{\mu}_b^{i-k}, k \geq 0, \tag{5}$$

$$\hat{g}_0 = \sum_{i=1}^{\infty} \hat{a}_i \bar{\varphi}^i \bar{\mu}_v^i, \tag{6}$$

$$\hat{g}_k = \sum_{i=max(1,k)}^{\infty} \hat{a}_i \sum_{l=1}^{i} \mu_v \bar{\mu}_v^{l-1} \bar{\varphi}^{l-1} \begin{pmatrix} i-l \\ k-1 \end{pmatrix} \mu_b^{k-1} \bar{\mu}_b^{i-l-k+1}, k \geq 1, \tag{7}$$

$$\hat{h}_k = \sum_{i=max(1,k)}^{\infty} \hat{a}_i \sum_{l=1}^{i} \bar{\mu}_v^l \varphi \bar{\varphi}^{l-1} \begin{pmatrix} i-l \\ k \end{pmatrix} \mu_b^k \bar{\mu}_b^{i-l-k}, k \geq 1. \tag{8}$$

The one-step transition probability matrix (TPM) $\mathscr{P}$ of dimension $(2N+1) \times (2N+1)$ is obtained by noticing the system state at two successive embedded points as

$$\mathscr{P} = \begin{bmatrix} \Delta & \Theta \\ \Upsilon & \Phi \end{bmatrix}_{(2N+1)\times(2N+1)},$$

where

$$\Theta_{(N+1)\times N} = \begin{cases} \beta_k \hat{h}_0, & if \; k = l-1, \\ \beta_k \left(\hat{g}_1 + \hat{h}_1\right) + \bar{\beta}_k \hat{h}_0, & if \; k = l, \\ \beta_k \left(\hat{g}_{k+1-l} + \hat{h}_{k+1-l}\right) + \bar{\beta}_k \left(\hat{g}_{k-l} + \hat{h}_{k-l}\right), & if \; k+1 > l, \; k-l \neq 0, \\ 0, & otherwise. \end{cases} \tag{9}$$

$$\Phi_{N\times N} = \begin{cases} \beta_k \hat{f}_0, & if\ k=l-1,\\ \beta_k \hat{f}_{k+1-l} + \bar{\beta}_k \hat{f}_{k-l}, & if\ k+1>l,\ k-l\neq 0,\\ 0, & otherwise.\end{cases}$$

$$\Delta_{(N+1)\times(N+1)} = \begin{cases} \beta_k \hat{g}_0, & if\ k=l-1,\\ \bar{\beta}_k \hat{g}_0, & if\ k=l,\ k\geq 1,\\ \psi(k), & if\ l=0,\\ 0, & otherwise.\end{cases}$$

$$\Upsilon_{N\times(N+1)} = \begin{cases} \gamma(k), & if\ l=0,\\ 0, & otherwise.\end{cases}$$

and $\psi(k) = 1-\sum_{l=1}^{N}\left(\Delta(k,l)+\Theta(k,l)\right), \gamma(k) = 1-\sum_{l=1}^{N}\Phi(k,l)$.

Note: One may note that in the above TPM for the continuous time model, the probabilities $\hat{f}_k, \hat{g}_k, \hat{h}_k$ are taken from (1) to (4) and for the discrete time model $\hat{f}_k, \hat{g}_k, \hat{h}_k$ are considered from (5) to (8).

Let $\Omega = (\omega_{0,0}^{-},\ \omega_{1,0}^{-},\ldots,\omega_{N,0}^{-},\omega_{1,1}^{-},\ldots,\omega_{N,1}^{-})$ and $\mathbf{e}$ be $(2N+1)$ dimensional unit column vector. To compute the probabilities $\omega_{n,j}^{-}, j=0,1; j\leq n\leq N$, we solve the system of equations $\Omega = \Omega\mathscr{P},\ \Omega\mathbf{e}=1$ using GTH algorithm given in Latouche and Ramaswami [9]

4 Steady-state probability distributions at arbitrary instants

Under this section, the probability distributions at arbitrary instants for both continuous and discrete time models are obtained using supplementary variable approach. The steady-state probabilities at outside observer's observation epoch for the discrete time model are also evaluated in this section. The difference equations are built utilising the remaining i.a.t as the supplementary variable for the evaluation of arbitrary epoch probabilities.

4.1 $GI/M/1/N/MWV$ queue with VI and balking

Let $\omega_{n,j}(x) = \lim_{t\to\infty} P\left\{\hat{N}(t)=n,\ x\leq \hat{X}(t)\leq x+dx,\ \varsigma(t)=j\right\},\ j=0,1; j\leq n\leq N,\ x\geq 0$ at steady-state. The respective rate probabilities are denoted as $\omega_{n,j}(0)$ with the remaining i.a.t. equal to zero indicating that a client is about to arrive. The Laplace transform of the probabilities are defined as

$$\omega_{n,j}^{*}(s) = \int_0^\infty e^{-sx}\omega_{n,j}(x)dx, j=0,1;\ j\leq n\leq N.$$

At an arbitrary instant, let the probability of n clients in the system while the server being in state j be denoted as $\omega_{n,j}, j=0,1; j\leq n\leq N$. The steady-state differential difference equations are obtained by relating the state of the system at two consecutive time instants t and $(t+dt)$ as:

$$
\begin{aligned}
-\omega_{0,0}^{(1)}(x) &= \mu_v \omega_{1,0}(x) + \mu_b \omega_{1,1}(x), \\
-\omega_{n,0}^{(1)}(x) &= \hat{a}(x)\left(\beta_{n-1}\omega_{n-1,0}(0) + \bar{\beta}_n \omega_{n,0}(0)\right) - (\varphi+\mu_v)\,\omega_{n,0}(x),\ 1 \le n \le N, \\
-\omega_{1,1}^{(1)}(x) &= \hat{a}(x)\bar{\beta}_1 \omega_{1,1}(0) + \mu_b \omega_{2,1}(x) + \varphi \omega_{1,0}(x) + \mu_v \omega_{2,0}(x) - \mu_b \omega_{1,1}(x), \\
-\omega_{n,1}^{(1)}(x) &= \hat{a}(x)\left(\beta_{n-1}\omega_{n-1,1}(0) + \bar{\beta}_n \omega_{n,1}(0)\right) + \varphi \omega_{n,0}(x) + \mu_v \omega_{n+1,0}(x) + \mu_b \omega_{n+1,1}(x) - \mu_b \omega_{n,1}(x), \\
&\quad 2 \le n \le N-1, \\
-\omega_{N,1}^{(1)}(x) &= \hat{a}(x)\left(\beta_{N-1}\omega_{N-1,1}(0) + \omega_{N,1}(0)\right) + \varphi \omega_{N,0}(x) - \mu_b \omega_{N,1}(x).
\end{aligned}
$$

The following set of equations are obtained by multiplying the aforementioned equations with e^{-sx} and integrating over x from 0 to ∞.

$$-s\omega_{0,0}^{*}(s) = \mu_v \omega_{1,0}^{*}(s) + \mu_b \omega_{1,1}^{*}(s) - \omega_{0,0}(0), \tag{10}$$

$$(\varphi+\mu_v-s)\,\omega_{n,0}^{*}(s) = A^{*}(s)\left(\beta_{n-1}\omega_{n-1,0}(0) + \bar{\beta}_n \omega_{n,0}(0)\right) - \omega_{n,0}(0), 1 \le n \le N, \tag{11}$$

$$(\mu_b-s)\,\omega_{1,1}^{*}(s) = A^{*}(s)\bar{\beta}_1 \omega_{1,1}(0) + \mu_b \omega_{2,1}^{*}(s) + \varphi \omega_{1,0}^{*}(s) + \mu_v \omega_{2,0}^{*}(s) - \omega_{1,1}(0), \tag{12}$$

$$
\begin{aligned}
(\mu_b-s)\,\omega_{n,1}^{*}(s) &= A^{*}(s)\left(\beta_{n-1}\omega_{n-1,1}(0) + \bar{\beta}_n \omega_{n,1}(0)\right) + \varphi \omega_{n,0}^{*}(s) \\
&\quad + \mu_v \omega_{n+1,0}^{*}(s) + \mu_b \omega_{n+1,1}^{*}(s) - \omega_{n,1}(0), 2 \le n \le N-1,
\end{aligned} \tag{13}
$$

$$(\mu_b-s)\,\omega_{N,1}^{*}(s) = A^{*}(s)\left(\beta_{N-1}\omega_{N-1,1}(0) + \omega_{N,1}(0)\right) + \varphi \omega_{N,0}^{*}(s) - \omega_{N,1}(0). \tag{14}$$

Summing up Eqs. (10) to (14), we obtain

$$\sum_{n=0}^{N} \omega_{n,0}^{*}(s) + \sum_{n=1}^{N} \omega_{n,1}^{*}(s) = \left(\frac{1-A^{*}(s)}{s}\right)\left(\sum_{n=0}^{N} \omega_{n,0}(0) + \sum_{n=1}^{N} \omega_{n,1}(0)\right).$$

As $s \to 0$ and utilizing the normalization condition $\sum_{n=0}^{N} \omega_{n,0} + \sum_{n=1}^{N} \omega_{n,1} = 1$, we get

$$\sum_{n=0}^{N} \omega_{n,0}(0) + \sum_{n=1}^{N} \omega_{n,1}(0) = \lambda. \tag{15}$$

This indicates that the mean arrival rate λ is equal to the mean number of arrivals into the system.

4.1.1 Arbitrary epoch probabilities in terms of pre-arrival epoch probabilities

For $j = 0,1;\ j \le n \le N$, the rate probabilities $\omega_{n,j}(0)$ and the pre-arrival instant probabilities $\omega_{n,j}^{-}$ are related as

$$\omega_{n,j}^{-} = \frac{\omega_{n,j}(0)}{\lambda}, \quad j = 0,1;\ j \le n \le N, \tag{16}$$

where λ is obtained from (15). Below is a theorem that demonstrates the relationship between pre-arrival and arbitrary epoch probabilities.

Theorem 1. *The arbitrary epoch probabilities $\omega_{n,j}, j=0,1;\ j\le n\le N$ and the pre-arrival epoch probabilities are related as:*

$$\omega_{n,0} = \frac{\lambda}{\varphi+\mu_v}\left(\beta_{n-1}\omega^-_{n-1,0} - \beta_n\omega^-_{n,0}\right), 1\le n\le N, \tag{17}$$

$$\omega_{1,1} = \frac{\lambda}{\mu_b(\varphi+\mu_v)}\left(\mu_v\beta_1\omega^-_{1,0} + \varphi\omega^-_{0,0}\right), \tag{18}$$

$$\omega_{n,1} = \frac{\lambda\beta_{n-1}}{\mu_b}\omega^-_{n-1,1} + \frac{\lambda}{\mu_b(\varphi+\mu_v)}\left(\mu_v\beta_n\omega^-_{n,0} + \varphi\beta_{n-1}\omega^-_{n-1,0}\right), 2\le n\le N, \tag{19}$$

$$\omega_{0,0} = 1 - \sum_{n=1}^{N}\left(\omega_{n,0}+\omega_{n,1}\right). \tag{20}$$

Proof. Setting $s=0$ in (11) to (14), using (16) and after simplification we obtain (17) to (19). Finally, to obtain the only unknown $\omega_{0,0}$, the normalization condition is utilized. □

4.2 $GI/Geo/1/N/MWV$ queue with VI and balking

Let the joint probabilities be denoted as

$$\omega_{n,j}(x) = \lim_{t\to\infty} P\left\{\hat{N}(t)=n,\ \hat{X}(t)=x,\ \varsigma(t)=j\right\},\ j=0,1;\ j\le n\le N, x\ge 0$$

at steady-state. The steady-state difference equations are obtained by relating the state of the system at two consecutive time instants t and $(t+dt)$ as:

$$\begin{aligned}
\omega_{0,0}(x-1) =& \omega_{0,0}(x) + \mu_v\omega_{1,0}(x) + \mu_b\omega_{1,1}(x) + \hat{a}_x\left(\mu_v\omega_{0,0}(0) + \bar{\beta}_1\left(\mu_v\omega_{1,0}(0)+\mu_b\omega_{1,1}(0)\right)\right),\\
\omega_{n,0}(x-1) =& \bar{\varphi}\bar{\mu}_v\left(\omega_{n,0}(x) + \hat{a}_x\left(\beta_{n-1}\omega_{n-1,0}(0) + \bar{\beta}_n\omega_{n,0}(0)\right)\right),\ 1\le n\le N,\\
\omega_{1,1}(x-1) =& \bar{\mu}_b\left(\omega_{1,1}(x) + \hat{a}_u\bar{\beta}_1\omega_{1,1}(0)\right) + \mu_b\left(\omega_{2,1}(x) + \hat{a}_x\left(\beta_1\omega_{1,1}(0) + \bar{\beta}_2\omega_{2,1}(0)\right)\right) + \varphi\bar{\mu}_v\left(\omega_{1,0}(x)\right.\\
&\left. + \hat{a}_x\left(\omega_{0,0}(0) + \bar{\beta}_1\omega_{1,0}(0)\right)\right) + \mu_v\left(\omega_{2,0}(x) + \hat{a}_x\left(\beta_1\omega_{1,0}(0) + \bar{\beta}_2\omega_{2,0}(0)\right)\right),\\
\omega_{n,1}(x-1) =& \bar{\mu}_b\left(\omega_{n,1}(x) + \hat{a}_x\left(\beta_{n-1}\omega_{n-1,1}(0) + \bar{\beta}_n\omega_{n,1}(0)\right)\right) + \mu_b\left(\omega_{n+1,1}(x) + \hat{a}_x\left(\beta_n\omega_{n,1}(0)\right.\right.\\
&\left.\left. + \bar{\beta}_{n+1}\omega_{n+1,1}(0)\right)\right) + \varphi\bar{\mu}_v\left(\omega_{n,0}(x) + \hat{a}_x\left(\beta_{n-1}\omega_{n-1,0}(0) + \bar{\beta}_n\omega_{n,0}(0)\right)\right) + \mu_v\left(\omega_{n+1,0}(x)\right.\\
&\left. + \hat{a}_x\left(\beta_n\omega_{n,0}(0) + \bar{\beta}_{n+1}\omega_{n+1,0}(0)\right)\right), 2\le n\le N-1,\\
\omega_{N,1}(x-1) =& \bar{\mu}_b\left(\omega_{N,1}(x) + \hat{a}_x\left(\beta_{N-1}\omega_{N-1,1}(0) + \omega_{N,1}(0)\right)\right) + \varphi\bar{\mu}_v\left(\omega_{N,0}(x) + \hat{a}_x\left(\beta_{N-1}\omega_{N-1,0}(0)\right.\right.\\
&\left.\left. + \omega_{N,0}(0)\right)\right).
\end{aligned}$$

The probability generating function of $\omega_{n,j}(x)$ be

$$\omega^*_{n,j}(z) = \sum_{x=0}^{\infty}\omega_{n,j}(x)z^x, j=0,1;\ j\le n\le N.$$

Further, at an arbitrary epoch, let $\omega_{n,j} = \omega^*_{n,j}(1), j=0,1; j\le n\le N$ where $\omega_{n,j}$ denotes the probability of n clients in the system with server state j. Multiplying the aforementioned set of equations with z^x

and adding over x, we get the following equations:

$$
\begin{aligned}
(z-1)\omega_{0,0}^{*}(z) =& \mu_v\omega_{1,0}^{*}(z)+\mu_b\omega_{1,1}^{*}(z)+A^{*}(z)\big(\mu_v\omega_{0,0}(0)+\bar{\beta}_1\left(\mu_v\omega_{1,0}(0)+\mu_b\omega_{1,1}(0)\right)\big)\\
&-\omega_{0,0}(0)-\mu_v\omega_{1,0}(0)-\mu_b\omega_{1,1}(0), \qquad (21)\\
(z-\bar{\varphi}\bar{\mu}_v)\omega_{n,0}^{*}(z) =& \bar{\varphi}\bar{\mu}_v\left(A^{*}(z)\left(\beta_{n-1}\omega_{n-1,0}(0)+\bar{\beta}_n\omega_{n,0}(0)\right)-\omega_{n,0}(0)\right), 1\le n\le N, \qquad (22)\\
(z-\bar{\mu}_b)\omega_{1,1}^{*}(z) =& \bar{\mu}_bA^{*}(z)\bar{\beta}_1\omega_{1,1}(0)+\mu_b\big(\omega_{2,1}^{*}(z)+A^{*}(z)\big(\beta_1\omega_{1,1}(0)+\bar{\beta}_2\omega_{2,1}(0)\big)\big)\\
&+\varphi\bar{\mu}_v\big(\omega_{1,0}^{*}(z)+A^{*}(z)\left(\omega_{0,0}(0)+\bar{\beta}_1\omega_{1,0}(0)\right)\big)+\mu_v\big(\omega_{2,0}^{*}(z)+A^{*}(z)\big(\beta_1\omega_{1,0}(0)\\
&+\bar{\beta}_2\omega_{2,0}(0)\big)\big)-\bar{\mu}_b\omega_{1,1}(0)-\mu_b\omega_{2,1}(0)-\varphi\bar{\mu}_v\omega_{1,0}(0)-\mu_v\omega_{2,0}(0), \qquad (23)\\
(z-\bar{\mu}_b)\omega_{n,1}^{*}(z) =& \bar{\mu}_bA^{*}(z)\left(\beta_{n-1}\omega_{n-1,1}(0)+\bar{\beta}_n\omega_{n,1}(0)\right)+\mu_b\big(\omega_{n+1,1}^{*}(z)+A^{*}(z)\big(\beta_n\omega_{n,1}(0)\\
&+\bar{\beta}_{n+1}\omega_{n+1,1}(0)\big)\big)+\varphi\bar{\mu}_v\big(\omega_{n,0}^{*}(z)+A^{*}(z)\big(\beta_{n-1}\omega_{n-1,0}(0)+\bar{\beta}_n\omega_{n,0}(0)\big)\big)\\
&+\mu_v\big(\omega_{n+1,0}^{*}(z)+A^{*}(z)\left(\beta_n\omega_{n,0}(0)+\bar{\beta}_{n+1}\omega_{n+1,0}(0)\right)\big)-\bar{\mu}_b\omega_{n,1}(0)\\
&-\varphi\bar{\mu}_v\omega_{n,0}(0)-\mu_v\omega_{n+1,0}(0)-\mu_b\omega_{n+1,1}(0),\ 2\le n\le N-1, \qquad (24)\\
(z-\bar{\mu}_b)\omega_{N,1}^{*}(z) =& \bar{\mu}_bA^{*}(z)\left(\beta_{N-1}\omega_{N-1,1}(0)+\omega_{N,1}(0)\right)+\varphi\bar{\mu}_v\big(\omega_{N,0}^{*}(z)+A^{*}(z)\big(\beta_{N-1}\omega_{N-1,0}(0)\\
&+\omega_{N,0}(0)\big)\big)-\bar{\mu}_b\omega_{N,1}(0)-\varphi\bar{\mu}_v\omega_{N,0}(0). \qquad (25)
\end{aligned}
$$

Adding the above equations yields

$$\sum_{n=0}^{N}\omega_{n,0}^{*}(z)+\sum_{n=1}^{N}\omega_{n,1}^{*}(z)=\left(\frac{A^{*}(z)-1}{z-1}\right)\left(\sum_{n=0}^{N}\omega_{n,0}(0)+\sum_{n=1}^{N}\omega_{n,1}(0)\right).$$

As $z\to 1$ and on using the condition $\sum_{n=0}^{N}\omega_{n,0}+\sum_{n=1}^{N}\omega_{n,1}=1$, the above equation reduces to

$$\sum_{n=0}^{N}\omega_{n,0}(0)+\sum_{n=1}^{N}\omega_{n,1}(0)=\lambda. \qquad (26)$$

4.2.1 Arbitrary epoch probabilities in terms of pre-arrival epoch probabilities

The probabilities $\omega_{n,j}^{-}$ and the rate probabilities $\omega_{n,j}(0)$ are related by

$$\omega_{n,j}^{-}=\frac{\omega_{n,j}(0)}{\lambda},\quad j=0,1;\,j\le n\le N, \qquad (27)$$

where λ is obtained from (26).

Theorem 2. *The arbitrary instant probabilities $\omega_{n,j}, j=0,1; j\le n\le N$, are given by*

$$
\begin{aligned}
\omega_{n,0} =& \frac{\lambda\bar{\varphi}\bar{\mu}_v}{1-\bar{\varphi}\bar{\mu}_v}\left(\beta_{n-1}\omega_{n-1,0}^{-}-\beta_n\omega_{n,0}^{-}\right), 1\le n\le N, \qquad (28)\\
\omega_{1,1} =& \lambda\beta_1\omega_{1,1}^{-}+\frac{1}{\mu_b}\left(\frac{\lambda\varphi\bar{\mu}_v\beta_1}{1-\bar{\varphi}\bar{\mu}_v}\omega_{0,0}^{-}+\frac{\lambda\mu_v\beta_1}{1-\bar{\varphi}\bar{\mu}_v}\omega_{1,0}^{-}\right), \qquad (29)\\
\omega_{n,1} =& \lambda\beta_n\omega_{n,1}^{-}+\frac{1}{\mu_b}\left(\lambda\bar{\mu}_b\beta_{n-1}\omega_{n-1,1}^{-}+\frac{\lambda\varphi\bar{\mu}_v\beta_{n-1}}{1-\bar{\varphi}\bar{\mu}_v}\omega_{n-1,0}^{-}+\frac{\lambda\mu_v\beta_n}{1-\bar{\varphi}\bar{\mu}_v}\omega_{n,0}^{-}\right), 2\le n\le N, \qquad (30)\\
\omega_{0,0} =& 1-\sum_{n=1}^{N}\left(\omega_{n,0}+\omega_{n,1}\right). \qquad (31)
\end{aligned}
$$

Proof. Setting $z = 1$ in (22) to (25), utilizing (27) yields (28) to (30) after simplifications. The unknown $\omega_{0,0}$ is derived from the normalization condition. □

4.2.2 Outside observer's observation epoch probabilities

For an EAS, an outside observer's observation instant falls in a time interval after a potential arrival and before a potential departure. At an outside observer's observation instant, for $j = 0,1; j \leq n \leq N, \omega^{o}_{n,j}$ denote the probability of n in the system with the server being in j state . Noticing the outside observer's observation instants and arbitrary instants displayed in Fig. 1 yields

$$\omega_{0,0} = \omega^{o}_{0,0} + \mu_v \omega^{o}_{1,0} + \mu_b \omega^{o}_{1,1}, \tag{32}$$

$$\omega_{n,0} = \bar{\varphi}\bar{\mu}_v \omega^{o}_{n,0}, 1 \leq n \leq N, \tag{33}$$

$$\omega_{n,1} = \bar{\mu}_b \omega^{o}_{n,1} + \mu_b \omega^{o}_{n+1,1} + \varphi\left(\bar{\mu}_v \omega^{o}_{n,0} + \mu_v \omega^{o}_{n+1,0}\right) + \bar{\varphi}\mu_v \omega^{o}_{n+1,0},\ 1 \leq n \leq N-1, \tag{34}$$

$$\omega_{N,1} = \bar{\mu}_b \omega^{o}_{N,1} + \varphi\bar{\mu}_v \omega^{o}_{N,0}. \tag{35}$$

Theorem 3. *The outside observer's observation epoch probabilities at steady-state are given as*

$$\omega^{o}_{n,0} = \frac{\lambda}{1-\bar{\varphi}\bar{\mu}_v}\left(\beta_{n-1}\omega^{-}_{n-1,0} - \beta_n \omega^{-}_{n,0}\right),\ 1 \leq n \leq N, \tag{36}$$

$$\omega^{o}_{1,1} = \frac{\lambda}{\mu_b\left(1-\bar{\varphi}\bar{\mu}_v\right)}\left(\varphi\bar{\mu}_v \omega^{-}_{0,0} + \mu_v \beta_1 \omega^{-}_{1,0}\right), \tag{37}$$

$$\omega^{o}_{n,1} = \frac{\lambda}{\mu_b}\left(\beta_{n-1}\omega^{-}_{n-1,1} + \frac{\varphi\bar{\mu}_v}{1-\bar{\varphi}\bar{\mu}_v}\beta_{n-1}\omega^{-}_{n-1,0} + \frac{\mu_v}{1-\bar{\varphi}\bar{\mu}_v}\beta_n \omega^{-}_{n,0}\right),\ 2 \leq n \leq N, \tag{38}$$

$$\omega^{o}_{0,0} = 1 - \sum_{n=1}^{N}\left(\omega^{o}_{n,0} + \omega^{o}_{n,1}\right). \tag{39}$$

Proof. Solving Eqs. (33) to (35) and using (28) to (30), one can easily obtain (36) to (38). $\omega^{o}_{0,0}$ is evaluated using $\sum_{n=0}^{N}\omega^{o}_{n,0} + \sum_{n=1}^{N}\omega^{o}_{n,1} = 1$. □

The evaluation of steady-state probability distributions at different epochs is thus complete.

5 Performance characteristics

The mean number of clients in the system $(E[L])$ for both the models and the mean number of clients in the system at an outside observer's observation epoch $(E[L^o])$ (only for the discrete time queue) are, respectively, given by

$$E[L] = \sum_{n=1}^{N} n\left(\omega_{n,0} + \omega_{n,1}\right); \quad E[L^o] = \sum_{n=1}^{N} n\left(\omega^{o}_{n,0} + \omega^{o}_{n,1}\right).$$

The blocking probability (P_l) and the mean balking rate (BR) are, respectively,

$$P_l = \omega^{-}_{N,0} + \omega^{-}_{N,1}; \quad BR = \sum_{n=1}^{N} \lambda\bar{\beta}_n\left(\omega^{-}_{n,0} + \omega^{-}_{n,1}\right).$$

Utilizing Little's rule, the mean sojourn time of a client in the system for the continuous time queue (W_c) and for the discrete time queue (W_d) are obtained as $W_c = \frac{E[L]}{\lambda'}, W_d = \frac{E[L^o]}{\lambda'}$, where $\lambda' = \lambda(JR)$ is the effective arrival rate and $JR = \sum_{n=0}^{N} \beta_n \omega_{n,0}^{-} + \sum_{n=1}^{N} \beta_n \omega_{n,1}^{-}$ is the mean joining rate.
Substituting $E[L]$ in W_c, using Eqs. (17) to (19) and after simplification, we get

$$W_c = \frac{1}{JR}\left(\sum_{n=1}^{N-1} \beta_n \omega_{n,1}^{-}\left(\frac{n+1}{\mu_b}\right) + \sum_{n=0}^{N-1} \beta_n \omega_{n,0}^{-}\left(\frac{(n+1)\varphi + n\mu_v + \mu_b}{\mu_b(\varphi+\mu_v)}\right)\right). \tag{40}$$

Similarly, substituting $E[L^o]$ in W_d, using (36) to (38) yields

$$W_d = \frac{1}{JR}\left(\sum_{n=1}^{N-1} \beta_n \omega_{n,1}^{-}\left(\frac{n+1}{\mu_b}\right) + \sum_{n=0}^{N-1} \beta_n \omega_{n,0}^{-}\left(\frac{(n+1)\varphi\bar{\mu}_v + n\mu_v + \mu_b}{\mu_b(1-\bar{\varphi}\bar{\mu}_v)}\right)\right). \tag{41}$$

5.1 Sojourn time analysis

The sojourn time analysis under the FCFS queue discipline is studied here. Upon arrival, a client may see the system in any of the conditions stated below.

- $n(1 \le n \le N-1)$ clients in the system with the server being busy with regular service. In such a situation, if an arriving client becomes a part of the queue with probability β_n then it has to wait until $(n+1)$ departures during regular service.
- $n(0 \le n \le N-1)$ clients in the system with the server in WV, then
  - if the service time during WV is shorter than the vacation duration, then a non-balking client has to wait until one departure in WV and n departures in regular service period due to VI.
  - if the service time during WV is not shorter than the vacation duration, then a non-balking client need to wait until $n+1$ departures in regular service due to vacation completion.

In the continuous time model, let $W_{ac}^*(s)$ and W_{ac} be the LST of the steady-state sojourn time distribution and the mean sojourn time in the system, respectively. Taking into account all of the aforementioned conditions, the LST of the sojourn time is obtained as

$$W_{ac}^*(s) = \frac{1}{JR}\left(\sum_{n-1}^{N-1} \beta_n \omega_{n,1}^{-}\left(\frac{\mu_b}{\mu_b+s}\right)^{n+1} + \sum_{n=0}^{N-1} \beta_n \omega_{n,0}^{-}\left(\left(\frac{\mu_v}{\varphi+\mu_v+s}\right)\left(\frac{\mu_b}{\mu_b+s}\right)^{n} + \left(\frac{\varphi}{\varphi+\mu_v+s}\right)\left(\frac{\mu_b}{\mu_b+s}\right)^{n+1}\right)\right).$$

Thus, the steady-state mean sojourn time is given by

$$W_{ac} = \frac{1}{JR}\left(\sum_{n=1}^{N-1} \beta_n \omega_{n,1}^{-}\left(\frac{n+1}{\mu_b}\right) + \sum_{n=0}^{N-1} \beta_n \omega_{n,0}^{-}\left(\frac{(n+1)\varphi + n\mu_v + \mu_b}{\mu_b(\varphi+\mu_v)}\right)\right).$$

It should be noticed that the mean sojourn time (W_{ac}) obtained through the analysis above and the mean sojourn time estimated using Little's rule (40) are identical.

For the discrete time model, let $W^*_{ad}(z)$ and W_{ad} be the z-transform of the sojourn time distribution and the mean sojourn time at steady-state. Combining all the possible cases discussed earlier, the z-transform of the sojourn time in the discrete time is given by

$$W^*_{ad}(z) = \frac{1}{JR}\left(\sum_{n-1}^{N-1}\beta_n\omega^-_{n,1}\left(\frac{\mu_b z}{1-\bar{\mu}_b z}\right)^{n+1} + \sum_{n=0}^{N-1}\beta_n\omega^-_{n,0}\left(\left(\frac{\mu_v z}{1-\bar{\varphi}\bar{\mu}_v z}\right)\left(\frac{\mu_b z}{1-\bar{\mu}_b z}\right)^n \right.\right.$$
$$\left.\left. + \left(\frac{\varphi\bar{\mu}_v z}{1-\bar{\varphi}\bar{\mu}_v z}\right)\left(\frac{\mu_b z}{1-\bar{\mu}_b z}\right)^{n+1}\right)\right).$$

Hence, at steady-state, the average sojourn time for the discrete time model is

$$W_{ad} = \frac{1}{JR}\left(\sum_{n=1}^{N-1}\beta_n\omega^-_{n,1}\left(\frac{n+1}{\mu_b}\right) + \sum_{n=0}^{N-1}\beta_n\omega^-_{n,0}\left(\frac{(n+1)\,\varphi\bar{\mu}_v + n\mu_v + \mu_b}{\mu_b\,(1-\bar{\varphi}\bar{\mu}_v)}\right)\right).$$

Note that the average sojourn time (W_{ad}) obtained through analysis for the discrete time model also matches with the one computed using Little's rule (41).

6 Numerical investigations

Various numerical simulations to demonstrate the impact of the model parameters on the performance characteristics are presented under this section. The capacity of the system is presumed to be $N = 10$ while the balking function is considered as $\beta_n = 1 - n/(N+1), 1 \leq n \leq N-1$ with the assumption that $\beta_0 = 1,\ \beta_N = 0$. The arbitrary choice of the model parameters for the continuous time queue is: $\lambda = 1.9,\ \mu_b = 2.5, \mu_v = 2.1, \varphi = 1.0$ and for the discrete time queue is: $\lambda = 0.4,\ \mu_b = 0.5, \mu_v = 0.2, \varphi = 0.1$. The distribution of number of clients in the system at different epochs for exponential and geometric i.a.t. distributions is presented in Table 1 along with the performance measures.

The influence of μ_b on the mean balking rate (BR) and on the mean joining rate (JR) is presented in Figure 2 for distinct values of λ when the i.a.t. follow Erlang-4 distribution. The figure illustrates that for fixed λ, as μ_b increases BR decreases whereas JR increases which is coherent with the fact that as μ_b increases, the mean system size decreases and hence, the arriving clients tend to join the queue resulting in the increase of JR and decrease of BR. Further, for fixed μ_b, rise in λ results in an increase in BR and a decrease in JR as expected.

In Figure 3, one may observe the effect of the arrival rate (λ) on $E[L]$ for the continuous time model with balking and without balking when the i.a.t. follow deterministic distribution. With the growth of λ an increase is observed in $E[L]$ in both the models. Further, the mean system size is more in the model without balking in contrast to the model with balking.

Figure 4 depicts the impact of N on P_l in the discrete time queueing model for different i.a.t. distributions. The i.a.t. distributions are taken as geometric, deterministic and arbitrary. From the figure, it is clear that P_l decreases with the rise in N for any i.a.t. distribution which agrees with our intuition.

The consequence of λ and φ on the mean sojourn time in the discrete time model is displayed in Figure 5 when the i.a.t. are geometrically distributed. The figure depicts that as λ rises the mean sojourn time also rises for any φ. Further, with the growth of φ, W_d decreases because of the fact that as φ increases the vacation time depletes and the server resumes regular service wherein service is provide with rate $\mu_b > \mu_v$.

Table 1: The distribution of the number of clients in the system.

n	$M/M/1/10$		$Geo/Geo/1/10$			
	$\omega^{-}_{n,0} = \omega_{n,0}$	$\omega^{-}_{n,1} = \omega_{n,1}$	$\omega^{-}_{n,0} = \omega_{n,0}$	$\omega^{-}_{n,1} = \omega_{n,1}$	$\omega^{o}_{n,0}$	$\omega^{o}_{n,1}$
0	0.315769	–	0.221463	–	0.132878	–
1	0.124286	0.135585	0.117717	0.175642	0.163496	0.111772
2	0.046122	0.140805	0.059772	0.172842	0.083016	0.180145
3	0.015998	0.102796	0.028776	0.112402	0.039966	0.136270
4	0.005130	0.061351	0.013010	0.056627	0.018069	0.074912
5	0.001499	0.030893	0.005455	0.023032	0.007576	0.032421
6	0.000392	0.013099	0.002086	0.007635	0.002896	0.011272
7	0.000089	0.004585	0.000709	0.002051	0.000985	0.003141
8	0.000017	0.001277	0.000207	0.000437	0.000288	0.000687
9	0.000002	0.000266	0.000049	0.000071	0.000068	0.000113
10	0.000000	0.000037	0.000009	0.000007	0.000013	0.000012
	$E[L] = 1.544810$, $P_l = 0.000037$, $BR = 0.266837, JR = 0.859559$, $W_{ac} = W_c = 0.945901$		$E[L] = 1.687150, E[L^o] = 2.025799$, $P_l = 0.000016$, $BR = 0.061351, JR = 0.846621$, $W_{ad} = W_d = 5.982010$			

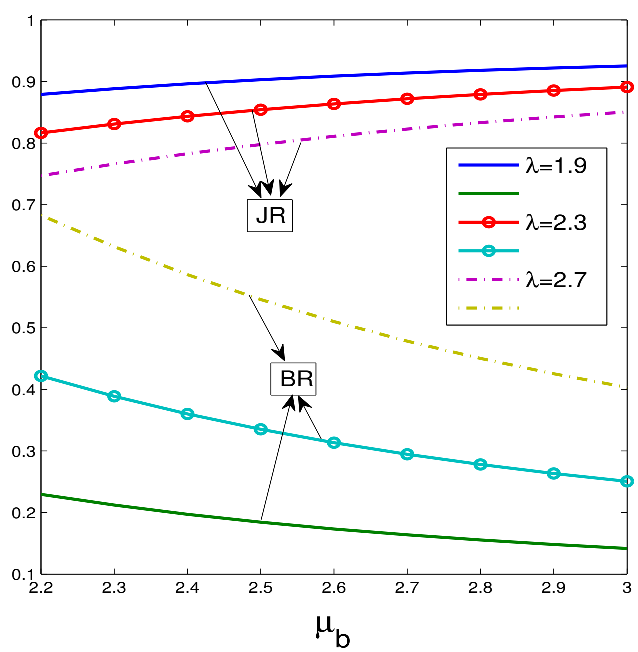

Figure 2: Influence of μ_b on BR and JR in the continuous time model.

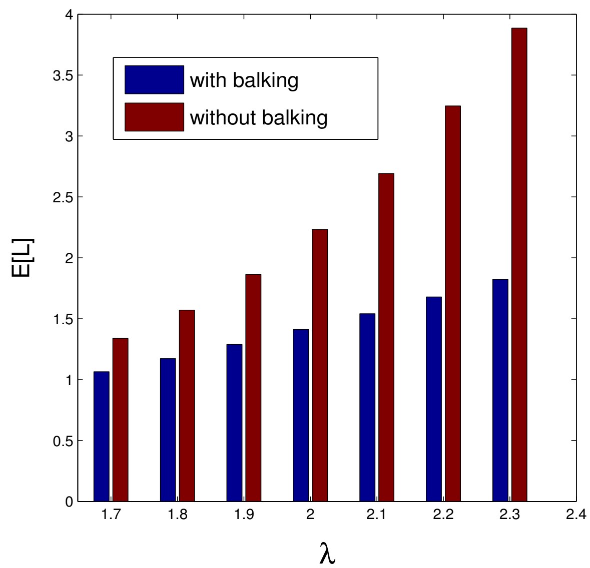

Figure 3: Effect of λ on $E[L]$ for the continuous time model.

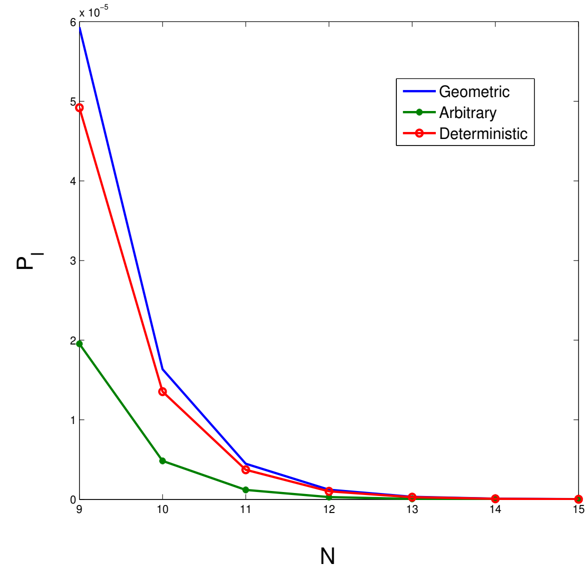

Figure 4: N versus P_l for the discrete time model.

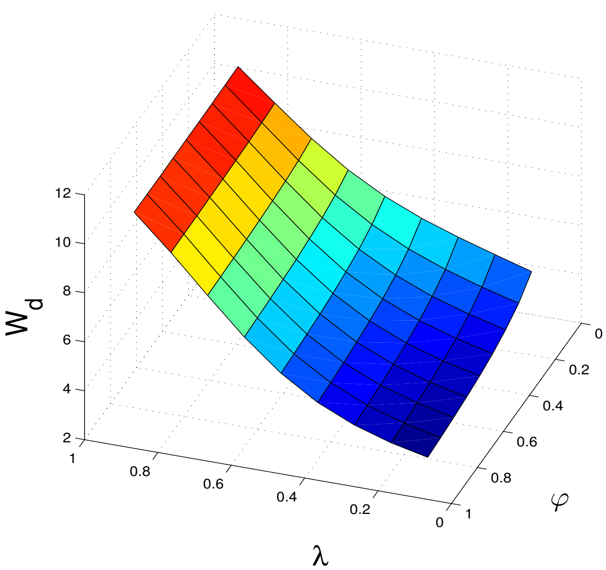

Figure 5: Impact of λ and φ on W_d.

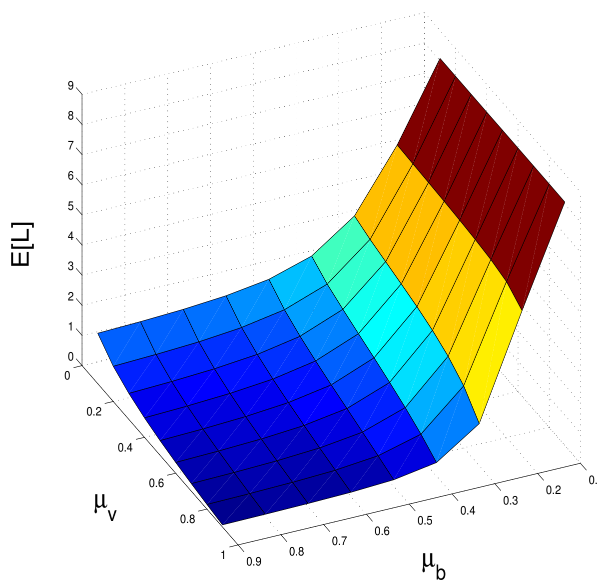

Figure 6: Effect of μ_b and μ_v on $E[L]$.

Figure 6 shows the effect of the service rates μ_b and μ_v on the mean system size ($E[L]$) when the i.a.t. are geometrically distributed. The mean system size decreases with the increase of both μ_b and μ_v. However, the decrease in $E[L]$ with the increase of μ_b is steep in comparison with the decrease in $E[L]$ with μ_v.

7 Conclusions

The current article presents the study of a finite buffer general input queue with balking and vacation interruption in both continuous and discrete time environments. The models under consideration might be applied to numerous real-time systems including contact centres, communications networks, etc. A transfer model of an email system (SMTP) is offered as a potential application of the queueing systems under consideration. The discrete time model has been discussed for the early arrival system. We have obtained the steady-state pre-arrival epoch probabilities in both the systems using embedded Markov chain technique. The arbitrary epoch probabilities at steady-state have been expressed explicitly in terms of the pre-arrival epoch probabilities using supplementary variable technique. The steady-state outside observer's observation epoch probabilities for the discrete time model are also expressed in terms of the pre-arrival epoch probabilities. Different performance characteristics such as the the mean system size, blocking probability, sojourn time using Little's rule, etc., have been presented. The study of the mean sojourn time in the system is also carried out and matched analytically with the one obtained using Little's rule. Further extensions of the present models can be continuous and discrete time queues with balking and variant working vacations.

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