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Bernoulli Wavelet Method for Numerical Solution of Fokker-Planck-Kolmogorov Time Fractional Differential Equations

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ABSTRACT

The purpose of this paper is to present a wavelet method for numerical solutions Fokker-Planck-Kolmogorov time-fractional differential equations with initial and boundary conditions. The authors was employed the Bernoulli wavelets for the solution of Fokker-Planck-Kolmogorov time-fractional differential equation. We calculated the Bernoulli wavelet fractional integral operation matrix of the fractional order and the upper error boundary for the Riemann-Liouville fractional integral operation matrix and the Bernoulli wavelet fractional integral operation matrix. The Fokker-Planck-Kolmogorov time-fractional differential equation is converted to the linear equation using the Bernoulli wavelet operation matrix in this technique. This method has the advantage of being simple to solve. The simulation was carried out using MATLAB software. Finally, the proposed strategy was used to solve certain problems. the Bernoulli wavelet and Bernoulli fraction of the fractional order, the Bernoulli polynomial, and the Bernoulli fractional functions were introduced. Explaining how functions are approximated by fractional-order Bernoulli wavelets as well as fractional-order Bernoulli functions. The Bernoulli wavelet fractional integral operational matrix was used to solve the Fokker-Planck-Kolmogorov fractional differential equations. The results for some numerical examples are documented in table and graph form to elaborate on the efficiency and precision of the suggested method. The results revealed that the suggested numerical method is highly accurate and effective when used to Fokker-Planck-Kolmogorov time fraction differential equations

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1. Introduction

Fractional Differential Equations are the generalization of ordinary differential equations of arbitrary order (non-integer). In recent years, the application of fractional differential equations has increased in many fields. Therefore, analyzing and solving these equations has become one of the researchers' concerns. It is not always possible to obtain a closed-form solution for these equations, and even in many cases, it is impossible. Therefore, researchers have tended to use approximate methods to solve this type of problem[3].In the Mid-Nineteenth Century, Riemann and Liouville introduced the concept of Differential Calculus. But Oldham and Spanier published the first book in this context in 1974. It did not take long that the number of publications on fractional calculus experienced a rapid increase. The reason is that many physical systems show fractional-order dynamics, meaning that their behavior is under fractional differential equations control [1]. Fractional differential equation is the generalization of ordinary differential equation to arbitrary order (non-integer). Everybody can find the history of fractional differential equations appearance in [2]. Many researchers are interested in fractional differential equations because these equations have a high ability to model complex phenomena such as economics [10], statistical and quantum mechanics [11], solid mechanics [12], and joint surface dynamics between rigid layers and soft nanoparticles [13]. Moreover, researchers are eager to improve numerical methods to solve them. These methods include Fourier transformation [14], eigenvector expansion [15], Laplace transforms [16], Edomian decomposition method [17], finite difference method [18], power series method [19], fractional differential conversion method [20], and homotopic analytical method [21]. Meanwhile, orthogonal functions have a particular place especially facing various problems of dynamical systems. Researchers have employed orthogonal functions to solve many fractional differential equations. The importance of orthogonal functions is that they can reduce a differential equation into an algebraic equation using derivative or integral operational matrices. Among orthogonal polynomials, the transferred Legender polynomials $(p_m(t), m = 0,1,2, \ldots, 0 \leq t \leq 1)$ have the best behavior and are more computationally efficient [22] and [23].Tyler and Bernoulli polynomials $(\beta_m(t), m = 0,1,2, \ldots, 0 \leq t \leq 1)$ are not orthogonal. However, it is possible to calculate their integral operational matrix. Since the integral of multiplying two Tyler vectors is a kind of Hilbert's bad matrix [24], the applications of the Tyler series are limited. In statistical mechanics, the Fokker-Planck equation is a partial differential equation that accounts for the time evolution of the density probability velocity function for the particles influenced by drag and random forces. Brownian motion is described by this equation, which may be extended to include observations expect for velocity [28, 29]. For the first time, a mathematician and physicist, Joseph Fourier, proposed the idea of representing a function in terms of a complete set of functions. Fourier proved that it is possible to represent a function $f(t)$ concisely using axes made up of a set of sine-like functions. In other words, Fourier showed that it is feasible to represent a function $f(t)$ by an infinite sum of sine and cosine functions in in the form of$\sin(at)$ and $\cos(at)$. Fourier bases became essential tools with many applications in science. But over time, the weakness of the Fourier foundations became apparent. Scientists found that Fourier bases and the representation of sine-like functions for complex theoretical image signals are not ideal. For instance, they cannot efficiently display transitory structures such as existing boundaries in images. They also observed that the Fourier transform is applicable only for elementary functions. In 1957, Har was the first to point out the wavelets. Generally, the goal of wavelet theory is to find new bases for $L^2(\mathbb{R})$. In this paper, we defined a new set of fractional functions. This set is called Bernoulli fractional-order wavelets and constructed on Bernoulli wavelets by changing t to x^{α}[30]. We show Bernoulli fractional-order wavelets $\psi_{n,m}(x^{\alpha})$ with$\psi_{n,m}^{\alpha}(x)$. Furthermore, we obtained the transformation matrix of Bernoulli fractional-order wavelets to Bernoulli fractional-order functions. Finally, we found the operational matrix of the Bernoulli fractional-order wavelet integral. Previous research has suggested a technique for solving two-dimensional Fokker-Planck equations for non-hybrid continuous systems using the finite difference approach, and the proposed method's stability and accuracy have been investigated. Many other articles are written on the numerical solution of Fokker-Planck equations [26, 27]. The Bernolii wavelet method is utilized in this study to solve the Fokker-Planck-Kolmogorov time-fractional differential equations in the following way[25]:

$$D_t^{\alpha}u-\frac{1}{2}\sigma^2x^2\frac{\partial^2u}{\partial x^2}+(\beta-2\sigma^2)x\frac{\partial u}{\partial x}+(\beta-\sigma^2)u=R(x,t) \tag{1}$$

Initial conditions:

$$\mathrm{u}(0,\mathrm{x})=f_0(x),\ u_t(0,x)=f_1(x),\quad 0\le x\le 1$$

Boundary conditions:

$$\mathrm{u}(t,0)=g_0(t),\quad u_t(t,1)=g_0(t),\quad 0\le t\le 1$$

$R(x,t)$ Is the right side function of the equation, which is given for each equation.

2. Preliminaries:

2.1. Fractional order integral and Fractional Order Derivative

The Riemann - Liouville fractional integral operator of order υ of the function $f(t)\in C_{\mu};\ \mu\ge -1$ is defined as follows [7]:

$$I^{\upsilon}f(t)=\begin{cases}\frac{1}{\Gamma(\upsilon)}\int_0^t\frac{f(s)}{(t-s)^{1-\upsilon}}ds=\frac{1}{\Gamma(\upsilon)}t^{\upsilon-1}*f(t) & ;\upsilon>0\\ f(t) & ;\upsilon=0,\end{cases} \tag{2}$$

where $t^{\upsilon-1}$*f(t) is the convolution product of the two functions $t^{\upsilon-1}$ and f(t).

The following formula is the definion of Riemann - Liouville fractional integral operator which is the generalization of the Cauchy's formula for integrals,

$$\int_a^{x_1}dx_1\int_a^{x_2}dx_2\ldots\int_a^{x_{n-1}}dx_{n-1}=\frac{1}{(n-1)!}\int_a^x\frac{f(t)}{(x-t)^{1-n}}dt. \tag{3}$$

For Riemann - Liouville fractional integral, we have [7]

$$(I^{\upsilon_1}I^{\upsilon_2}f)(t)=I^{\upsilon_1+\upsilon_2}f(t);\quad \upsilon_1,\upsilon_2\ge 0$$

And

$$(I^{\upsilon_1}I^{\upsilon_2}f)(t)=(I^{\upsilon_2}I^{\upsilon_1}f)(t),$$

$$I^{\upsilon}t^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+\upsilon+1)}t^{\upsilon+\beta};\quad \beta>0, \tag{4}$$

Riemann - Liouville fractional integral is a linear operator, i.e

$$I^{\upsilon}\big(\lambda_1f(t)+\lambda_2g(t)\big)=\lambda_1I^{\upsilon}f(t)+\lambda_2I^{\upsilon}g(t),$$

where λ_1 and λ_2 are constants. The Caputo fractional derivative of order υ of the function f(t)$\in C_{-1}^{n}$is ([8-9])

$$D^{\upsilon}f(t) = \frac{1}{\Gamma(n-\upsilon)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\upsilon+1-n}}ds; \quad n-1 < \upsilon \leq n, t > 0, n \in \mathbb{N}.$$

Caputo fractional derivative satisfies in two relations below:

$$(D^{\upsilon}I^{\upsilon}f)(t) = f(t)$$

$$(D^{\upsilon}I^{\upsilon}f)(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0)\frac{t^i}{i!}, \tag{5}$$

equation is a special state of the following equation

$$D^{\upsilon}I^{\upsilon}f(t) = I^{\alpha}I^{m-\alpha}f^{(m)}(t) = I^{\alpha-\beta}\left(I^n f^{(n)}(t)\right) = I^{\alpha-\beta}f(t) - \sum_{i=0}^{n-1}\frac{f^{(i)}(0)}{\Gamma(\alpha-\beta+i-)\beta}, \ \alpha > \beta, \tag{6}$$

where $n-1 < \alpha \leq n \ and \ m-1 < \beta < m.$

Some properties of the Caputo fractional derivatives for $f(x) \in C^1[0,1], 0 < \upsilon \leq 1$ for are listed below and you can find their proofs in [4].

1. $(I^{\upsilon}D^{\upsilon}f)(t) = f(t) - f(0),$
2. $D^n\left(D^{\upsilon}f(t)\right) = D^{\upsilon}\left(D^n f(t)\right),$
3. $D^n\left(D^{\upsilon}f(t)\right) = D^{n+\upsilon}f(t),$
4. $D^{\beta}f(t) = I^{m-\beta}D^m f(t); \ m-1 < \beta \leq m,$
5. $D^{\upsilon}C = 0,$
6. $I^{\upsilon}t^{\beta} = 0$ when $\upsilon \in \mathbb{N}_0$ and $\beta < \upsilon$, otherwise $I^{\upsilon}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\upsilon+1)}t^{\upsilon+\beta}$,
7. $D^{\upsilon}\left(\lambda_1 f(t) + \lambda_2 g(t)\right) = \lambda_1 D^{\upsilon}f(t) + \lambda_2 D^{\upsilon}g(t),$

C, λ_1 and λ_2 are constants.

3- Research method

3-1- Bernolii wavelets and

Bernolii wavelets on the interval [0.1) is defined as follows[4]:

$$\psi_{m,n}(t) = \begin{cases} 2^{\frac{k-1}{2}}\bar{\beta}_m(2^{k-1}t\hat{n}) & t\epsilon[\xi_1, \xi_2) \\ 0 & otherwise \end{cases} \tag{7}$$

Or

$$\bar{\beta}_m(t) = \begin{cases} \frac{1}{\sqrt{\frac{(-1)^{m-1}m!}{(2m)!}\alpha_{2m}}}\beta_m(t) & m > 0 \\ 1 & m = 0 \end{cases} \tag{8}$$

where$\xi_1 = \frac{\hat{n}}{2^{k-1}}$ ، $\xi_2 = \frac{\hat{n}+1}{2^{k-1}}$ ، $m = 0,1,2, \ldots M-1$ and n=1,2,…, 2^{k-1},

Bernolii polynomials are defined as follows[5]:

$$\beta_m(t) = \sum_{i=0}^{m}\binom{m}{i}t^i\alpha_{m-i} \text{ ,i=0,1,…,m,} \tag{9}$$

$$\frac{t}{e^t-1} = \sum_{i=0}^{\infty}\frac{t^i}{i!}\alpha_i, \ \alpha_0 = 1, \alpha_1 = \frac{-1}{2}, \ \alpha_2 = \frac{1}{6}, \alpha_4 = \frac{-1}{30}, \alpha_{2i+1} = 0, \ i = 1,2,3, \ldots \tag{10}$$

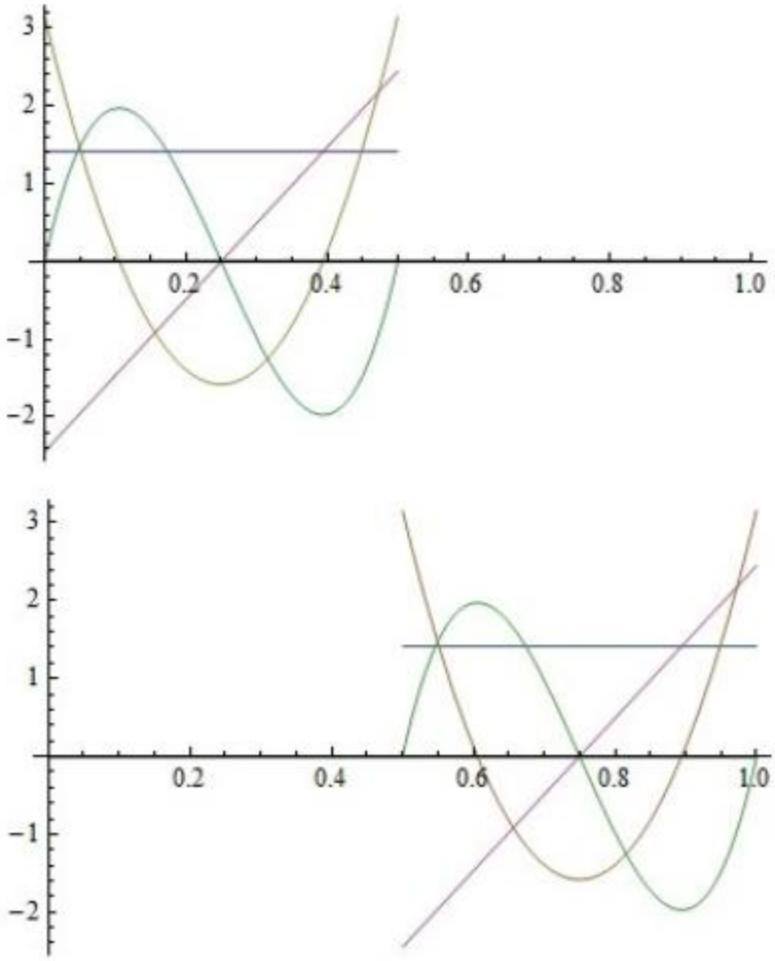

Figure1. Bernoulli wavelet with M=4 ,k=2

3-2-Functions Approximation

If $\{\psi_{10}^{\alpha}(x), \psi_{11}^{\alpha}(x), \ldots, \psi_{2^{k-1}M-1}^{\alpha}(x)\} \subset L^2[0,1]$ is a set of fractional-order Bernolii wavelets, then

$$Y = Span\{\psi_{10}^{\alpha}(x), \psi_{11}^{\alpha}(x), \ldots, \psi_{1\,M-1}^{\alpha}(x), \psi_{20}^{\alpha}(x), \ldots, \psi_{2\,M-1}^{\alpha}(x), \ldots, \psi_{2^{k-1}0}^{\alpha}(x), \psi_{2^{k-1}1}^{\alpha}(x), \ldots, \psi_{2^{k-1}M-1}^{\alpha}(x)\} \quad (11)$$

is a finite dimensional vector space.

Since Y is finite dimensional vector space, there is the best approximation for $f(x)$ in Y like $f_0(x)$ i.e

$$\forall y(x) \in Y, \quad \|f(x) - f_0(x)\| \leq \|f(x) - y(x)\|.$$

From the last relation, we can conclude that

$$\forall y(x) \in Y, \quad < f(x) - f_0(x), y(x) >= 0, \quad (12)$$

where $<,>$ shows inner product.

Because $f_0(x) \in Y$, there are unique coefficients such as $c_{10}, c_{11}, \ldots, c_{2^{k-1}M-1}$ that

$$f(x) \simeq f_0(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = C^T \Psi^{\alpha}(x) \quad (13)$$

Where t represents transpose of matrix, C and $\Psi^{\alpha}(x)$ are matrices of order $2^{k-1}M \times 1$ and

$$C = [c_{10}, c_{11}, \ldots, c_{1\,M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}M-1}]^T \quad (14)$$

$$\Psi^{\alpha}(x)=[\psi^{\alpha}_{10}(x),\psi^{\alpha}_{11}(x),\dots,\psi^{\alpha}_{1\,M-1}(x),\psi^{\alpha}_{20}(x),\dots,\psi^{\alpha}_{2\,M-1}(x),\dots,\psi^{\alpha}_{2^{k-1}0}(x),\psi^{\alpha}_{2^{k-1}1}(x),\dots,\psi^{\alpha}_{2^{k-1}M-1}(x)]^T \quad (15)$$

By the use of equation (3.14), we obtain

$$< f(x)-C^T\Psi^{\alpha}(x),\Psi^{\alpha}_i >= 0, i=0,1,\dots,2^{k-1}M \quad (16)$$

For simplicity, we write

$$C^T < \Psi^{\alpha}(x),\Psi^{\alpha}(x) >=< f(x),\Psi^{\alpha}(x) >, \quad (17)$$

where

$$D =< \Psi^{\alpha}(x),\Psi^{\alpha}(x) >= \int_0^1 \Psi^{\alpha}(x)\Psi^{\alpha T}(x)\,\mathrm{x}^{\alpha-1}dx \quad (18)$$

is a matrix of order $2^{k-1}M\times 2^{k-1}M$.

Matrix D in equation (18) can be calculated by equation (2.10) in every interval $n=1,\dots,2^{k-1}$. For example, when $k=2$ and $M=3$, matrix D is Identity matrix and for $k=2$ and $M=4$, we get:

$$D=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&-\sqrt{\frac{7}{10}}&0&0&0&0\\0&0&1&0&0&0&0&0\\0&-\sqrt{\frac{7}{10}}&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&-\sqrt{\frac{7}{10}}\\0&0&0&0&0&0&1&0\\0&0&0&0&0&-\sqrt{\frac{7}{10}}&0&1\end{bmatrix}$$

Also,

$$F=[f_{1,0},f_{1,1},\dots,f_{1,M-1},f_{2,0},f_{2,1},\dots,f_{2,M-1},\dots,f_{2^{k-1},0},\dots,f_{2^{k-1},M-1}]^T, \quad (19)$$

Where

$$f_{i,j} =< f(x),\psi^{\alpha}_{i,j}(x) >= \int_0^1 f(x)\psi^{\alpha}_{i,j}(x)x^{\alpha-1}dx, \quad (20)$$

$$i=1,2,\dots,2^{k-1}, j=0,1,\dots,M-1.$$

Using the above equations, we get coefficient vector C as follow

$$C^T=F^TD^{-1}, \quad (21)$$

3.3. Transformation matrix of Bernoulli wavelet-fraction to fractional-order Bernoulli functions

Assume that $y(x)\in L^2[0,1]$. Then, it can be expressed in terms of Bernoulli functions as follows

$$y(x)\simeq \sum_{i=0}^{M-1} a_i\beta^{\alpha}_i(x)=A^TB^{\alpha}(x)\,, \quad (22)$$

where matrices A and $B^{\alpha}(x)$ are

$$B^{\alpha}(x)=[\beta^{\alpha}_0(x),\beta^{\alpha}_1(x),\dots,\beta^{\alpha}_{M-1}(x)]^T, \quad A=[a_0,a_1,\dots,a_{M-1}]^T, \quad (23)$$

Similar to Equation (3.21), we can write

$$A^T = Y^T D^{*-1}, \tag{24}$$

Where

$$D^* =< B^\alpha, B^\alpha >= \int_0^1 B^\alpha(x) B^{\alpha T}(x) x^{\alpha-1} dx,\ Y = [y_0, y_1, \dots, y_{M-1}]^T \quad , \tag{25}$$

And

$$y_i = \int_0^1 y(x) \beta_i^\alpha(x) x^{\alpha-1} dx,\ i = 0,1,\dots,M-1 \tag{26}$$

Fractional-order Bernoulli wavelets can be expressed in terms of a fractional Bernoulli function as follows

$$\Psi^\alpha_{2^{k-1}M\times 1}(x) = \Theta_{2^{k-1}M\times M} B^\alpha_{M\times 1}(x), \tag{27}$$

where Θ is the transformation matrix of Bernoulli fraction-order wavelet to fractional-order Bernoulli function. For example, suppose $k = 2$ and $M = 3,$ then

$$\Psi^\alpha(x) = [\psi^\alpha_{10}(x), \psi^\alpha_{11}(x), \psi^\alpha_{12}(x), \psi^\alpha_{20}(x), \psi^\alpha_{21}(x), \psi^\alpha_{22}(x)]^T \tag{28}$$

$$B^\alpha(x) = [\beta^\alpha_0(x), \beta^\alpha_1(x), \beta^\alpha_2(x)]^T, \tag{29}$$

which for $0 \le x < (\frac{1}{2})^{\frac{1}{\alpha}}$

$$\psi^\alpha_{10}(x) = \sqrt{2} = \sqrt{2}\beta^\alpha_0(x), \tag{29}$$

$$\psi^\alpha_{11}(x) = \sqrt{6}(-1 + 4x^\alpha) = \sqrt{6}\beta^\alpha_0(x) + 4\sqrt{6}\beta^\alpha_1(x),$$

$$\psi^\alpha_{12}(x) = \sqrt{10}(1 - 12x^\alpha + 24x^{2\alpha}) = 3\sqrt{10}\beta^\alpha_0(x) + 12\sqrt{10}\beta^\alpha_1(x) + 24\sqrt{10}\beta^\alpha_2(x),$$

and for $(\frac{1}{2})^{\frac{1}{\alpha}} < x \le 1$

$$\psi^\alpha_{20}(x) = \sqrt{2} = \sqrt{2}\beta^\alpha_0(x), \tag{30}$$

$$\psi^\alpha_{21}(x) = \sqrt{6}(-3 + 4x^\alpha) = -\sqrt{6}\beta^\alpha_0(x) + 4\sqrt{6}\beta^\alpha_1(x),$$

$$\psi^\alpha_{22}(x) = \sqrt{10}(13 - 36x^\alpha + 24x^{2\alpha}) = 3\sqrt{10}\beta^\alpha_0(x) - 12\sqrt{10}\beta^\alpha_1(x) + 24\sqrt{10}\beta^\alpha_2(x).$$

Using the definition of fractional-order Bernoulli wavelet, we get $\psi^\alpha_{11}(x)$ for $k = 2, m = 1,$ and $n = 1$ as follow

$$\psi^\alpha_{11}(x) = \sqrt{2}\sqrt{12}\beta_1(2x^\alpha) = 2\sqrt{6}\beta_1(2x^\alpha) = 2\sqrt{6}\left(2x^\alpha - \frac{1}{2}\right) = \sqrt{6}(4x^\alpha - 1)$$

According to equation (2.9), it is easy to obtain

$$\beta^\alpha_1(x) = x^\alpha - \frac{1}{2} \Longrightarrow x^\alpha = x^\alpha + \frac{1}{2}$$

$$\beta^\alpha_0(x) = 1,$$

Therefore, we have

$$\psi^\alpha_{11}(x) = \sqrt{6}(4\beta^\alpha_1(x) + 1) = 4\sqrt{6}\beta^\alpha_1(x) + \sqrt{6}\beta^\alpha_0(x).$$

The other wavelets can be calculated in the same way.

Consider

$$\Theta = \begin{cases} \Phi = [a_{i,j}]_{6\times 3}, & 0 \le x < (\frac{1}{2})^{\frac{1}{\alpha}} \\ \Phi' = [a'_{i,j}]_{6\times 3}, & (\frac{1}{2})^{\frac{1}{\alpha}} \le x < 1 \end{cases} \tag{31}$$

And

$$\Phi = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{6} & 4\sqrt{6} & 0 \\ 3\sqrt{10} & 12\sqrt{10} & 24\sqrt{10} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Phi' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ -\sqrt{6} & 4\sqrt{6} & 0 \\ 3\sqrt{10} & -12\sqrt{10} & 24\sqrt{10} \end{bmatrix}$$

To calculate inverses of matrices Φ and Φ', we proceed as follow

$$\Phi = \begin{bmatrix} A \\ \dots \\ 0 \end{bmatrix}_{2^{k-1}M\times M} \qquad \Phi^{-1} = [A^{-1} \ \vdots \ 0]_{M\times 2^{k-1}M}$$

$$\Phi' = \begin{bmatrix} 0 \\ \dots \\ B \end{bmatrix}_{2^{k-1}M\times M} \qquad {\Phi'}^{-1} = [0 \ \vdots \ B^{-1}]_{M\times 2^{k-1}M}.$$

Generally, for $k = 2$ and arbitrary M, we get

$$\Theta = \begin{cases} \Phi = [a_{i,j}]_{2^{k-1}M\times M}, & 0 \le x < (\frac{1}{2})^{\frac{1}{\alpha}} \\ \Phi' = [a'_{i,j}]_{2^{k-1}M\times M}, & (\frac{1}{2})^{\frac{1}{\alpha}} \le x < 1 \end{cases}$$

where

$$a_{i,j} = \frac{1}{2^{\frac{k-1}{2}}} \begin{cases} 2^{i}\frac{1}{\lambda_{i-1}}, & i = j \\ 2^{j-1}\binom{i-1}{i-j}\frac{1}{\lambda_{i-1}}, & j < i \le M \\ 0, & else \end{cases}$$

and

$$a'_{i,j} = \begin{cases} 0, & 1 \le i \le M \\ (-1)^{i+j-M} a_{i-M,j}, & M+1 \le i \le 2^{k-1}M \end{cases}, \quad j = 1,2,\dots,M,$$

and

$$\lambda_i = \sqrt{\frac{(-1)^{i-1}(i!)^2}{(2i)!}\beta_{2i}},\ i = 1,2,\dots,M-1,\ \lambda_0 = 1$$

3.4. Fractional Integral Operational Matrix of Bernoulli Fractional-Order Wavelets

Riemann-Liouville fractional integral of vector $B^{\alpha}(x)$ in equation is given by

$$I^{\nu}B^{\alpha}(x) \simeq F^{(\upsilon,\alpha)}(x)B^{\alpha}(x) \tag{32}$$

Where $\boldsymbol{F}^{(\boldsymbol{v},\boldsymbol{\alpha})}$ is the operational matrix of Riemann-Liouville fractional integral of order ν, which is $M \times M$.

If we use equation (2.9) and properties of operator I^{ν} for $i = 0,1,\dots,M-1$, we get

$$I^{\nu}\beta_i^{\alpha}(x) = I^{\nu}\left(\sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}x^{\alpha r}\right) = \sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}I^{\nu}x^{\alpha r}$$
$$= \sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1+\upsilon)} \tag{33}$$
$$= \sum_{r=0}^{i} b_{i,r}^{(\upsilon,\alpha)}x^{\alpha r+\upsilon}$$

where

$$b_{i,r}^{(\upsilon,\alpha)} = \binom{i}{r}\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1+\upsilon)}\beta_{i-r}. \tag{34}$$

Suppose that it is possible to expand $x^{\alpha r+\upsilon}$ by the Bernoulli fractional-order polynomials with M sentences as follow

$$x^{\alpha r+\upsilon} \simeq \sum_{r=0}^{i} c_{r,j}^{(\upsilon,\alpha)}x^{\alpha r+\upsilon}\,\beta_j^{\alpha}(x). \tag{35}$$

By placing equation (3.34) in equation (3.32) for $i = 0,1,\dots,M$, we get

$$I^{\nu}\beta_i^{\alpha}(x) \simeq \sum_{r=0}^{i} b_{i,r}^{(\upsilon,\alpha)}\sum_{r=0}^{i} c_{r,j}^{(\upsilon,\alpha)}\,\beta_j^{\alpha}(x) = \sum_{j=0}^{M-1}\left(\sum_{r=\lceil\frac{\upsilon}{\alpha}\rceil}^{i} w_{i,j,r}^{(\upsilon,\alpha)}\right)\beta_j^{\alpha}(x), \tag{36}$$

where

$$w_{i,j,r}^{(\upsilon,\alpha)} = b_{i,r}^{(\upsilon,\alpha)}c_{r,j}^{(\upsilon,\alpha)}. \tag{37}$$

Equation (3.35) can be written as follow

$$I^{\nu}\beta_i^{\alpha}(x) \simeq \left[\sum_{r=0}^{i} w_{i,0,r}^{(\upsilon,\alpha)},\sum_{r=0}^{i} w_{i,1,r}^{(\upsilon,\alpha)},\dots,\sum_{r=0}^{i} w_{i,M-1,r}^{(\upsilon,\alpha)},\right]B^{\alpha}(x), \quad i = 0,1,\dots,M-1.$$

Thus, we have

$$F^{(\upsilon,\alpha)} = \begin{bmatrix} w_{i,0,r}^{(\upsilon,\alpha)} & w_{i,1,r}^{(\upsilon,\alpha)} & \cdots & w_{i,M-1,r}^{(\upsilon,\alpha)} \\ \sum_{r=0}^{1} w_{i,0,r}^{(\upsilon,\alpha)} & \sum_{r=0}^{1} w_{i,1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=0}^{1} w_{i,M-1,r}^{(\upsilon,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=0}^{M-1} w_{M-1,0,r}^{(\upsilon,\alpha)} & \sum_{r=0}^{M-1} w_{M-1,1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=0}^{M-1} w_{M-1,M-1,r}^{(\upsilon,\alpha)} \end{bmatrix}.$$

We calculate $w_{i,0,r}^{(\upsilon,\alpha)}, w_{i,0,r}^{(\upsilon,\alpha)}, w_{i,0,r}^{(\upsilon,\alpha)}$ for $\alpha = 2, M = 3,$ and $\upsilon = 2$:

According equation (3.36), we can write

$$w_{i,j,r}^{(\upsilon,\alpha)} = w_{0,0,0}^{(\upsilon,\alpha)} = b_{0,0}^{(\upsilon,\alpha)}c_{0,0}^{(\upsilon,\alpha)}.$$

Based on equation (3.33) for $b_{0,0}^{(\upsilon,\alpha)}$, we also have

$$b_{0,0}^{(\upsilon,\alpha)} = b_{0,0}^{(\upsilon,\alpha)} = \binom{0}{0}\frac{\Gamma(1)}{\Gamma(3)}\beta_0 = \frac{1}{2},$$

where is the first Bernoulli’s number.

We calculate $c_{0,0}^{(\upsilon,\alpha)}$ from equation (3.34)

$$c_{r,j}^{(\upsilon,\alpha)} = c_{0,0}^{(\upsilon,\alpha)} = \frac{1}{\int_0^1 \beta_0^\alpha(x).\beta_0^\alpha(x).x^{\alpha-1}dx}\int_0^1 x^2.\beta_0^\alpha(x).x^{\alpha-1}dx = \frac{1}{\int_0^1 xdx}\int_0^1 x^3 dx = 2\times\frac{1}{4} = \frac{1}{2},$$

Therefore, $w_{0,0,0}^{(\upsilon,\alpha)} = \frac{1}{4}$.

Similarly, we have for $w_{i,j,r}^{(\upsilon,\alpha)} = w_{0,1,0}^{(\upsilon,\alpha)} = b_{0,0}^{(\upsilon,\alpha)}c_{0,1}^{(\upsilon,\alpha)}$:

$$b_{i,r}^{(\upsilon,\alpha)} = b_{0,0}^{(\upsilon,\alpha)} = \binom{0}{0}\frac{\Gamma(1)}{\Gamma(3)}\beta_0 = \frac{1}{2},$$

$$c_{r,j}^{(\upsilon,\alpha)} = c_{0,2}^{(\upsilon,\alpha)} = \frac{1}{\int_0^1 \beta_2^\alpha(x).\beta_2^\alpha(x).x^{\alpha-1}dx}\int_0^1 x^2.\beta_2^\alpha(x).x^{\alpha-1}dx = \frac{1}{\int_0^1 (x^4-x^2+\frac{1}{6})^2.xdx}\int_0^1 x^3(x^4-x^2+\frac{1}{6})dx = \frac{1}{360}\times 0 = 0.$$

So, $w_{0,1,0}^{(\upsilon,\alpha)} = \frac{1}{2}\times 0 = 0$.

The other components of this matrix are calculated in the same way. In this section, we evaluate the fractional integral operational matrix of Bernoulli fractional-order wavelets:

$$I^\upsilon\Psi^\alpha(x) = I^\upsilon\Theta B^\alpha(x) = \Theta I^\upsilon B^\alpha(x) \simeq \Theta F^{(\upsilon,\alpha)}B^\alpha(x), \tag{38}$$

From equations (3.37) and (3.38), it is concluded that

$$P^{(\upsilon,\alpha)}\Psi^\alpha(x) = P^{(\upsilon,\alpha)}\Theta B^\alpha(x) \simeq \Theta F^{(\upsilon,\alpha)}B^\alpha(x).$$

Therefore, fractional integral operational matrix of FBWs is acquired as follow

$$P^{(\upsilon,\alpha)} \simeq \Theta F^{(\upsilon,\alpha)}\Theta^{-1} \tag{39}$$

3.5. Fractional Derivative Operational Matrix of Bernoulli Fractional-Order Wavelets

In this section, we apprise the fractional derivative operational matrix of order υ. At first, we describe the following lemma.

Lemma 3.6 Suppose that $\beta_i^\alpha(x)$ is a Bernoulli fractional-order function, then

$$D^\nu\beta_i^\alpha(x) = 0, \quad i = 0,1,\dots,\left\lceil\frac{\upsilon}{\alpha}\right\rceil - 1,\ \nu > 0,$$

Proof. This claim can be proved by properties of Caputo fractional derivative and equation (2.9).

Riemann-liouvill fractional derivative of vector $B^\alpha(x)$ in equation (3.22) is described by

$$D^\nu B^\alpha(x) \simeq G^{(\nu,\alpha)}B^\alpha(x) \tag{40}$$

where $G^{(\nu,\alpha)}$ is fractional derivative operational matrix of order $M\times M$.

We find by using equation (2.9) and properties of Caputo fractional derivative for $i = \left\lceil\frac{\upsilon}{\alpha},\dots,M-1\right\rceil$ that

$$D^\nu\beta_i^\alpha(x) = D^\nu\left(\sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}x^{\alpha r}\right) = \sum_{r=0}^{i}\binom{i}{r}\beta_{i-r}D^\nu x^{\alpha r} = \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\binom{i}{r}\beta_{i-r}\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-\upsilon)}x^{\alpha r-\nu} == \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\eta_{i,r}^{(\nu,\alpha)}x^{\alpha r-\nu} \tag{41}$$

where

$$\eta_{i,r}^{(\nu,\alpha)} = \binom{i}{r}\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-\upsilon)}\,\beta_{i-r}.$$

Imagine that we can expand $x^{\alpha r+\nu}$ by M sentence of Bernoulli fractional-order functions as follow

$$x^{\alpha r-\nu} \simeq \sum_{j=0}^{M-1} k_{r,j}^{(\upsilon,\alpha)}\,\beta_j^{\alpha}(x). \tag{42}$$

Applying equations (3.41) and (3.42), we find out

$$D^{\nu}\beta_i^{\alpha}(x) \simeq \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\eta_{i,r}^{(\upsilon,\alpha)}\sum_{j=0}^{i} k_{r,j}^{(\upsilon,\alpha)}\,\beta_j^{\alpha}(x) = \sum_{j=0}^{M-1}\left(\sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\theta_{i,j,r}^{(\upsilon,\alpha)}\right)\beta_j^{\alpha}(x), \tag{43}$$

$$\theta_{i,j,r}^{(\upsilon,\alpha)} = \eta_{i,r}^{(\upsilon,\alpha)}k_{r,j}^{(\upsilon,\alpha)},$$

If we rewrite the equation (3.43) as a vector, we obtain

$$D^{\nu}\beta_i^{\alpha}(x) \simeq \left[\sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\theta_{i,0,r}^{(\upsilon,\alpha)},\sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\theta_{i,1,r}^{(\upsilon,\alpha)},\dots,\sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\theta_{i,M-1,r}^{(\upsilon,\alpha)}\right]B^{\alpha}(x),\ i=\left\lceil\frac{\upsilon}{\alpha}\right\rceil,\dots,M-1. \tag{44}$$

Hence, we have

$$G^{(\upsilon,\alpha)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \theta^{(\upsilon,\alpha)}_{\left\lceil\frac{\upsilon}{\alpha}\right\rceil,0,\left\lceil\frac{\upsilon}{\alpha}\right\rceil} & \theta^{(\upsilon,\alpha)}_{\left\lceil\frac{\upsilon}{\alpha}\right\rceil,1,\left\lceil\frac{\upsilon}{\alpha}\right\rceil} & \cdots & \theta^{(\upsilon,\alpha)}_{\left\lceil\frac{\upsilon}{\alpha}\right\rceil,M-1,\left\lceil\frac{\upsilon}{\alpha}\right\rceil} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\theta_{i,0,r}^{(\upsilon,\alpha)} & \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\theta_{i,1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{i}\theta_{i,M-1,r}^{(\upsilon,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{M-1}\theta_{M-1,0,r}^{(\upsilon,\alpha)} & \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{M-1}\theta_{M-1,1,r}^{(\upsilon,\alpha)} & \cdots & \sum_{r=\left\lceil\frac{\upsilon}{\alpha}\right\rceil}^{M-1}\theta_{M-1,M-1,r}^{(\upsilon,\alpha)} \end{bmatrix}.$$

In this section, we obtain the fractional integral operational matrix of Bernoulli fractional-order wavelets:

$$D^{\nu}\Psi^{\alpha}(x) \simeq D^{(\nu,\alpha)}\Psi^{\alpha}(x), \tag{45}$$

where $D^{(\nu,\alpha)}$ is called the fractional integral operational matrix of Bernoulli fractional-order wavelets.

Using equations (3.26) and (3.40), we find that

$$D^{\nu}\Psi^{\alpha}(x) = D^{\nu}\Theta B^{\alpha}(x) = \Theta D^{\nu}B^{\alpha}(x) \simeq \Theta G^{(\upsilon,\alpha)}B^{\alpha}(x), \tag{46}$$

From (3.45) and (3.46), the following relations can be concluded

$$D^{(\nu,\alpha)}\Psi^{\alpha}(x) = D^{(\nu,\alpha)}\Theta B^{\alpha}(x) \simeq \Theta G^{(\upsilon,\alpha)}B^{\alpha}(x).$$

Consequently, fractional derivative operational matrix of FBWs obtain as follow

$$D^{(\nu,\alpha)} \simeq \Theta G^{(\upsilon,\alpha)}\Theta^{-1}. \tag{47}$$

3.7. The upper limit of error for the fractional integral operational matrix of fractional-order Bernoulli wavelets

In this section, we obtain an upper error bound for the operational matrix of the fractional integrals $P^{(\nu,\alpha)}$and $F^{(\nu,\alpha)}$. Besides, we show that the error vectors $E_I^{(\nu)}$and $\tilde{E}_I^{(\nu)}$ approach zero when the number of Bernoulli fractional-order functions increases. To find these errors, we repeat the following theorems.

Theorem 3.8 Imagine $f \in L^2[0,1]$. It is possible to write f by infinitive series of Bernoulli fractional-order wavelets and uniformly convergent series as follow

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}^{\alpha}(x).$$

Since the reduced series of Bernoulli fractional-order wavelets is an approximate solution of a system, the error function $E(x)$ for $f(x)$ exists as follow:

$$E(x) = \left| f(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}^{\alpha}(x) \right|$$

By placing $x = x_j \in [0,1]$, we can determine the absolute value of the error in x_j.

The following theorem gives an error bound for the approximate solution by using series of $FBWs$. Before that, we have to provide the following definition.

Taylor's original formula [32]. Assume that $D^{i\alpha} f(x) \in (0,1]$ for $i = 1,2, \dots, m$, so we have:

$$f(t) = \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha} f(\xi), \tag{48}$$

where $0 < \xi \le t, \forall t \in (0,1]$. Also, we have

$$\left| f(x) - \sum_{i=0}^{m-1} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) \right| \le M_\alpha \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}, \tag{49}$$

Where $sup_{\xi \in (0,1]} |D^{m\alpha} f(\xi)| \le M_\alpha$. When $\alpha = 1$, the original Taylor formula is reduced to the Taylor Classic Formula.

Theorem 3.9. Suppose that $D^{i\alpha} f(x) \in (0,1]$ for $i = 1,2, \dots, m$, $(2M+1)\alpha \ge 1, (\hat{m} = 2^{k-1}M)$ and $Y_M^\alpha = span\{\beta_0^\alpha(x), \beta_1^\alpha(x), \dots, \beta_{M-1(x)}^\alpha\}$. If $f_M(x) = A^T B^\alpha(x)$ is the best approximation derived from Y_M^α on the interval $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$, then approximate solution error bound of $f_{\hat{m}}(x)$ can be obtained by $FBWs$ series on [0,1] as follow [6]

$$\|f - f_{\hat{m}}\| \le \frac{sup_{x \in [0,1]} |D^{M\alpha} f(x)|}{\Gamma(M\alpha+1)\sqrt{(2M+1)\alpha}} \tag{50}$$

Proof. We define

$$f_1(x) = \sum_{i=0}^{M-1} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+).$$

Based on the above definition, we get

$$|f(x) - f_1(x)| \le \frac{x^{M\alpha}}{\Gamma(M\alpha+1)} sup_{x \in I_{k,n}} |D^{M\alpha} f(x)|$$

where

$$I_{k,n} = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right].$$

Since $f_M(x) = A^T B^\alpha(x)$, the best approximation derived from Y_M^α on the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ and $\sum_{i=0}^{M-1} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) \in Y_M^\alpha$, thus

$$\|f - f_{\widehat{m}}\|_{L^2[0,1]}^2 = \|f - C^T\Psi^\alpha\|_{L^2[0,1]}^2 = \sum_{n=1}^{2^{k-1}} \|f - A^T B^\alpha\|_{L^2\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]}^2 \le \sum_{n=1}^{2^{k-1}} \|f - f_1\|_{L^2\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]}^2 \le \sum_{n=1}^{2^{k-1}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left[\frac{x^{M\alpha}}{\Gamma(M\alpha+1)} sup_{x\in I_{k,n}} |D^{M\alpha} f(x)|\right]^2 x^{\alpha-1} dx \le \int_0^1 \left[\frac{x^{M\alpha}}{\Gamma(M\alpha+1)} sup_{x\in I_{k,n}} |D^{M\alpha} f(x)|\right]^2 x^{\alpha-1} dx \le \frac{1}{\Gamma(M\alpha+1)^2 (2M+1)\alpha} (sup_{x\in[0,1]} |D^{M\alpha} f(x)|)^2$$

The proof is complete if we take the second root.

The last theorem proves that the approximations of Bernoulli fractional-order wavelets $f(x)$ is convergent. Now, we try to find the upper bound of $P^{(\nu,\alpha)}$. Furthermore, we show that the error vector of $E_I^{(\nu)}$ approaches zero when the number of FBws increases. At first, we explain the following theorems.

Theorem 3.10. Suppose Y is a subspace of the Hilbert space H so that $dimY < \infty$ and $y_1, y_2, \dots, y_n$ is a basis for Y. Also, imagine z is an arbitrary member of H and y^* is the best approximation of z derived from Y. Based on [13], we have

$$\|y - y^*\|_2^2 = \frac{G(z, y_1, y_2, \dots, y_n)}{G(y_1, y_2, \dots, y_n)},$$

where

$$G(z, y_1, y_2, \dots, y_n) = \begin{vmatrix} < z, y_1 > & < z, y_1 > & \dots & < z, y_n > \\ < y_1, z > & < y_1, y_1 > & \dots & < y_1, y_n > \\ \vdots & \vdots & \vdots & \vdots \\ < y_n, z > & < y_n, y_1 > & \dots & < y_n, y_n > \end{vmatrix}$$

Theorem 3.11. Assume that $g \in L^2[0,1]$ is approximated by $g_M(x)$ as follow

$$g(x) \simeq g_M(x) = \sum_{i=0}^{M-1} a_i \beta_i^\alpha = A^T B^\alpha(x).$$

Remember that A^T and $B^\alpha(x)$ were defined in equation (3.22). By considering

$$L_M(g) = \int_0^1 [g(x) - g_M(x)]^2 \, dx,$$

we obtain

$$\lim_{M\to\infty} L_M(g) = 0.$$

The error vector $E_I^{(\nu)}$ of the operational matrix $P^{(\nu,\alpha)}$ can be calculated as follow

$$E_I^{(\nu)} = P^{(\nu,\alpha)}\Psi^\alpha - I^\alpha\Psi^\alpha, \quad E_I^{(\nu)} = \begin{bmatrix} e_{I0} \\ e_{I1} \\ \vdots \\ e_{I2^{k-1}(M-1)} \end{bmatrix} \tag{51}$$

From equation (3.43) by the assumption $x^{\alpha r+\upsilon}$, we conclude

$$x^{\alpha r+v} \simeq \sum_{j=0}^{M-1} c_{r,j}^{(\nu,\alpha)} \beta_j^{\alpha}(x), \tag{52}$$

where $c_{r,j}^{(\nu,\alpha)}$ was calculated with the best approximation. Using theorem (3.10), we find

$$\left\| x^{\alpha r+v} - \sum_{j=0}^{M-1} c_{r,j}^{(\nu,\alpha)} \beta_j^{\alpha}(x) \right\|_2 = \left(\frac{G(x^{\alpha r+v}, \beta_0^{\alpha}(x), \beta_1^{\alpha}(x), \dots, \beta_{M-1}^{\alpha}(x))}{G(\beta_0^{\alpha}(x), \beta_1^{\alpha}(x), \dots, \beta_{M-1}^{\alpha}(x))} \right)^{\frac{1}{2}} \tag{53}$$

Based on (3.32)-(3.35) for $0 \le i \le M-1$, we get

$$\|\tilde{e}_{Ii}\|_2 = \left\| I^{v} \beta_i^{\alpha}(x) - \sum_{j=0}^{M-1} \left(\sum_{r=0}^{i} w_{i,j,r}^{(\nu,\alpha)} \right) \beta_j^{\alpha}(x) \right\|_2 \le \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1+v)} \beta_{i-r} \left\| x^{\alpha r+v} - \sum_{j=0}^{M-1} c_{r,j}^{(\nu,\alpha)} \beta_j^{\alpha}(x) \right\|_2 \le \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1+v)} \beta_{i-r} \left(\frac{G\left(x^{\alpha r+v}, \beta_0^{\alpha}(x), \beta_1^{\alpha}(x), \dots, \beta_{M-1}^{\alpha}(x)\right)}{G\left(\beta_0^{\alpha}(x), \beta_1^{\alpha}(x), \dots, \beta_{M-1}^{\alpha}(x)\right)} \right)^{\frac{1}{2}},$$

Now, we can find error vector $E_I^{(\nu)}$ of fractional integral operational matrix of Bernoulli fractional-order wavelets. Using equations (3.26) and (3.39), we find

$$E_I^{(\nu)} = P^{(\nu,\alpha)} \Psi^{\alpha} - I^{v} \Psi^{\alpha} = \Theta F^{(\nu,\alpha)} \Theta^{-1} \Theta B^{\alpha} - I^{\alpha} \Theta B^{\alpha} = \Theta F^{(\nu,\alpha)} B^{\alpha} - \Theta I^{v} B^{\alpha} = \Theta \tilde{E}_I^{(v)}.$$

Therefore, we obtain

$$E_I^{(\nu)} = \Theta \tilde{E}_I^{(v)}. \tag{55}$$

According to the above discussion and Theorem 2, we can conclude that the bases of vectors $E_I^{(\nu)}$ and $\tilde{E}_I^{(v)}$ approach zero when the number of Bernoulli fractional-order functions increases.

4. The wavelets method for solving differential equations of Fokker-Planck-Kolmogorov fractional order

For the approximate solution of the Fokker-Planck-Kolmogorov fractional differential equation, the Bernoulli wavelet method is explained as follows:

$$D_t^{\alpha} u - \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + (\beta - 2\sigma^2) x \frac{\partial u}{\partial x} + (\beta - \sigma^2) u = R(x,t) \tag{56}$$

Initial conditions:

$$u(0,x) = f_0(x), \qquad u_t(0,x) = f_1(x), \quad 0 \le x \le 1$$

Boundary conditions:

$$u(t,0) = g_0(t), \quad u_t(t,1) = g_0(t), \quad 0 \le t \le 1$$

$R(x,t)$ Is the right-side function of the equation given for each equation.
Consider:

$$\frac{\partial^4 u(t,x)}{\partial x^2 \, \partial t^2} \approx \Psi_{m\times m}^{T}(x) C_{m\times m} \Psi_{m\times m}(t) \tag{57}$$

By twice integrating with t from both sides of equation (57) we have:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx f_0''(x) + t f_1''(x) + \Psi_{m\times m}^{T}(x) C_{m\times m} (I^2 \Psi_{m\times m}(t)) \tag{58}$$

By twice integrating with x from both sides of equation (58) we have:

$$\frac{\partial\, u(t,x)}{\partial x} = \frac{\partial\, u(t,x)}{\partial x}|_{x=0} + f_0'(x) + f_0'(0) + t\big(f_1'(x) - f_1'(0)\big) + (IH^T_{m\times m}(x))^T \Psi_{m\times m}(I^2\Psi_{m\times m}(t)) \quad (59)$$

$$u(t,x) \approx u(t,0) + x\frac{\partial\, u(t,x)}{\partial x}|_{x=0} + \big(f_0(x) - f_0(0) - xf_0'(0)\big) + t\big(f_1(x) - f_1(0) - xf_1'(0)\big) + (I^2\Psi_{m\times m}(x))^T C_{m\times m}(I^2\Psi_{m\times m}(t)) \quad (60)$$

Now by applying the boundary conditions and putting x = 1, we will have:

$$u(t,1) \approx u(t,0) + x\frac{\partial\, u(t,x)}{\partial x}|_{x=0} + \big(f_0(1) - f_0(0) - xf_0'(0)\big) + t\big(f_1(1) - f_1(0) - f_1'(0)\big) + (I^2\Psi_{m\times m}(1))^T C_{m\times m}(I^2\Psi_{m\times m}(t)) \quad (61)$$

Therefor:

$$\frac{\partial\, u(t,x)}{\partial x}|_{x=0} \approx g_1(t) - g_0(t) - \big(f_0(1) - f_0(0) - f_0'(0)\big) - t\big(f_1(1) - f_1(0) - f_1'(0)\big) - \big(I^2\Psi_{m\times m}(1)\big)^T C_{m\times m}(I^2\Psi_{m\times m}(t)) = K(t) \quad (62)$$

Now by placing K(t) in Equation (60) we have:

$$u(t,x) \approx g_0(t) + xK(t) + \big(f_0(x) - f_0(0) - xf_0'(0)\big) + t\big(f_1(x) - f_1(0) - xf_1'(0)\big) + \big(I^2\Psi_{m\times m}(x)\big)^T C_{m\times m}(I^2\Psi_{m\times m}(t)) \quad (63)$$

Now we need the fraction derivative u (t, x) according to Equation (56). From Equation (62) we derive the order fraction α with respect to t:

$$D_t^\alpha u(t,x) \approx D_t^\alpha g_0(t) + xD_t^\alpha K(t) + \big(I^2\Psi_{m\times m}(x)\big)^T C_{m\times m}(I^{2-\alpha}\Psi_{m\times m}(t)) \quad (64)$$

And we will have:

$$D_t^\alpha K(t) = D_t^\alpha g_1(t) - D_t^\alpha g_0(t) - \big(I^2\Psi_{m\times m}(1)\big)^T C_{m\times m}(I^{2-\alpha}\Psi_{m\times m}(t)) \quad (65)$$

Now convert all approximations (≈) to equals (=), and place equations (58), (59), (63) and (65) in Equation (45), the following linear equation is obtained:

$$D_t^\alpha g_0(t_j) + x_i D_t^\alpha K(t_j) + \big(I^2\Psi_{m\times m}(x_i)\big)^T C_{m\times m}(I^{2-\alpha}\Psi_{m\times m}(t)) - \frac{1}{2}\sigma^2 {x_i}^2 (f_0''(x_i) + t_j f_1''(x_i) + {\Psi^T}_{m\times m}(x_i)C_{m\times m}({\Psi^2}_{m\times m}(t_j))) + (\beta - 2\sigma^2)x_i(K(t_j) + f_0'(x_i) - f_0'(0) + t_j\big(f_1'(x_i) - f_1'(0)\big) + (I\Psi_{\,m\times m}(x_i))^T C_{m\times m}({\Psi^2}_{m\times m}(t_j))) + (\beta - \sigma^2)(g_0(t_j) + x_i K(t_j) + \big(f_0(x_i) - f_0(0) - x_i f_0'(0)\big) + t_j\big(f_1(x_i) - f_1(0) - x_i f_1'(0)\big) + \big(I^2\Psi_{m\times m}(x_i)\big)^T C_{m\times m}(I^2\Psi_{m\times m}(t_j)) = R(x_i, t_j) \quad (66)$$

5- Solving numerical examples

Numerical solutions and errors are calculated, evaluated, and provided in tables after evaluating certain numerical instances with conditions of varying initial values. The MATLAB software is used to solve all of the examples.

Example 1: In equation 1, by placing, $\alpha = 1.1\ , \beta = 1\ , \sigma = 0.2, m = 3\ , k = 2\ ,$

Initial conditions:

$$u(0,x) = 0, \qquad u_t(0,x) = 0, \quad 0 \le x \le 1$$

Boundary conditions:

$$u(t,0) = 0, \quad u_t(t,1) = 0, \quad 0 \le t \le 1$$

The right-side functions of the equation:

$$R(t,x)=\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}+\frac{1}{2}(\sigma x t\pi)^2+(\beta-\sigma^2)t^2\right)sin(\pi x)+(\beta-2(\sigma)^2)t^2x\pi\, cos(\pi x)$$

The accurate answer of this equation in example (1) is $u(t,x)=t^2\sin(\pi x)$. Example (1) is solved by the Bernoulli wavelet method for , $\alpha=1.1$, $\beta=1$, $\sigma=0.2, m=3$, $k=2$and its error is presented in Table 1.

Table 1: example 1 error,by placing, $\alpha=1.1$, $\beta=1$, $\sigma=0.2, m=3$, $k=2$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.11*10^{-10}$	$2.18*10^{-10}$	$5.10*10^{-11}$	$6.32*10^{-11}$	$2.08*\mathbf{10^{-11}}$
(3.13,3.13)	$4.02*10^{-8}$	$1.84*10^{-8}$	$2.34*10^{-8}$	$5.47*10^{-8}$	$4.06*\mathbf{10^{-8}}$
(5.13,5.13)	$3.64*10^{-6}$	$1.52*10^{-6}$	$1.82*10^{-6}$	$4.19*10^{-6}$	$5.28*\mathbf{10^{-6}}$
(7.13,7.13)	$2.74*10^{-5}$	$2.01*10^{-5}$	$3.01*10^{-5}$	$2.94*10^{-5}$	$3.64*\mathbf{10^{-5}}$
(9.13,9.13)	$3.19*10^{-4}$	$3.84*10^{-4}$	$4.62*10^{-4}$	$6.38*10^{-4}$	$4.15*\mathbf{10^{-4}}$
(11.13,11.13)	$1.59*10^{-7}$	$1.04*10^{-7}$	$2.14*10^{-7}$	$3.55*10^{-7}$	$2.07*\mathbf{10^{-7}}$

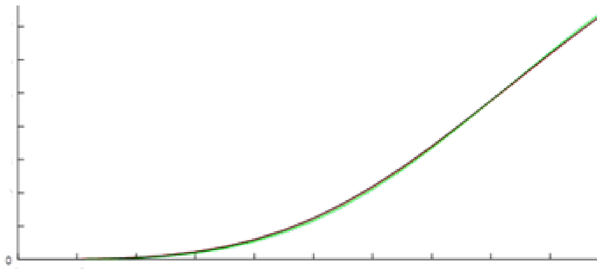

Figure 2: Relation of B and error for example 1 for $\alpha=1.1$, $\beta=1$, $\sigma=0.2, m=3$, $k=2$

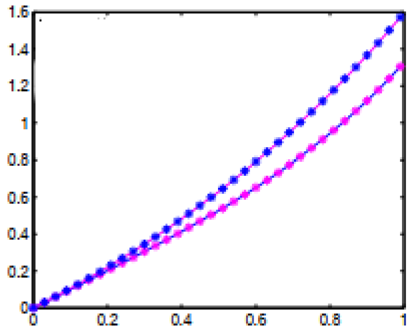

Figure 3: Approximate and exact solution, respectively for example 1 for $\alpha=1.1$, $\beta=1$, $\sigma=0.2, m=3$, $k=2$

The method of numerical solution for $\alpha=1.1$, $\beta=1$, $\sigma=0.2, m=3$, $k=2$is presented in Table (2).

Table 2. the numerical solution of example 1 py placing , $\alpha=1.1$, $\beta=1$, $\sigma=0.2, m=3$, $k=2$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$2.55*10^{-4}$	$2.54*10^{-4}$	$2.54*10^{-4}$	$2.53*10^{-4}$	$2.53*\mathbf{10^{-4}}$
(3.13,3.13)	$3.84*10^{-3}$	$3.83*10^{-3}$	$3.83*10^{-3}$	$3.82*10^{-3}$	$3.82*\mathbf{10^{-3}}$
(5.13,5.13)	$1.64*10^{-2}$	$1.63*10^{-2}$	$1.63*10^{-2}$	$1.62*10^{-2}$	$1.62*\mathbf{10^{-2}}$
(7.13,7.13)	$1.79*10^{-2}$	$1.78*10^{-2}$	$1.78*10^{-2}$	$1.76*10^{-2}$	$1.76*\mathbf{10^{-2}}$
(9.13,9.13)	$2.13*10^{-2}$	$2.12*10^{-2}$	$2.12*10^{-2}$	$2.11*10^{-2}$	$2.11*\mathbf{10^{-2}}$
(11.13,11.13)	$3.18*10^{-2}$	$3.18*10^{-2}$	$3.17*10^{-2}$	$3.17*10^{-2}$	$3.16*\mathbf{10^{-2}}$

Example 2: the numerical solution of the following equation:

In equation (1), by placing $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$,
Initial conditions:

$$u(0,x) = 0, \quad u_t(0,x) = 0, \quad 0 \le x \le 1$$

Boundary conditions:

$$u(t,0) = t^3, \quad u_t(t,1) = et^3, \quad 0 \le t \le 1$$

The right-side functions of the equation:

$$R(t,x) = \left(\frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} - \frac{1}{2}\sigma^2 x^2 t^3 + (\beta - 2\sigma^2) x t^3 + (\beta - \sigma^2) t^3\right) e^x$$

The accurate response to this equation in example (2) is$u(t,x) = t^3 e^x$. Example (2) is solved by the Bernoulli wavelet method for $\boldsymbol{\alpha} = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$ and its error has been shown in Table (3).

Table 3. The error of example 2, by placing $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$452*10^{-8}$	$5.41*10^{-8}$	$5.19*10^{-8}$	$4.99*10^{-8}$	$5.55*\mathbf{10^{-8}}$
(3.13,3.13)	$3.74*10^{-6}$	$4.12*10^{-6}$	$4.63*10^{-7}$	$4.47*10^{-6}$	$4.34*\mathbf{10^{-6}}$
(5.13,5.13)	$2.91*10^{-9}$	$3.04*10^{-9}$	$5.33*10^{-8}$	$3.12*10^{-8}$	$3.17*\mathbf{10^{-8}}$
(7.13,7.13)	$2.74*10^{-4}$	$2.14*10^{-4}$	$3.55*10^{-4}$	$2.39*10^{-4}$	$2.37*\mathbf{10^{-4}}$
(9.13,9.13)	$3.21*10^{-4}$	$3.01*10^{-4}$	$4.12*10^{-4}$	$4.19*10^{-4}$	$4.20*\mathbf{10^{-4}}$
(11.13,11.13)	$1.18*10^{-3}$	$1.44*10^{-3}$	$2.01*10^{-3}$	$1.67*10^{-3}$	$1.33*\mathbf{10^{-3}}$

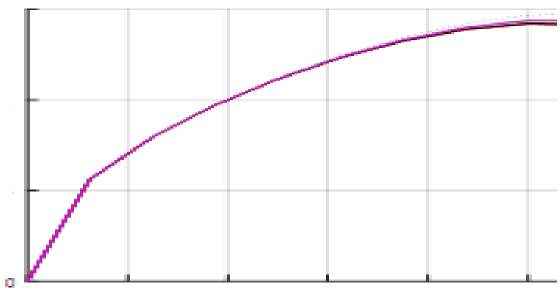

Figure 4: Relation of *B* and error for example2 for, $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

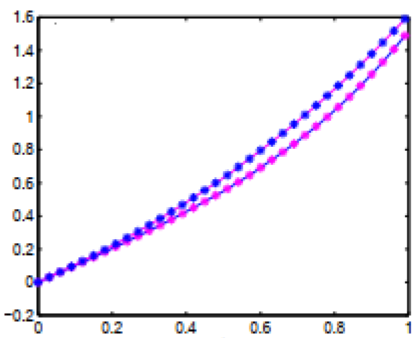

Figure 5: Approximate and exact solution, respectively example2 for, $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

In Table (4), the numerical solution method for $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2, m = 3}$, $\boldsymbol{k = 2}$ has been shown.

Table 4: the numerical solution of example 2 by placing $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2, m = 3$, $k = 2$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.55*10^{-4}$	$5.54*10^{-4}$	$5.53*10^{-4}$	$5.53*10^{-4}$	$5.52*\mathbf{10^{-4}}$
(3.13,3.13)	$2.84*10^{-3}$	$2.83*10^{-3}$	$2.82*10^{-3}$	$2.82*10^{-3}$	$2.81*\mathbf{10^{-3}}$
(5.13,5.13)	$1.55*10^{-2}$	$1.54*10^{-2}$	$1.53*10^{-2}$	$1.53*10^{-2}$	$1.51*\mathbf{10^{-2}}$
(7.13,7.13)	$3.28*10^{-2}$	$3.27*10^{-2}$	$3.26*10^{-2}$	$3.25*10^{-2}$	$3.25*\mathbf{10^{-2}}$
(9.13,9.13)	$1.13*10^{-2}$	$1.12*10^{-2}$	$1.11*10^{-2}$	$1.11*10^{-2}$	$1.10*\mathbf{10^{-2}}$
(11.13,11.13)	$7.18*10^{-2}$	$7.17*10^{-2}$	$7.16*10^{-2}$	$3.16*10^{-2}$	$3.15*\mathbf{10^{-2}}$

Example 3: the numerical solution of the following equation:

In equation (1), by placing , $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2, m = 3$, $k = 2$
Initial conditions:

$$\mathrm{u}(0,x) = 0, \qquad u_t(0,x) = 0, \quad 0 \le x \le 1$$

Boundary conditions:

$$\mathrm{u}(t,0) = 0, \quad u_t(t,1) = t^3 sin^2 x \,, \quad 0 \le t \le 1$$

The right-side functions of the equation:

$$R(t,x) = (\frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + (\beta - \sigma^2)t^3) sin^2 x - \sigma^2 x^2 t^3 cos\,(2x) + (\beta - \sigma^2) x t^3 sin\,(2x)$$

The accurate response to this equation in example (3) is$u(t,x) = t^3 sin^2 x$. Example (3) is solved by the Bernoulli wavelet method for $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2, m = 3$, $k = 2$ and its error has been shown in Table (5).

Table5. The error of example 3, by placing $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2, m = 3$, $k = 2$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$352*10^{-8}$	$5.85*10^{-8}$	$5.19*10^{-8}$	$4.84*10^{-8}$	$5.60*\mathbf{10^{-8}}$
(3.13,3.13)	$3.18*10^{-6}$	$4.24*10^{-6}$	$4.63*10^{-7}$	$4.55*10^{-6}$	$4.23*\mathbf{10^{-6}}$
(5.13,5.13)	$2.91*10^{-9}$	$3.11*10^{-9}$	$5.02*10^{-8}$	$3.11*10^{-8}$	$3.27*\mathbf{10^{-8}}$
(7.13,7.13)	$2.45*10^{-4}$	$2.20*10^{-4}$	$3.44*10^{-4}$	$2.44*10^{-4}$	$2.21*\mathbf{10^{-4}}$
(9.13,9.13)	$3.39*10^{-4}$	$3.21*10^{-4}$	$4.19*10^{-4}$	$4.45*10^{-4}$	$4.37*\mathbf{10^{-4}}$
(11.13,11.13)	$1.71*10^{-3}$	$1.59*10^{-3}$	$2.11*10^{-3}$	$1.55*10^{-3}$	$1.47*\mathbf{10^{-3}}$

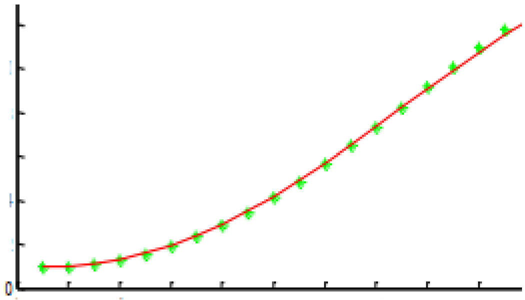

Figure 6: Relation of *B* and error for example3 for, $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2, m = 3$, $k = 2$

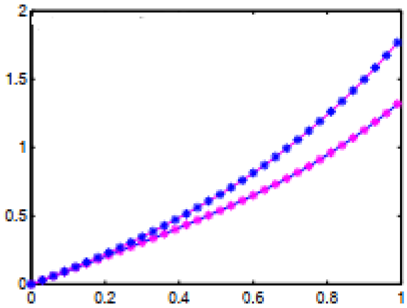

Figure 7: Approximate and exact solution, respectively for example3 for, $\alpha = 1.1$,$\beta = 0.5$,$\sigma = 0.2, m = 3$,$k = 2$

In Table (6), the numerical solution method for $\alpha = 1.1$,$\beta = 0.5$,$\sigma = 0.2, m = 3$,$k = 2$has been shown.

Table 6: the numerical solution of example 3by placing $\alpha = 1.1$,$\beta = 0.5$,$\sigma = 0.2, m = 3$,$k = 2$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.47*10^{-4}$	$5.42*10^{-4}$	$5.77*10^{-4}$	$5.78*10^{-4}$	$5.44*\mathbf{10^{-4}}$
(3.13,3.13)	$2.75*10^{-3}$	$2.76*10^{-3}$	$2.75*10^{-3}$	$2.74*10^{-3}$	$2.79*\mathbf{10^{-3}}$
(5.13,5.13)	$1.62*10^{-2}$	$1.67*10^{-2}$	$1.54*10^{-2}$	$1.72*10^{-2}$	$1.47*\mathbf{10^{-2}}$
(7.13,7.13)	$3.31*10^{-2}$	$3.33*10^{-2}$	$3.36*10^{-2}$	$3.36*10^{-2}$	$3.33*\mathbf{10^{-2}}$
(9.13,9.13)	$1.19*10^{-2}$	$1.28*10^{-2}$	$1.01*10^{-2}$	$1.22*10^{-2}$	$1.19*\mathbf{10^{-2}}$
(11.13,11.13)	$7.63*10^{-2}$	$7.37*10^{-2}$	$7.42*10^{-2}$	$3.43*10^{-2}$	$3.01*\mathbf{10^{-2}}$

6. Discussion and conclusion

In this paper, we employ the Bernoulli wavelet method to solve the Fokker-Planck-Kolmogorov time fractional-order differential equations. Accordingly, it is essential to be familiar with fractional calculus and wavelets. So, we initially introduced the Bernoulli and Bernoulli fractional-order wavelets, the Bernoulli polynomial, and Bernoulli fractional-order functions. In the following, we continued by introducing fractional integrals and derivatives and described approximation of functions by Bernoulli fractional-order wavelets and functions. Then, we obtained the transformation matrix of the Bernoulli fractional-order wavelet to Bernoulli fractional-order functions. Finally, we specified the operational matrix of the fractional integral and derivative Bernoulli fractional-order wavelet and the upper error bound for the operational matrix of Riemann-Leouville fractional integral and operational matrix of fractional integral of Bernoulli fractional-order wavelet. After introducing the operational matrix of the Bernoulli wavelet fractional integral, we used it to solve the Fokker-Planck-Kolmogorov fractional differential equations. After numerically solving the equation, we analyzed the error between the exact answer and the approximate answer obtained from the numerical method

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