Journal of Algebra and Related Topics
Vol. 2, No 1, (2014), pp 1-14

PRIMAL STRONG CO-IDEALS IN SEMIRINGS

S. EBRAHIMI ATANI *, S. DOLATI PISH HESARI AND M. KHORAMDEL

Abstract. In this paper, we introduce the notion of primal strong co-ideals and give some results involving them. It is shown that subtractive strong co-ideals are intersection of all primal strong co-ideals that contain them. Also we prove that the representation of strong co-ideals as reduced intersections of primal strong co-ideals is unique.

1. Introduction

As a generalization of rings, semirings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. They play an important role in studying optimization theory, graph theory, theory of discrete event dynamical systems, generalized fuzzy computation, automata theory, coding theory, cryptography theory.

Primal ideals in a commutative ring with non-zero identity have been introduced and studied by Ladislas Fuchs in [10] and continued to primal ideals over semirings in [5]. Moreover, the theory of primal decomposition of ideals is studied extensively in [1]. This paper is concerned with generalizing some results of primal ideals and from commutative rings theory and commutative semiring theory to primal strong co-ideals in commutative semirings theory. We introduce the notion of primal strong co-ideals and give some result involving

MSC(2010):16Y60
Keywords: Prime strong co-ideals, primal strong co-ideals, subtractive strong co-ideals.
Received: 15 December 2013, Accepted: 17 February 2014.
*Corresponding author .

them such subtractive strong co-ideals are intersection of all primal strong co-ideals that contain them and give examples of primal strong co-ideals(subtractive irreducible strong co-ideals and primary strong co-ideals). Also Intersection of primal strong co-ideals are considered in Sec.3.

For the sake of completeness, we state some definitions and notations used throughout. A commutative semiring R is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b+c) = ab+ac$ for all $a, b, c \in R$, and there exists $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity.

Definition 1.1. Let R be a semiring.

(1) A non-empty subset I of R is called a *co-ideal*, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$. A co-ideal I in R is called strong co-ideal provided that $1 \in I$. (clearly, $0 \in I$ if and only if $I = R$) [2], [8], [11] and [12].

(2) A strong co-ideal I of a semiring R is called *subtractive* if $x, xy \in I$, then $y \in I$ [8].

(3) A proper strong co-ideal P of a semiring R is called *prime* if $x + y \in P$, then $x \in P$ or $y \in P$ [2] and [8].

(4) A proper strong co-ideal I of a semiring R is said to be maximal if J is a strong co-ideal in R with $I \subseteq J$ and $I \neq J$, then $J = R$ [2] and [8].

(5) A proper co-ideal I of a semiring R is called *primary* if $a+b \in I$, then $a \in I$ or $b \in \mathrm{co}-\mathrm{rad}(I) = \{r \in R : \ nr \in I \ for \ some \ n \in \mathbb{N}\}$ [8].

(6) A strong co-ideal I of a semiring R is called a *partitioning strong co-ideal* (= Q-strong co-ideal) if there exists a subset Q of R such that

(1) $R = \bigcup\{qI : q \in Q\}$, where $qI = \{qt : t \in I\}$.

(2) If $q_1, q_2 \in Q$, then $(q_1I) \cap (q_2I) \neq \emptyset$ if and only if $q_1 = q_2$. [8]

If A and B are nonempty subsets of the semiring R, then we define $A + B = \{a + b : a \in A, b \in B\} \subseteq R$. Moreover, if I is a Q-strong co-ideal of R and $q, q' \in Q$, then $q + q' \in qI + q'I$; so $qI + q'I \neq \emptyset$.

Let $R/I = \{qI : q \in Q\}$. Define the binary operations $\oplus$ and $\odot$ on R/I as follows:

(1) $(q_1I) \oplus (q_2I) = q_3I$, where q_3 is the unique element in Q such that $(q_1I + q_2I) \subseteq q_3I$; and

(2) $(q_1I) \odot (q_2I) = q_3I$, where q_3 is the unique element in Q such that $(q_1q_2)I \subseteq q_3I$ (note that $q_1I = q_2I$ if and only if $q_1 = q_2$) [8].

We need the following propositions, proved in [2], [8] and [9], respectively.

Proposition 1.2. *(a) If I is a non-zero proper co-ideal of a semiring R, then I is contained in a maximal co-ideal of R.*

(b) If P is a maximal co-ideal of R, then P is a prime co-ideal of R.

(c) If P is a maximal co-ideal of R, then P is subtractive strong co-ideal of R and $R - P$ is an ideal of R.

(d) Let I be a Q-strong co-ideal in the semiring R. If $x \in R$, then there exists a unique $q \in Q$ such that $xI \subseteq qI$. In particular, $x = qa$ for some $a \in I$.

(e)If I is a Q-strong co-ideal in the semiring R, then $(R/I, \oplus, \odot)$ is a commutative semiring with identity.

(f) If I is a Q-strong co-ideal of a semiring R, then there exists a unique $q_e \in Q$ such that $I = q_eI = q_e^2I$.

(g)Let I be a Q-strong co-ideal of a semiring R, and let L be a subtractive strong co-ideal of R with $I \subseteq L$. Then $L/I = \{qI : q \in Q \cap L\}$ is a subtractive strong co-ideal of R/I.

(h) Let I be a Q-strong co-ideal of a semiring R, and let P be a subtractive strong co-ideal of R with $I \subseteq P$. Then P is a prime co-ideal of R if and only if P/I is a prime strong co-ideal of R/I.

(l)Let $I_1, I_2, I_3, ..., I_n$ be subtractive co-ideals of a semiring R such that at most two of the I_r are not prime. If I is a co-ideal of R such that $I \subseteq \bigcup_{i=1}^{r} I_i$, then $I \subseteq I_r$ for some r.

2. Primal strong co-ideals

The main purpose of this section is to investigate some properties of strong primal co-ideal.

Definition 2.1. (i) Let R be a semiring and I be a strong co-ideal of R. An element $r \in R$ is called *prime to* I if $a + r \in I$, then $a \in R$, that is $(I : r) = I$ $((I : r) = \{a \in R : a + r \in I\})$.

(ii) A strong co-ideal I of R is called primal if the set of all elements of R that are not prime to I forms a strong co-ideal.

Lemma 2.2. *Let I be proper strong co-ideal of a semiring R and P be the set of all elements of R that are not prime to I. Then*

(i) $I \subseteq P$;

(ii) If P is a strong co-ideal of R, then P is a prime strong co-ideal.

Proof. (i) Let $b \in I$. As $b + 0 \in I$ and $0 \notin I$, b is not prime to I (because I is proper). Hence $b \in P$. Therefore $I \subseteq P$.

(ii) Let $a, b \in R$ such that $a + b \in P$. Thus $a + b$ is not prime to I. So there exists $r \in R \setminus I$ such that $a + b + r \in I$. If $a \notin P$, then a is prime to I, hence $a + b + r \in I$ implies that $b + r \in I$. If $b \notin P$, then b

is prime to I. Therefore $r \in I$, a contradiction. Therefore $b \in P$. Thus P is prime. $\square$

Let I be a primal strong co-ideal of a semiring R and P be the set of all elements of R that are not prime to I. Then we say I is P-primal or Primal strong co-ideal with adjoint prime P.

Proposition 2.3. *Let Q be a P-primary strong co-ideal of a semiring R. Then Q is P-primal.*

Proof. We claim that the set of elements of R that are not prime to Q is equal to P. Assume that a is not prime to Q, hence $a + r \in Q$ for some $r \in R \setminus Q$. Thus $a \in P$, because Q is P-primary.

Let $a \in P$, we have two possibilities.

case 1: $a \in Q$. It is clear that a is not prime to Q.

case 2: $a \notin Q$. Then $a \in P$ implies that $na \in Q$ for some $n \in \mathbb{N}$. Therefore a is not prime to Q. $\square$

Lemma 2.4. *Let I be a subtractive strong co-ideal of a semiring R. Then $(I : x)$ is a subtractive strong co-ideal of R for each $x \in R$.*

Proof. Let $a \in (I : x)$. Then $a + x \in I$. Thus for each $r \in R$, $r + a + x \in I$. Hence $r + a \in (I : x)$ for each $r \in R$.
Let $a, b \in (I : x)$. Then $a+x \in I$ and $b+x \in I$. Thus $ab+x^2+ax+bx \in I$. Therefore

$$(ab + x)(a + 1)(x + 1)(b + 1) = c + ab + x^2 + ax + bx \in I.$$

where $c \in R$. As I is subtractive, $ab+x \in I$. Therefore $(I : x)$ is strong co-ideal of R. Clearly $(I : x)$ is subtractive. $\square$

Note that the condition "I is subtractive " is necessary in Lemma 2.4, as the following example shows.

Example 2.5. Assume that $R = \{0, 1, 2, 3, 4, 5\}$. Define

$$a + b = \begin{cases} 5 & \text{if} \quad a \neq 0,\ b \neq 0, a \neq b, \\ a & \text{if} \quad a = b, \\ b & \text{if} \quad a = 0, \\ a & \text{if} \quad b = 0. \end{cases}$$

and

$$a * b = \begin{cases} 0 & \text{if a=0 or b=0,} \\ 3 & \text{if a=b=2,} \\ b & \text{ if a=1,} \\ a & \text{if b=1,} \\ 5 & \text{Otherwise} \end{cases}$$

Then $(R,+,*)$ is easily checked to be a commutative semiring. An inspection will show that $I=\{1,4,5\}$ is a strong co-ideal of R, but $(I:3)=\{1,2,4,5\}$ is not a strong co-ideal of R since $2*2=3\notin(I:3)$.

Theorem 2.6. *Let I be a strong co-ideal of a semiring R such that $(I:a)$ is strong co-ideal. If I is irreducible, then I is primal.*

Proof. It suffices to show that if $a,b\in R$ are not prime to I, then ab is not prime to I. We claim that $(I:a)\bigcap(I:b)\subseteq(I:ab)$. Let $c\in(I:a)\bigcap(I:b)$, then $c+a\in I$ and $c+b\in I$. By hypothesis $(I:c)$ is a strong co-ideal of R. Since $a,b\in(I:c)$, $ab\in(I:c)$. Thus $c+ab\in I$ and hence $c\in(I:ab)$. Therefore $(I:a)\bigcap(I:b)\subseteq(I:ab)$. If ab is prime to I, then $(I:ab)=I$. As $I\subseteq(I:a)$ and $I\subseteq(I:b)$, $I\subseteq(I:a)\bigcap(I:b)$. Therefore $I=(I:a)\bigcap(I:b)$. Since I is irreducible, $(I:a)=I$ or $(I:b)=I$. Hence a or b is prime to I, a contradiction. Thus ab is not prime to I and so I is primal. □

Corollary 2.7. *Let I be subtractive irreducible strong co-ideal of a semiring R, then I is primal.*

Proof. it is clear by Lemma 2.4 and Theorem 2.6. □

Example 2.8. (i) Let T be the set of all non-negative integers. Define $a+b=\gcd(a,b)$ and $a\times b=\mathrm{lcm}(a,b)$,(take $0+0=0$ and $0\times 0=0$). Then $(T,+,\times)$ is easily checked to be a commutative semiring. Let $J=\{1\}$. Then it is clear that J is a strong co-ideal of T. An inspection will shows that the set of elements that are not prime to J is equal to $T\setminus\{0\}$. Since $T\setminus\{0\}$ is a strong co-ideal of T, J is primal. As $J=J_1\bigcap J_2$ where $J_1=\{1,2\}$ and $J_2=\{1,3\}$(J_1 and J_2 are strong co-ideal of T), J is not irreducible.

Also, Since $2+3\in J$ and $2,3\notin J=co-rad(J)$, J is not a primary strong co-ideal of T.

(ii) Let X={a,b,c} and $R=(P(X),\bigcup,\bigcap)$ a semiring, where P(X)= the set of all subsets of X. Let $I=\{X,\{a,b\}\}$. It is clear that I is a strong co-ideal of R. Let Q denotes the set of all elements of R that are not prime to I, then it is equal to $\{X,\{a,b\},\{b,c\},\{a,c\},\{b\},\{a\},\{c\}\}$. Since $\{a\}\bigcap\{b\}=\emptyset\notin Q$, Q is not a strong co-ideal of R. Hence I is not primal.

We need the following definition to push the theory further.

Definition 2.9. Let R be a semiring. We say that R is co-valuation semiring if its strong co-ideals are linearly ordered by inclusion.

Proposition 2.10. *Let I be a strong co-ideals of a co-valuation semiring R such that $(I:a)$ is strong co-ideal for each $a\in R$. Then I is primal with adjoint prime $P=\{r\in R:\ (I:r)\neq I\}$.*

Proof. Let $I = J \bigcap K$ for strong co-ideals J, K of R. As R is co-valuation, either $J \subseteq K$ or $K \subseteq J$. Hence $I = J$ or $I = K$. Therefore I is irreducible and so it is primal by Theorem 2.6. The other statement is clear. □

Example 2.11. Let $T = (\mathbb{Z}^+ \cup \{\infty\}, max, min)$. An inspection will show that the list of strong co-ideals of T are T, $I_n = \{k : k \geq n\}$. It is clear each T is co-valuation semiring and and its strong co-ideals are subtractive. Hence its strong co-ideals are primal.

The following proposition offer several characterization of P, for some P-primal strong co-ideal I of a semiring R.

Proposition 2.12. *Let I be P-primal and subtractive strong co-ideal of a semiring R. Then the following are hold.*
(i) If $ab \in P$ $(a, b \in R)$, then $a \in P$ and $b \in P$.
(ii) P is subtractive.
(iii) $R - P$ is a prime ideal of R.

Proof. (i) Let $a, b \in R$ such that $ab \in P$. Thus $ab + c \in I$ for some $c \in R \setminus I$. Therefore $(b+1)(a+c) = ab+c+a+bc \in I$. Since $b+1 \in I$ and I is subtractive, $a + c \in I$. As $c \notin I$, $a \in P$. Similarly, we can show $b \in P$.
(ii) It is clear from (i).
(iii) Let $a, b \in R - P$. Hence a, b are prime to I. We show $a + b$ is prime to I. Let $r \in R$ and $a+b+r \in I$. Since a is prime to I, $b+r \in I$. Therefore $r \in I$, because b is prime to I. Hence $a+b$ is prime to I and so $a + b \notin P$.
Let $a \in R - P$ and $r \in R$. We show that $ra \in R - P$. If $ra \in P$, then $r \in P$ and $a \in P$ by (ii), a contradiction. Thus $R - P$ is an ideal of R. We show $R - P$ is a prime ideal of R. Since $1 \in P$, $R - P$ is proper. Let $ab \in R - P$ and $a \notin R - P, b \notin R - P$. Hence $a, b \in P$. Thus $ab \in P$. This contradicts $ab \notin P$. Therefore $R - P$ is a prime ideal of R. □

By Proposition 1.2(c), If P is a maximal co-ideal of a semiring R, then P is subtractive strong co-ideal of R such that $R \setminus P$ is a prime ideal of R. By considering Proposition 2.12, we may suspect that if I is a subtractive primal strong co-ideal of R, then the adjoint prime P of I is maximal. However, the following example erase this possibility.

Example 2.13. Let R be the set of all non-negative integers. Define $a + b = \gcd(a, b)$ and $a \times b = \mathrm{lcm}(a, b)$,(take $0 + 0 = 0$ and $0 \times 0 = 0$). Then $(R, +, \times)$ is easily checked to be a commutative semiring. Let $I = \{2k+1 : k \in R\}$, then I is a prime strong co-ideal of R and so it is primal. An inspection shows that $R-I$ is an ideal of R. But I is not a maximal co-ideal of R, because I is properly is contained in $R-\{0\}$ and $R-\{0\}$ is maximal co-ideal of R.

The next theorem investigate the relationship between the primal co-ideals of semirings R and R/I, for some Q-strong co-ideal J of R containing I.

Theorem 2.14. *Let J be a Q-strong co-ideal of a semiring R, I a proper subtractive strong co-ideal of R and $J \subseteq I$. Then I is a primal strong co-ideal of R if and only if I/J is a primal strong co-ideal of R/J. In particular, there is a bijective correspondence between the primal strong co-ideals of R containing J and the primal strong co-ideals of R/J.*

Proof. Let I be a primal strong co-ideal of R with adjoint prime P and $J \subseteq I$. Since I is subtractive, P is subtractive by Proposition 2.12. Therefore by Proposition 1.2(h), P/J is a prime strong co-ideal of R/J. It suffices to show that P/J is is the set of all elements of R/J that are not prime to I/J. Let $qJ \in P/J$, so $q \in Q \bigcap P$ by Proposition 1.2(g). Since P is adjoint prime strong co-ideals of I, there exists $b \in R-I$ such that $a+b \in I$. Let q_e be unique element of Q that $1 \in q_eJ$. By Proposition 1.2(f), $J = q_eJ$. Let $q'J$ be unique element of R/J that $b \in q'J$. Since $b \notin I$, $b \notin J$. Thus $b \notin q_eJ$. If $q'J \in I/J$, then $q'I \bigcap Q$ by Proposition 1.2(g). Therefore $J \subseteq I$ gives $q'J \subseteq q'I \subseteq I$. As $b \in q'J$, $b \in I$, a contradiction. Hence $q'J \notin I/J$. It remains to show that $qJ \oplus q'J \in I/J$. Let $qJ \oplus q'J = q''J$ where $qJ + q'J \subseteq q''J$. Since $a \in qJ$ and $b \in q'J$, $a+b \in qJ+q'J \subseteq q''J$. Hence $a+b = q''j$ for some $j \in J$. Since $j \in J$, $a+b \in I$ and I is subtractive, $q'' \in I$. Thus $q''J \in I/J$. Hence every element of P/J is not prime to I/J. Now assume that $qJ \in R/J$ is not prime to I/J. Thus $qJ \oplus q'J \in I/J$ for some $q'J \in R/J - I/J$. Let $q''J \in R/J$ such that $qJ + q'J \subseteq q''J$. Hence $qJ \oplus q'J = q''J$ and $q''J \in I/J$. Therefore $q'' \in I \cap Q$. Since $q'J \notin I/J$, $q' \notin I$. As $q+q' \in qJ+q'J \subseteq q''J$, $q+q' = q''j$ for some $j \in J$. Therefore $j \in J \subseteq I$ and $q'' \in I$ gives $q+q' \in I$. Since $q' \notin I$, q is not prime to I. Therefore $q \in P$ and hence $qJ \in P/J$. Hence P/J is equal to the set of elements of R/J that are not prime to I/J and so I/J is primal.

Conversely, suppose that I/J is P/J-primal strong co-ideal of R/J; we show that I is a primal strong co-ideal of R with adjoint prime P. By Proposition 1.2(h), P is a prime strong co-ideal of R. It is enough to show that P is equal to the set of elements of R that are not prime to I. Let $a \in P$. Then there is a unique element $qJ \in R/J$ such that $a \in qJ$. Thus $a = qj$ for some $j \in J$. Since $j \in J \subseteq P$, $a \in P$ and P is subtractive, $q \in P$. Thus $qJ \in P/J$. Since P/J is adjoint prime of I/J, $qJ \oplus q'J \in I/J$ for some $q'J \in R/J - I/J$. Let $qJ \oplus q'J = q''J \in I/J$ where $qJ + q'J \subseteq q''J$ and $q'' \in I \cap Q$. Since $a + q' \in qJ + q'J \subseteq q''J$, $a + q' \in q''J$. As $J \subseteq I$, $q''J \subseteq q''I$. Therefore $q'' \in I$ gives $q''I \subseteq I$ and so $a + q' \in I$. If $q' \in I$, then $q'J \in I/J$ which is a contradiction. Therefore $q' \notin I$ and a is not prime to I. Now, suppose that a is not prime to I; we show $a \in P$. Since a is not prime to I, $a+b \in I$ for some $b \in R - I$. Let $q'' \in Q$ be unique element of Q that $a + b \in q''J$. Since I is subtractive and $J \subseteq I$, $q'' \in I$. Let $q,\ q' \in Q$ be unique element of Q such that $a \in qJ$ and $b \in q'J$. Since I is subtractive, $q' \in I$ gives $b \in I$, which is a contradiction. Hence $q' \notin I$ and $q'J \notin I/J$. Let $qJ \oplus q'J = q'''J$ where $qJ + q'J \subseteq q'''J$. As $a + b \in q''J$ and $a + b \in qJ + q'J \subseteq q'''J$, $q''J \bigcap q'''J \neq \emptyset$. Thus $q''J = q'''J$. Hence $qJ \oplus q'J \in I/J$. Since $q'J \notin I/J$, qJ is not prime to I/J. Hence $qJ \in P/J$ and so $q \in P$ and $qJ \subseteq P$. As $a \in qJ$, $a \in P$, as required. □

The following theorem shows every strong co-ideal I of a semiring R is an intersection of all primal strong co-ideals which contain I.

Theorem 2.15. *Let I be a subtractive strong co-ideal of semiring R. Then I is intersection of all primal strong co-ideals of R that contains I.*

Proof. Let I be a subtractive strong co-ideal of R and $\{P_\alpha\}_{\alpha\in\Gamma}$ be collection of all primal strong co-ideals that contain I. We show that $I = \bigcap_{\alpha\in\Gamma} P_\alpha$. Clearly, $I \subseteq \bigcap_{\alpha\in\Gamma} P_\alpha$. For the reverse of inclusion, let $a \notin I$, set

$$\Sigma = \{J :\ J\ is\ subtractive\ strong\ co-ideal,\ I \subseteq J\ and\ a \notin J\}.$$

It is clear that $(\Sigma, \subseteq)$ is a poset. By Zorn' Lemma, Σ has a maximal element. Let K be a maximal element of Σ, we claim that Σ is irreducible. If $K = K_1 \bigcap K_2$ where $K_1 \subset K$ and $K_2 \subset K$, maximality of K implies that $a \in K_1$ and $a \in K_2$. Therefore $a \in K$, a contradiction. This shows that K is irreducible. Since K is subtractive, K is primal by corollary 2.7. Hence $a \notin K$ implies that $a \notin \bigcap_{\alpha\in\Gamma} P_\alpha$. Therefore $\bigcap_{\alpha\in\Gamma} P_\alpha \subseteq I$ and so $\bigcap_{\alpha\in\Gamma} P_\alpha = I$. □

In the following theorem, it is shown that under Noetherian property of semirings, every subtractive strong co-ideal is a finite intersection of primal strong co-ideals.

Theorem 2.16. *Let R be Noetherian semiring. Then every strong co-ideal is an intersection of finitely many primal strong co-ideals.*

Proof. we show every strong co-ideal is a finite intersection of subtractive irreducible strong co-ideals. Let Σ denotes the set of all proper strong co-ideals I of R such that I is not a finite intersection of subtractive irreducible strong co-ideals. we claim that $\Sigma = \emptyset$. For if not, Σ has a maximal element K. But K is not irreducible and so $K = K_1 \bigcap K_2$ where K_1 and K_2 are strong co-ideals of R. Thus K_1 and K_2 are finite intersection of irreducible strong co-ideals and so is I, a contradiction. By Corollary 2.7, every irreducible strong co-ideal is primal. Hence every strong co-ideal is an intersection of finitely many primal strong co-ideals. □

In Theorem 2.15, it is shown that every subtractive strong co-ideal is an intersection of a primal strong co-ideals. The following example shows that this intersection may be infinite for some semirings.

Let A be any nonempty subset of a semiring R. Then the set $F(A)$ consisting of all elements of R of the form $a_1a_2...a_n + r$ (with $a_i \in A$ for all $1 \leq i \leq n$ and $r \in R$) is a co-ideal of R containing A [8] and [11]. If A is a subset of R that $1 \in A$, then $F(A)$ is a strong co-ideal of R.

Example 2.17. Let $X = \{x_i : \ i \in \mathbb{N}\}$ and $R = (P(X), +, \times)$ a semiring, where P(X)= the set of all subsets of X and + and $\times$ means $\bigcup$ and $\bigcap$, respectively. In six steps, we show that there is a subtractive strong co-ideals I of R such that I is not an intersection of finite primal strong co-ideals.(co-spec(R) denotes the set of all prime strong co-ideals of R).

Step 1: For each $t \in R$ where $0, 1 \neq t$ there exists $t' \neq 0, 1$ such that $t \times t' = 0$ and $t + t' = 1$.

Proof: It is clear.

Step 2: Strong co-ideal P of R is prime if and only if it is maximal.

Proof: Let P be a prime strong co-ideal of R and $P \subset Q$ where $Q \neq P$ is a strong co-ideal of R. Hence there exists $q \in Q$ such that $q \notin P$. By Step 1, there exists $q' \in R$ such that $q + q' = 1$ and $q \times q' = 0$. Thus $q + q' \in P$ and $q' \notin P$ gives $q' \in P$ and so $q' \in Q$. Thus $0 = q \times q' \in Q$. Therefore $Q = R$. Hence P is maximal. The converse is clear.

Step 3: Strong co-ideal P is primal if and only if P is prime.

Proof: Let P be a primal strong co-ideal of R with adjoint co-ideal Q. We will show that $Q = P$. If not, then there exists $q \in Q$ such that $q \notin P$. By Step 1, $q + q' = 1$ and $q \times q' = 0$ for some $0, 1 \neq q' \in R$. Since $q \notin P$, q' is not prime to P. This implies that $q' \in Q$. Hence $0 \in Q$. This is a contradiction. Therefore $Q = P$ and so P is prime. The converse is clear.

Step 4: The subset $\{1\}$ is a subtractive strong co-ideal of R and every strong co-ideal is subtractive.

Proof: It is clear.

Step 5: $\{1\} = \bigcap_{P \in co-spec(R)} P$.

Proof: It is clear that $\{1\} \subseteq \bigcap_{P \in co-spec(R)} P$. For the reverse inclusion, let $x \in \bigcap_{P \in co-spec(R)} P$. Assume that $x \neq 1$. Set $\Sigma = \{I \ : \ x \notin I,\ I\ is\ a\ strong\ co-ideal\ of\ R\ \}$. Since $\{1\} \in \Sigma$, $\Sigma \neq \emptyset$. An inspection shows that $(\Sigma, \subseteq)$ is a poset and every chain in Σ has a upper bound. By Zorn' Lemma, Σ has a maximal element K. Since $x \notin K$, $K \neq R$. We claim that K is prime. Let $a+b \in K$ and $a \notin K, b \notin k$. Since K is properly contained in $F(K \bigcup \{a\})$ and $F(K \bigcup \{b\})$, $x \in F(K \bigcup \{a\})$ and $x \in F(K \bigcup \{b\})$. Hence $x = r_1 + k_1 \times a^n = r_2 + k_2 \times b^m$ for some $r_1, r_2 \in R$, $k_1, k_2 \in K$, $n, m \in \mathbb{N}$. Since $k_1 \times (a+b)^n = k_1 \times a^n + b \times t \in K$, $x + b \times t = r_1 + k_1 \times a^n + b \times t \in K$ $(t \in R)$. Hence $b \times t \in (K : x)$. By Step 4 and Lemma 2.4, $(K : x)$ is a subtractive co-ideal of R. Hence $b \in (K : x)$. Therefore $k_2 \times b^m \in (K : x)$, because $k_2 \in K \subseteq (K : x)$. So $x = r_2 + k_2 \times b^m \in (K : x)$, hence $x = x + x \in K$, a contradiction. Therefore K is prime implies $x \in K$. This contradicts $x \notin K$. Hence $x = 1$.

Step 6: $\{1\}$ is not an intersection of finite primal strong co-ideals of R.

Proof: Let $\{1\} = \bigcap_{i=1}^{n} P_i$ for some prime strong co-ideals P_i of R. Let $r_i = X \setminus \{x_i\}$. Then for each $i \neq j$, $x_i + x_j = 1$. Since $x_i \neq 1$ for each $i \in \mathbb{N}$, we have $x_i \in P_t$ for some $1 \leq t \leq n$. As $\mathbb{N}$ is infinite, there exists $1 \leq t \leq n$ such that $x_i, x_j \notin P_t$ for some $i \neq j$. But $x_i + x_j = 1 \in P_t$ gives a contradiction. Hence $\{1\}$ is not an intersection of finite prime(primal) strong co-ideals of R.

Definition 2.18. Let I be a strong co-ideal of semiring R and P be a prime strong co-ideal of R that contains I. The isolated P-component of I, $U(I, P)$, is the intersection of all strong co-ideals which contain I and are such that every element not in P is prime to them.

Lemma 2.19. *If I is a primal strong co-ideal of semiring R with adjoint prime P, then $I = U(I, P)$.*

Proof. Clearly, $I \subseteq U(I, P)$. Since $U(I, P)$ is the intersection of all strong co-ideals J which $I \subseteq J$ and if $x \notin P$, then x is prime to J and I is itself such an strong co-ideal, $U(I, P) \subseteq I$. Hence $I = U(I, P)$. □

Theorem 2.20. *Let I be a subtractive strong co-ideal of R, then $I = \bigcap_{\alpha \in \Gamma} U(I, P_\alpha)$ where $P_a^{,}s$ are the adjoint of all primal strong co-ideals I_α that contains I.*

Proof. By Theorem 2.15, $I = \bigcap_{\alpha\in\Gamma} I_\alpha$ where I_α is a primal strong co-ideals that contains I. Let P_α be adjoint prime of I_α for each $\alpha \in \Gamma$. By Lemma 2.19 , $I_\alpha = U(I_\alpha, P_\alpha)$. Since $I \subseteq I_\alpha$ and if $x \notin P_\alpha$, then x is prime to I_α, we have $U(I, P_\alpha) \subseteq I_\alpha = U(I_\alpha, P_\alpha)$. Also it is clear that $I \subseteq U(I, P_\alpha)$ for each $\alpha \in \Gamma$. Hence $I \subseteq \bigcap_{\alpha\in\Gamma} U(I, P_\alpha) \subseteq \bigcap_{\alpha\in\Gamma} U(I_\alpha, P_\alpha) = \bigcap_{\alpha\in\Gamma} I_\alpha = I$ and the equality follows. □

3. Intersection of primal strong co-ideals

We now inquire when the intersection of primal strong co-ideals is again primal. It will seem no doubt somewhat surprising at first glance that the intersection of two primal strong co-ideals is not necessarily primal as the following example shows. But if we restrict ourselves to some conditions, we may state the following theorem (e.g. Theorem 3.4).

Example 3.1. Let X={a,b,c} and $R = (P(X), \cup, \cap)$ a semiring, where P(X)= the set of all subsets of X. Let

$$P_1 = \{\{a\}, \{a, b\}, \{a, c\}, X\},$$

$$P_2 = \{\{b\}, \{a, b\}, \{b, c\}, X\}.$$

An inspection shows that P_1 and P_2 are Primal. Now $P_1 \bigcap P_2 = \{X, \{a, b\}\}$ is not primal by Example 2.8.

Definition 3.2. Let R be a semiring and I be a strong co-ideal of R. The representation $I = I_1 \bigcap ... \bigcap I_n$ of I by strong co-ideals I_i is called reduced, if none of the components may be replaced by a larger strong co-ideal without changing the intersection.

We continue, as follows, to study the properties of the co-ideal I of the semiring R in terms of it' s reduced representation.

Lemma 3.3. *Let $I = I_1 \bigcap I_2 \bigcap ... \bigcap I_n$ be a reduced representation of I by subtractive primal strong co-ideals* I$_i$ *with adjoint prime strong P_i. Then $r \in R$ is not prime to I if and only if r is not prime to I_i for some $1 \le i \le n$.*

Proof. Assume that r is not prime to I. Then $r + a \in I$ for some $a \notin I$. Since $a \notin I$, $a \notin I_i$ for some $1 \le i \le n$. Thus $r + a \in I \subseteq I_i$ and $a \notin I_i$ gives r is not prime to I_i.

Conversely, assume that $r \in R$ is not prime to I_i where $\le i \le n$. Then there is $a \in R - I_i$ such that $r + a \in I_i$. Therefore $J = F(\{a\} \bigcup I_i)$ properly contains I_i. Since $I = I_1 \bigcap I_2 \bigcap ... \bigcap I_n$ is reduced,

$$I_1 \bigcap I_2 \bigcap ... \bigcap I_{i-1} \bigcap J \bigcap I_{i+1} \bigcap I_2 \bigcap ... \bigcap I_n \nsubseteq I.$$

Hence there is $x \in I_1 \bigcap I_2 \bigcap ... \bigcap I_{i-1} \bigcap J \bigcap I_{i+1} \bigcap I_2 \bigcap ... \bigcap I_n \not\subseteq I$ such that $x \notin I$. As $x \in J$, $x = b + c_1...c_s a^m$ for some $c_t \in I_i (1 \leq t \leq s)$, $b \in R$ and $m \in \mathbb{N}$. Since $x \in I_j$, for each $i \neq j$, we have $r + x \in I_j$, for each $i \neq j$. By Lemma 2.4, (I_i, r) is a strong co-ideal of R. As $a \in (I_i : r)$ and $c_t \in (I_i : r)$ $(1 \leq t \leq s)$, we have $c_1...c_s a^m \in (I_i : r)$. Hence $c_1...c_s a^m + r \in I_i$ and so $r + x \in I_i$. As $r + x \in I_k$ for each $1 \leq k \leq n$, $r + x \in I$. Since $x \notin I$, r is not prime to I. □

Theorem 3.4. *Let $I = I_1 \bigcap I_2 \bigcap ... \bigcap I_n$ be a reduced representation of I by subtractive P_i-primal strong co-ideals I_i. Then I is primal if and only if there is a $1 \leq j \leq n$ such that $P_i \subseteq P_j$ for all $1 \leq i \leq n$.*

Proof. Assume that there is a $1 \leq j \leq n$ such that $P_i \subseteq P_j$ for all $1 \leq i \leq n$. By Lemma 3.3, r is not prime to I if and only if $r \in \bigcup_{i=1}^{n} P_i$. Since for all $1 \leq i \leq n$, $P_i \subseteq P_j$, $\bigcup_{i=1}^{n} P_i = P_j$. Therefore the set of all elements of R that are not prime to I is a strong co-ideal of R. Hence I is primal.

Conversely, suppose that I is a primal strong co-ideal of R with adjoint prime P. Let $r \in P$. Since r is not prime to I, $r \in \bigcup_{i=1}^{n} P_i$, by Lemma 3.3. Hence $P \subseteq \bigcup_{i=1}^{n} P_i$. By Proposition 1.2(l), $P \subseteq P_j$ for some $1 \leq j \leq n$. On the other hand, for each i, $P_i \subseteq P$ by Lemma 3.3. Thus $P = P_j$ and $P_i \subseteq P_j$ for each $1 \leq j \leq n$. □

Definition 3.5. Let I be a strong co-ideal of a semiring R. A strong co-ideal J is called is not prime to I, if every element of J is not prime to I.

Lemma 3.6. *Let $I = I_1 \bigcap I_2 \bigcap ... \bigcap I_n$ be a reduced representation of I by subtractive P_i-primal strong co-ideals I_i. Then J is not prime to I if and only if $J \subset P_i$ for some $1 \leq i \leq n$.*

Proof. It is clear from Lemma 3.3. □

Definition 3.7. Let I be a strong co-ideal of a semiring R. The maximal prime of I is a strong co-ideal which is maximal in the poset $(\Sigma, \subseteq)$ where

$$\Sigma = \{P : P \textit{ is prime strong co-ideal which is not prime to } I \textit{ and } I \subseteq P\}.$$

Proposition 3.8. *Let $I = I_1 \bigcap ... \bigcap I_n$ be a reduced representation of I as an intersection of P_i-primal strong co-ideals I_i of R. Then the maximal primes of I are the maximal elements of the "inclusion ordered" set $\{P_1, P_2, ..., P_n\}$.*

Proof. Let P be a maximal prime of I. By Lemma 3.6, there exists $1 \leq i \leq n$ such that $P \subseteq P_i$. Moreover, by Lemma 3.3, P_i is not prime to I, hence $P = P_i$, because P is maximal prime of I. This gives P is a prime strong co-ideal of R.

Conversely, let P_j be maximal in the set $\{P_1, ..., P_n\}$ with respect the inclusion. We claim that P_j is maximal prime of I. Otherwise, let Q be a prime strong co-ideals of R that is not prime to I and $P_j \subset Q$. As Q is not prime to I, $Q \subseteq P_i$ for some $1 \leq i \leq n$. Hence $P_j \subset P_i$, a contradiction, as needed. □

Definition 3.9. If $I = I_1 \bigcap ... \bigcap I_n$ is an irredundant representation of I by primal strong co-ideals I_i, and is such that $I_i \cap I_j$ is not primal if $i \neq j$, then it will be called a short representation of I by primal strong co-ideals.

Theorem 3.10. *Let $I = I_1 \bigcap ... \bigcap I_n$ be a reduced representation of I by primal strong co-ideals with prime adjoints P_i. Then I has a short representation by primal strong co-ideals whose adjoints are the maximal primes of I.*

Proof. Let $I = I_1 \bigcap ... \bigcap I_n$ be a reduced representation of I. Hence we can assume that this representation is irredundant. Otherwise, if $I = I_1 \bigcap ... \bigcap I_n$ is not irredundant, we can eliminate some I_i's and the remaining intersection is again reduced. Let the indexing be such that $P_1, P_2, ..., P_r$ are the maximal elements of the set $\{P_1, ..., P_n\}$. Let $I'_1 = \bigcap\{I_i : \ P_i \subseteq P_1\}$ and $I'_j = \bigcap\{I_i : \ P_i \subseteq P_j \ and \ P_i \not\subseteq P_t \ if \ t < j\}$. Each of $I'_1, ..., I'_r$ satisfies the condition of Theorem 3.4, and so they are all primal with prime adjoints $P_1, ..., P_r$. Also $I = I'_1 \bigcap ... \bigcap I'_r$. For each $i \neq j$, $I'_i \bigcap I'_j$ is a reduced representation by primal strong co-ideals I'_i and I'_j, not all of whose adjoints are contained in any one adjoint, hence by Theorem 3.4, $I'_i \bigcap I'_j$ is not primal and so $I = I'_1 \bigcap ... \bigcap I'_r$ is short. By Proposition 3.8, $P_1, ..., P_r$ are the maximal primes of I. □

Theorem 3.11. *Let I be a strong co-ideals of semiring R. In any short reduced representation of I by primal ideals with prime adjoints, the adjoints and the number of primal components are uniquely determined.*

Proof. Let $I = I_1 \bigcap ... \bigcap I_n$ with adjoint prime ideals $P_1, \ P_2, ..., P_n$ and $I = I'_1 \bigcap ... \bigcap I'_m$ with adjoint prime ideals $P'_1, \ P'_2, ..., P'_m$ be two short primal reduced representation of I. Since both representations are short, neither P_i properly contains the another P_j nor P'_i properly contains another P'_j. Thus by Proposition 3.8, both sets $\{P_1, P_2, ..., P_n\}$ and $\{P'_1, P'_2, ..., P'_m\}$ are the set of maximal primes of I. Therefore $n = m$ and $\{P_1, P_2, ..., P_n\} = \{P'_1, P'_2, ..., P'_m\}$. □

Acknowledgments

The authors would like to thank the referee(s) for careful reading of the manuscript.

References

1. W. E. Barnes, *Primal ideals and isolated components in noncommutative rings*, Trans. Amer. Math. Soc., **82**(1956), 1–16.
2. J. N. Chaudhari and K. J. Ingale, *Prime avoidance theorem for co-ideals in semirings*, Research Journal of Pure Algebra, (9) **1** (2011), 213–216.
3. C. W. Curtis, *On additive ideal theory in general rings*, Amer. J. Math. **76** (1952), 687–700.
4. S. Ebrahimi Atani, *The ideal theory in quotients of commutative semirings*, Glas. Math. **42** (2007), 301-308.
5. S. Ebrahimi Atani, *On Primal and weakly Primal ideals over commutative semirings*, Glas. Math. **43** (2008), 13–23.
6. S. Ebrahimi Atani, R, Ebrahimi Atani, *A Zariski topology for k-semirings*, Quasigroups Related Systems, **20** (2012), 29–36.
7. S. Ebrahimi Atani, A Yousefian Darani, *Some remarks on primal submodules*, Sarajevo J. Math., **17** (2008), 181–190.
8. S. Ebrahimi Atani, S. Dolati Pish Hesari, M. Khoramdel, *Strong Co-ideal theory in quotients of semirings*, J. Adv. Res. Pure Math., (3) **5** (2013), 19–32
9. S. Ebrahimi Atani, S. Dolati Pish Hesari, M. Khoramdel, *A Fundemental theorem of co-homomorphism for semirings*, To appear in Thai J. Math.
10. L. Fuchs, *On primal ideals*, Proc. Amer. Math. Soc., **1** (1950), 1–6.
11. J. S. Golan, *Semirings and their Applications*, Kluwer Academic Publisher Dordrecht, 1999.
12. H. Wang, *On rational series and rational language*, Theor. Comput. Sci., **205** (1998), 329–336.

Shahabaddin Ebrahimi Atani
Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.
Email: ebrahimi@guilan.ac.ir

Saboura Dolati Pish Hesari
Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.
Email: saboura_dolati@yahoo.com

Mehdi Khoramdel
Department of Mathematics, University of Guilan, P.O.Box 1914, Rasht, Iran.
Email: mehdikhoramdel@gmail.com