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CO-INTERSECTION GRAPH OF SUBACTS OF AN ACT

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Abstract. In this paper, we define the co-intersection graph $G(A)$ of an S-act A which is a graph whose vertices are non-trivial subacts of A and two distinct vertices B_1 and B_2 are adjacent if $B_1 \cup B_2 \neq A$. We investigate the relationship between the algebraic properties of an S-act A and the properties of the graph $G(A)$.

1. Introduction and preliminaries

The notion of an S-act over a monoid S is a fundamental concept in algebra, theoretical computer science and a variety of applications like automata theory and mathematical linguistics. Assigning graphs to algebraic structures is an approach to study algebraic properties via graph-theoretic properties. We investigate the relationship between the algebraic properties of an S-act A and the properties of the graph $G(A)$. The studing a classe of graphs associated with subacts of an S-act has been extensively investigated by Rasouli et. al. [1, 3, 2, 8], where extended the intersection graph to acts over semigroupes. The Zero divisor graphs for S-act studied by Estaji and Haghdadi in [4]. Recently co-intersection graph of submodules of a module interoduced by L. A. Mahdavi and Y. Talebi in [6, 7]. Motivated by these ideas, in this paper we define co-intersection graph of subacts of an act. We associate a graph $G(A)$ to an S-act A, called the co-intersection graph

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of A, whose vertices are non-trivial subacts of A in such a way that two distinct vertices B_1, B_2 are adjacent if $B_1 \cup B_2 \neq A$.

In the following, we give some basic definitions on the S-acts and associated graphs which are used in the main results.

Let S be a semigroup. A non-empty set A is said to be a (*left*) *S-act* if there is a mapping $\lambda : S \times A \to A$, denoting $\lambda(s, a)$ by sa, satisfying $(st)a = s(ta)$ and, if S is a monoid with 1, $1a = a$, for all $a \in A$, $s, t \in S$. An element $\theta \in A$ is said to be a *fixed element* if $s\theta = \theta$ for all $s \in S$. A non-empty subset B of A is called a *subact* of A if it is closed under the action, that is $sb \in B$, for every $s \in S$, $b \in B$. A *non-trivial* subact of an S-act A is a (non-empty) proper subact of A. The set of all non-trivial subacts of A is denoted by $\mathrm{Sub}(A)$. Clearly, S is an S-act with its operation as the action and so subacts of S are exactly the *left ideals* of S, the non-empty subsets I of S satisfying $SI \subseteq I$.

A non-trivial subact M of an S-act A is called a *minimal* subact if it properly contains no subact of A. We denote the set of all minimal subacts of A by $\mathrm{Min}(A)$. A *maximal* subact of A is a non-trivial subact N for which there is no subact of A properly contained between N and A. The set of all maximal subacts of A by $\mathrm{Max}(A)$. The *coproduct* of a family $\{A_i \mid i \in I\}$ of S-acts, denoted by $\coprod_{i\in I} A_i$, is the disjoint union $\bigcup_{i\in I}(A_i \times \{i\})$ with the action $s(a, i) = (sa, i)$ for every $s \in S$ and $a \in A_i, i \in I$. The reader is referred to [5] for more details on S-acts.

Let G be a simple and undirected graph with a vertex set $V(G)$. For distinct elements x and y of $V(G)$, the length of the shortest (x, y)-path is denoted by $d(x, y)$. If G has no such a path, then $d(x, y)=\infty$. The number of vertices which are adjacent to x is called the *degree* of x and denoted by $\deg(x)$. The *girth(G)* of a graph G is the length of its shortest cycle and denoted by girth(G). A graph with no cycle has infinite girth. A graph G is *connected* if there is a path between every two distinct vertices. A *complete graph* with n vertices, denoted by K_n, is a graph in which every pair of distinct vertices are adjacent. A cycle graph with n vertices denoted by C_n is a graph that consists of a single cycle. A *path graph* denoted by P_n where n refers to the number of vertices of the path graph. A graph is said to be null, if it has no edge. The reader is referred to [9] for more details on graph.

2. Basic notations

In this section, we proceed with the study of some facts about the co-intersection graphs of S-acts.Throughout S stands for a semigroup unless otherwise stated.

Definition 2.1. Let A be an S-act. The *co-intersection graph* of A, $G(A)$, is a graph whose vertices are all non-trivial subacts of A such that two distinct vertices B_1 and B_2 are adjacent if and only if $B_1 \cup B_2 \neq A$.

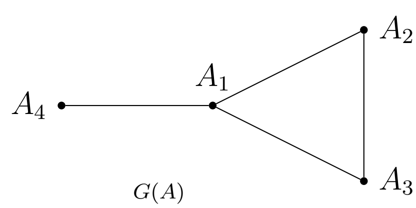
Example 2.2. Take the monoid $S = \{1, s\}$, where $s^2 = s$. Consider the S-act $A = \{a, b, c\}$ given by the following action table:

	a	b	c
1	a	b	c
s	a	b	a

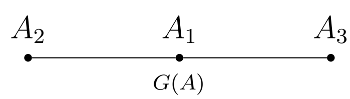
The non-trivial subacts of A are:

$$A_1 = \{a\}, A_2 = \{b\}, A_3 = \{a, b\}, A_4 = \{a, c\}$$

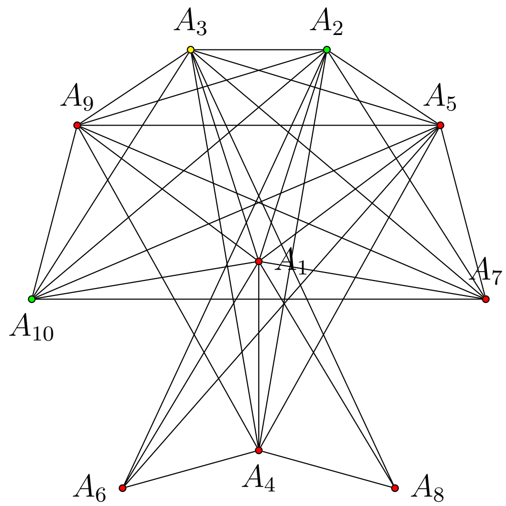
Thus $G(A)$ is the following graph:



Example 2.3. Let $S = \{0, 1\}$. The S-act $A = \{a, b, c\}$ with the action $0x = a$ for every $x \in A$ has three non-trivial subacts $A_1 = \{a\}, A_2 = \{a, b\}$ and $A_3 = \{a, c\}$. Thus $G(A)$ is the following graph:



Example 2.4. Let S be a non-trivial monoid. Take the S-act $A = \{a, b, c, d\}$ where a, b, c are fixed elements and $sd = a$ for all $1 \neq s \in S$. Then $A_1 = \{a\}, A_2 = \{b\}, A_3 = \{c\}, A_4 = \{a, d\}, A_5 = \{a, b\}, A_6 = \{a, b, d\}, A_7 = \{a, b, c\}, A_8 = \{a, c, d\}, A_9 = \{a, c\}$, and $A_{10} = \{b, c\}$ are all of non-trivial subacts of A.
Thus $G(A)$ is the following graph:



$G(A)$

In the following, we show that, for some graphs G, there is no S-act A for which $G(A) = G$. A *bipartite* graph is one whose vertex-set is partitioned into two (not necessarily non-empty) disjoint subsets in such a way that the two end vertices for each edge lie in disjoint partitions.

Theorem 2.5. *Let G be a non-null bipartite graph. Then G is a co-intersection graph of an S-act if and only if $G = P_i$, where $i \in \{2, 3\}$.*

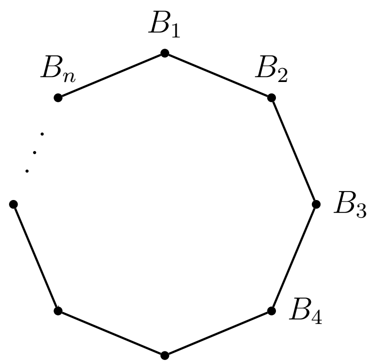
Proof. Let A be an S-act and $G = G(A)$ and $W_1 = \{B_1, B_2, \dots\}$, $W_2 = \{C_1, C_2, \dots\}$ be two components of G. Suppose that B_1 is adjacent to C_1 and $B_1 \cup C_1 \in W_1$. Then it follows that $B_1 \cup C_1 = B_1$, because if $B_1 \cup C_1 = B_i, i \neq 1$, then $B_1 \subset B_i$ and B_1 is adjacent to B_i, which is a contradiction. Thus $B_1 \cup C_1 = B_1$ that is $C_1 \subset B_1$. If $B_1 \cup C_1 \in W_2$, then $B_1 \cup C_1 = C_1$, because if $B_1 \cup C_1 = C_i, i \neq 1$, then $C_1 \subset C_i$, a contradiction. Thus $B_1 \cup C_1 = C_1$ that is $B_1 \subset C_1$. Hence, either $B_1 \subset C_1$ or $C_1 \subset B_1$. Without loss of generality assume that $C_1 \subset B_1$. Now, we show that B_1 is an endpoint vertex. Suppose that B_1 is adjacent to $C_2 \in W_2$, then $B_1 \cup C_2 \neq A$ and $C_1 \cup C_2 \neq A$. Therefore, C_1 and C_2 are adjacent which is a contradiction. Hence, B_1 is not adjacent to another element of W_2, so B_1 is an endpoint.

If C_1 is not adjacent to any element of W_1, then $G = P_2$. If C_1 is adjacent to another element, say B_2, then $C_1 \subset B_2$, because otherwise $B_2 \cup C_1 \in V(G) = W_1 \cup W_2$, if $B_2 \cup C_1 = B_i \in W_1$, then $B_2 \subset B_i$ and if $B_2 \cup C_1 = C_i \in W_2$, then $C_1 \subset C_i$, a contradiction in both cases. Now, we show that B_2 is an endpoint vertex and G is the path $B_1 - C_1 - B_2$. Assume on the contrary that B_2 is adjacent to $C_2 \in W_2$, then $B_2 \cup C_2 \in V(G) = W_1 \cup W_2$. If $B_2 \cup C_2 = B_i \in W_1$, then $B_2 \subset B_i$

and if $B_2 \cup C_2 = C_i \in W_2$, then $C_2 \subset C_i$, which are contradictions in both cases. For converesly see 2.3 and 2.6 when $n = 2$ □

Theorem 2.6. *The cycle graph C_n is a co-intersection graph of an S-act if and only if $n = 3$.*

Proof. Let $n > 3$ and suppose that there exists an S-act A with non-trivial subacts $B_1, B_2, B_3, \ldots, B_n$ such that the co-intersection graph $G(A)$ is the following cycle graph C_n:



Since $B_1 \cup B_2 \neq A$, $B_1 \cup B_2 = B_i$ for some $1 \leq i \leq n$. If $B_1 \cup B_2 = B_1$, then $B_2 \subset B_1$. Thus $B_2 \cup B_n \subset B_1 \cup B_n \neq A$. If $B_1 \cup B_2 = B_2$, then $B_1 \subset B_2$ so that $B_1 \cup B_3 \subset B_2 \cup B_3 \neq A$. If $B_1 \cup B_2 = B_i$, then $B_1 \subset B_i$ and $B_2 \subset B_i$. Hence, $B_1 - B_3 - B_2 - B_1$ is a cycle. In each case, we have a contradiction. □

It is clear that if A and B are isomorphic S-acts, then the graphs $G(A)$ and $G(B)$ are isomorphic. The converse is not true in general. This result is illustrated in the following example.

Example 2.7. Take the monoid $S = \{1, s\}$, where $s^2 = 1$. Consider two S-acts $A = \{a, b, c\}$ with trivial action and $B = \{a, b, c, d\}$ presented by the following action table:

	a	b	c	d
1	a	b	c	d
s	a	b	d	c

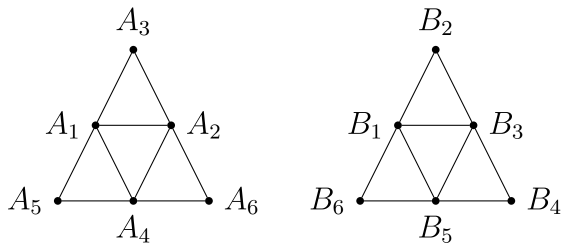
The non-trivial subacts of A and B are:

$$A_1 = \{a\}, A_2 = \{a, b\}, A_3 = \{b\}, A_4 = \{b, c\}, A_5 = \{c\}, A_6 = \{a, c\}$$

and

$$B_1 = \{a\}, B_2 = \{a, b\}, B_3 = \{b\}, B_4 = \{b, c, d\}, B_5 = \{c, d\}, B_6 = \{a, c, d\},$$

respectively. Then $G(A)$ and $G(B)$ are isomorphic which are given in the following:



$G(A) \cong G(B)$ whereas A and B are not isomorphic S-acts.

In the following, we give some conditions on two S-acts A, B under which A and B are isomorphic S-acts when $G(A) \cong G(B)$.

Recall that an S-act A is *free* if A has a basis and in this case, $A \cong S \times X$ where X is a non-empty set and $S \times X$ is a right S-act with the action $(s, x)t = (st, x)$ for all $(s, x) \in S \times X, t \in S$.

Lemma 2.8. *Let A be a free S-act with a basis X where S is a group. Then $G(A) \cong G(X)$ in which X is considered as an S-act with trivial action.*

Proof. Using the assumption, A is isomorphic to the S-act $S \times X$. Since S is a group, non-trivial subacts of A (if exist) are of the forms $S \times Y$ where $Y \subset X$. Consider the set X as an S-act with trivial action. We prove that the graphs $G(A)$ and $G(X)$ are isomorphic. For this, we define the map $f : G(A) \to G(X)$ by $f(S \times Y) = Y$, for any $Y \subset X$. Now, it is easy to see that f is a graph isomorphism. □

Theorem 2.9. *Let A and B be two free S-acts and $G(A) \cong G(B)$. Then $A \cong B$ under each of the following conditions:*

(i) *S is a group.*

(ii) *S has only finitely many left ideals, and A and B have finite bases.*

Proof. (i) Assume that X and Y are bases of free S-acts A and B, respectively. Using Lemma 2.8, $G(A) \cong G(X)$ and $G(B) \cong G(Y)$, where X and Y are considered as S-acts with trivial actions. From the assumption we have $G(X) \cong G(Y)$. Thus $2^{|X|} - 2 = |\mathrm{Sub}(X)| = |\mathrm{Sub}(Y)| = 2^{|Y|} - 2$. This implies that $|X| = |Y|$ and hence $A \cong B$.

(ii) This is trivial. □

The following example shows that for any complete graph K_n, there exists an S-act A whose co-intersection graph $G(A)$ is isomorphic to K_n.

Example 2.10. Let S be a *cyclic* (*monogenic*) *semigroup* of order $n + 1$, that is, $S = \{s, s^2, s^3, ..., s^{n+1}\}$, with $s^{n+2} = s^{n+1}$. It can be easily shown that, all distinct non-trivial ideals of S form the chain:

$$\langle s^n \rangle \subset \langle s^{n-1} \rangle \subset \cdots \subset \langle s^2 \rangle \subset \langle s \rangle$$

where $\langle s^k \rangle = \{s^i \mid k+1 \leq i \leq n+1\}$, for every $1 \leq k \leq n$. Since $\langle s^k \rangle \cup \langle s^l \rangle = \langle s^l \rangle$ for $l < k$, the graph $G(S)$ is complete with n distinct vertices. Clearly this graph is isomorphic to the complete graph K_n.

Example 2.11. The bicyclic monoid $S = \langle u, v \mid uv = 1 \rangle = \{v^m u^n : m, n \geq 0\}$ has a complete co-intersection graph. To see this, let I and J be two non-trivial left ideals of S such that $v^m u^n \notin I$ and $v^k u^l \notin J$ for some non-negative integers m, n, k and l. First, suppose that $n \geq l$. We show that $v^m u^l \notin I \cup J$. Assume on the contrary that $v^m u^l \in I \cup J$, then either $v^m u^l \in I$ or $v^m u^l \in J$. If $v^m u^l \in I$, then $(v^m u^{m+n-l})(v^m u^l) = v^m u^n \in I$ and if $v^m u^l \in J$, then $(v^k u^m)(v^m u^l) = v^k u^l \in J$, which are contradictions. Therefore, $v^m u^l \notin I \cup J$ and $I \cup J \neq S$. Now suppose that $n < l$. We show that $v^k u^n \notin I \cup J$. Let $v^k u^n \in I \cup J$, then either $v^k u^n \in I$ or $v^k u^n \in J$. If $v^k u^n \in I$, then $(v^m u^k)(v^k u^n) = v^m u^n \in I$ and if $v^k u^n \in J$, then $(v^k u^{l+k-n})(v^k u^n) = v^k u^l \in J$, which are contradictions in both cases. Therefore, $v^k u^n \notin I \cup J$ and $I \cup J \neq S$. Hence, the graph $G(S)$ is complete.

In the following, we give a necessary and sufficient condition for an S-act A to have a co-intersection complete graph. Recall that an S-act A is *Artinian* (*Noetherian*) if every descending (ascending) chain of subacts of A terminates.

Theorem 2.12. *Let A be a Noetherian S-act. Then $G(A)$ is complete if and only if A contains a unique maximal subact.*

Proof. Since A is a Noetherian S-act, then A has at least one maximal subact and every non-empty subact of A is contained in a maximal subact. First assume that A contains a unique maximal subact, say M, and B_1, B_2 are two non-trivial subacts of A. Since $B_1, B_2 \subset M$, $B_1 \cup B_2 \subset M$ and so the graph $G(A)$ is complete. Conversely, suppose that $G(A)$ is complete. If M_1 and M_2 are two maximal subacts of A, then $M_1 \cup M_2 = A$ and so these vertices are not adjacent, a contradiction. □

3. Connectivity, diameter and girth

In this section, we characterize all S-acts A for which the associated co-intersection graphs are connected. Using these results, the diameter and the girth of co-intersection graphs of S-acts are obtained.

Theorem 3.1. *Let A be an S-act. Then the graph $G(A)$ is disconnected if and only if A is a coproduct of two simple subacts.*

Proof. Let $G(A)$ be disconnected. Then there exist two vertices B and C with no path between them in $G(A)$. We show that $A = B \sqcup C$. It is clear that $A = B \cup C$. If $B \cap C \neq \emptyset$, then $B \cap C$ is a non-trivial subact of A since $B \cap C \subset B$ and $B \cap C \subset C$. Thus $B - B \cap C - C$ is a path between B and C, which is a contradiction. Hence, $B \cap C = \emptyset$.

Now, we show that B and C are simple subacts of A. If $D \subset B$, since $B \cap C = \emptyset$, $D \cup C \subset B \cup C = A$ and so D and C are adjacent, so $B - D - C$ is a path between B and C, a contradiction. Hence, B is a simple subact. A similar way can be applied to show that C is also a simple subact.

Conversely, suppose that there exists a subact D of A such that $B \cup D \neq A$, that is, B and D are adjacent in $G(A)$. Since B is simple, $B \cap D = \emptyset$, so $D \subseteq C$. But C is simple, then, $D = C$ and so $B \cup C = B \cup D \neq A$ which is a contradiction. Thus B is an isolated vertex, similarly it is shown that C is also an isolated vertex and hence A is disconnected. □

Corollary 3.2. *Let A be an S-act and $G(A)$ be connected. Then $B \cap C \neq \emptyset$ for any two maximal subacts B and C of A.*

Proof. Let $B \cap C = \emptyset$. It is clear that $B \cup C = A$. Now we show that B and C are simple. Let $D \subset B$, then $C \subset C \cup D$ and since C is a maximal subact of A so $C \cup D = A$ and $B = D$. Hence, B is simple. Similarly, C is also simple. Using Theorem 3.1, the graph $G(A)$ is disconnected, which is a contradiction. Hence, $B \cap C \neq \emptyset$. □

Corollary 3.3. *Let A be an S-act and $G(A)$ have at least one edge. Then $G(A)$ is connected.*

Proof. It is straightforward. □

Theorem 3.4. *Let A be an S-act. Then the following assertions hold:*

(i) *If $G(A)$ is connected, then diam $(G(A)) \leq 3$.*

(ii) *If $G(A)$ contains a cycle, then girth $(G(A)) = 3$.*

Proof. (i) Suppose that B and C are two non-trivial subacts of A. If B and C are adjacent vertices of $G(A)$, then $d(B, C) = 1$. Now, let B and C be non-adjacent, then $B \cup C = A$. There are two cases:

Case (1). One of the subacts B or C is not maximal. We suppose that B is not maximal and $B \subset D$, where D is a non-trivial subact of A. If $C \cup D \neq A$, then $B - D - C$ is a path and so $d(B, C) = 2$. If $C \cup D = A$, then $C \cap D \neq \emptyset$, otherwise $B \cap C \subset D \cap C$ implies that $B \cap C = \emptyset$ and since $A = B \cup C = D \cup C$ and $\emptyset = B \cap C = D \cap C$, $B = D$, which is a contradiction. Therefore, $C \cap D \neq \emptyset$ and the path $B - D - D \cap C - C$ implies that $d(B, C) = 3$.

Case (2). If both of B and C are maximal, then by Theorem 3.1 we have $B \cap C \neq \emptyset$ and since $B \cap C \neq B$ and $B \cap C \neq C$, $B - B \cap C - C$ is a path between B and C. Hence, $d(B, C) = 2$.

(ii) Let $n \geq 4$ and $B_1 - B_2 - \cdots - B_n$ be a cycle in $G(A)$. If $B_1 \cap B_2 = B_1$, then $B_1 \subset B_2$ and $B_1 \cup B_3 \subset B_2 \cup B_3 \neq A$, where B_1 and B_3 are adjacent, and $B_1 - B_2 - B_3 - B_1$ is a cycle. Thus $\mathrm{girth}(G(A)) = 3$.

Now, suppose that $B_1 \cap B_2 = B_2$, then $B_2 \subset B_1$, $B_2 \cup B_n \subset B_1 \cup B_n \neq A$ and B_2, B_n are adjacent, and $B_1 - B_2 - B_n - B_1$ is a cycle and so $\mathrm{girth}(G(A)) = 3$.

Finally let $B_1 \cap B_2 \neq B_1$, B_2. Then we have the cycle $B_1 - B_1 \cup B_2 - B_2 - B_1$ so that $\mathrm{girth}(G(A)) = 3$.

□

4. Finiteness conditions

In this section, we study finiteness conditions of some parameters of co-intersection graphs of S-acts such as clique number, chromatic number, independence number and domination number. It is useful to recall the following definitions from graph theory before we describe the results that are proved in this section. A *clique* of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G let $\chi(G)$ denote the *chromatic number* of G, i.e. the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors.

Theorem 4.1. *Let A be an S-act. Then the following are equivalent:*
(i) $\deg(B) < \infty$ *for each vertex B in $G(A)$.*
(ii) $\deg(B) < \infty$ *for some vertex B in $G(A)$.*
(iii) $|G(A)| < \infty$.
(iv) $\chi(G(A)) < \infty$.
(v) $\omega(G(A)) < \infty$.

Proof. (i) ⇒ (ii), (iii) ⇒ (iv) and (iv) ⇒ (v) are straightforward.

(ii) ⇒ (iii). Suppose that $|G(A)|$ is infinite and B be a vertex in $G(A)$ of finite degree. Let $W = \{B_1, B_2, \dots\}$ be an infinite set of non-trivial subacts of A such that $B \cup B_i = A$, for all $i \in \{1, 2, 3, \dots\}$. The set $W_1 = \{B_i \cap B | B_i \in W, B_i \cap B \neq \emptyset\}$ is finite, because all of these vertices are adjacent to B. Thus there exists an infinite subset $W_2 = \{B_{j_1}, B_{j_2}, \dots\}$ of W such that $B_{j_m} \cap B = B_{j_n} \cap B$ for all B_{j_m}, $B_{j_n} \in W_2$ which is a contradiction, because $|W_2| \leq 1$. Indeed let B_{j_n} and B_{j_m} be two distinct elements of W_2. Take any $x \in B_{j_n}$ and $x \notin B_{j_m}$, then $x \in A = B_{j_m} \cup B$ and so $x \in B$. Hence, $x \in B \cap B_{j_n} = B \cap B_{j_m}$ whence $x \in B_{j_m}$, which is a contradiction.

(v) $\Rightarrow$ (i). Suppose that there exists a vertex B and an infinite set $W = \{B_i | i \in I\}$ such that B_i is adjacent to B for all $i \in I$. By using *Infinite Ramsey's Theorem*, the subgraph of $G(A)$ induced by W contains either an infinite clique or an infinite set of pairwise disjoint subacts. The latter case yields also an infinite clique in $G(A)$, which contradicts the assumption. $\square$

The following corollary is direct consequence of the above theorem.

Corollary 4.2. *Let A be an S-act and B be non-trivial subact of A with* $\deg(B) < \infty$. *Then A is both Artinian and Noetherian.*

Let G be a graph. The (open) neighborhood $N(x)$ of a vertex $x \in V(G)$ is the set of vertices adjacent to x. For a subset T of vertices, we put $N(T) = \bigcup_{x\in T} N(x)$ and $N[T] = N(T) \cup T$. If $N[T] = V(G)$, then T is said to be a *dominating set*. It is clear that every vertex not in a dominating set T is adjacent to a vertex in T. The *domination number* of G, $\gamma(G)$, is the minimum cardinality of a dominating set of G. An *independent set* in a graph is a set of pairwise non-adjacent vertices. The *independence number* of G, denoted by $\alpha(G)$, is the maximum size of an independent set.

Theorem 4.3. *Let A be a Noetherian S-act. Then the following assertions hold:*
(i) $\mathrm{Max}(A)$ *is both independent and dominating set in $G(A)$.*
(ii) $\alpha(G(A)) = |\mathrm{Max}(A)|$.
(iii) $\gamma(G(A)) \le \alpha(G(A))$.

Proof. (i) Trivial.
(ii) Using (i), $\alpha(G(A)) \ge |\mathrm{Max}(A)|$. Suppose that $|\mathrm{Max}(A)| = n$. Let $\alpha(G(A)) = m > n$ and $W = \{C_1, C_2, \ldots, C_m\}$ be an independent set in $G(A)$ of size m. It follows from the hypothesis that there are distinct subacts C_i and C_j in W contained in a same maximal subact. Thus $C_i \cup C_j \neq A$ which is a contradiction.
(iii) Follows from (i) and (ii).

$\square$

Theorem 4.4. *Let A be an Artinian S-act. Then $\gamma(G(A)) = 1$ or 2.*

Proof. If A has only one minimal subact, say M, then $\{M\}$ is a dominating set and so $\gamma(G(A)) = 1$. If $\mathrm{Min}(A) = \{M_i : i \in I\}$ with $|I| \ge 2$, then the set $\{\bigcup_{i\in I, i\neq j} M_i, M_j\}$ forms a dominating set in the graph $G(A)$ and hence $\gamma(G(A)) \le 2$. $\square$

Recall that a *cut edge* of a graph is a edge whose deletion (the endpoints stay in place) from the graph increases the number of components.

Theorem 4.5. *Let A be an S-act and e be a cut edge with end-point B_1 and B_2. Then one end-point is a minimal subact and the other one is a maximal subact .*

Proof. It is clear that $B_1 \cup B_2 \neq A$. If $B_1 \cup B_2 \neq B_1, B_2$, then $B_1 - B_1 \cup B_2 - B_2$ is a path, which is a contradiction. Thus either $B_1 \cup B_2 = B_1$ or $B_1 \cup B_2 = B_2$. Suppose that $B_1 \cup B_2 = B_2$, then $B_1 \subset B_2$. If B_1 is not minimal and $C \subset B_1$, then $C \cup B_2 \neq A$ whence $B_1 - C - B_2$ is a path, a contradiction. If B_2 is not maximal and $B_2 \subset C$, then $B_2 \cup C = C \neq A$, that is, C and B_2 are adjacent. Moreover, since $B_1 \subset B_2$, B_1 and C are adjacent. Therefore, $B_1 - C - B_2$ is a path which is a contradiction. □

Note that the 2.2 shows that the converse is not true.

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