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# ON THE $m$-EXTENSION DUAL COMPLEX FIBONACCI $p$-NUMBERS

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Abstract. In this paper, we introduced $m$-extension dual complex Fibonacci $p$-numbers. We established the properties of $m$-extension dual complex Fibonacci $p$-numbers. They are connected to complex Fibonacci numbers, complex Fibonacci $p$-numbers and dual complex Fibonacci $p$-numbers.

## 1. Introduction

The Fibonacci numbers, $F_n$ are defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 2$$

with initial terms

$$F_1 = F_2 = 1.$$

The Fibonacci numbers, $F_n$ and golden mean (golden ratio),

$$\tau = \lim_{n \longrightarrow \infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$$

have materialized in several stream of arts, sciences, computer sciences, high energy physics, information and coding theory [15, 16, 4, 6, 7].

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The Fibonacci $p$-numbers [15] are defined by the recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

with $n > p+1$, for a given integer $p = 0,\, 1,\, 2,\, 3,\, \cdots$ and initial terms

$$F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1.$$

The Fibonacci $p$-numbers, $F_p(n)$ coincide with classical Fibonacci numbers, $F_n$ for $p = 1$. e.g. $F_1(n) = F_n$.
E. Kocer EG et al., [9] introduced the $m$-extension of Fibonacci $p$-numbers which satisfy the recurrence relation

$$F_{p,m}(n) = mF_{p,m}(n-1) + F_{p,m}(n-p-1)$$

with initial terms

$$F_{p,m}(1) = a_1, F_{p,m}(2) = a_2, F_{p,m}(3) = a_3, \cdots, F_{p,m}(p+1) = a_{p+1}$$

where $p(\geq 0)$ is integer, $m(> 0)$, $n > p+1$ and $a_1,\, a_2,\, a_3,\, \cdots$ ,$\mathrm{a}_{p+1}$ are arbitrary real or complex numbers.
A complex number is of the form $a+ib$ where $a$ and $b$ are real numbers and $i^2 = -1$. It is usually represented by $z$. The value $a$ is called the real part which is denoted by $Re(z)$ and $b$ is called the imaginary part and denoted by $Im(z)$.
The complex Fibonacci numbers [12] are defined by the recurrence relation:

$$F^{\star}_n = F^{\star}_{n-1} + F^{\star}_{n-2} \text{ for } n \geqslant 2$$

with initial terms

$$F^{\star}_0 = i, F^{\star}_1 = 1+i$$

where $i$ is the imaginary unit which satisfies $i^2 = -1$ and

$$F^{\star}_n = F_n + iF_{n+1}.$$

Clifford Algebra is a powerful mathematical tool that offers a natural and direct way to model geometric objects and their transformations. It makes geometric objects (points, lines and planes) into basic elements of computation and defines few universal operators that are applicable to all types of geometric elements. In 19th century, Cliffored [5] defined new number system by using the notion $\varepsilon^2 = 0$ and $\varepsilon \neq 0$ i.e. $\varepsilon$ is an nilpotent number. This number system is known as dual number system and dual number are represented in the form $A = a + \varepsilon a^{\star}$ where $a$ and $a^{\star}$ are real numbers and $\varepsilon$ is an nilpotent number. Kotelnikov [10] and Study [17] generalized the first applications of dual numbers in mechanics. It has also applications in algebraic geometry, kinematics,

number theory, theory of relativity etc. [11, 2, 13].
Dual complex numbers [11] are defined as follows:

$$W = \{w = z_1 + \varepsilon z_2 : z_1, z_2 \in \mathbf{C} \text{ where } \varepsilon^2 = 0 \text{ and } \varepsilon \neq 0 \}.$$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then any dual complex number can be written as follows:
$w = x_1 + iy_1 + \varepsilon x_2 + i\varepsilon y_2$ where $i^2 = -1$, $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $(i\varepsilon)^2 = 0$.
The real and dual quaternions form a division ring but dual complex numbers form a commutative ring with characteristics zero. The multiplication of dual complex numbers gives the structure of two dimensional complex Clifford Algebra and four dimensional real Clifford Algebra. The base elements of dual complex numbers satisfies the commutative multiplication scheme which are shown in the **Table 1.**

**Table 1: Multiplication scheme of dual complex numbers**

| x | 1 | i | $\varepsilon$ | i$\varepsilon$ |
|---|---|---|---|---|
| 1 | 1 | i | $\varepsilon$ | i$\varepsilon$ |
| i | i | -1 | i$\varepsilon$ | -$\varepsilon$ |
| $\varepsilon$ | $\varepsilon$ | i$\varepsilon$ | 0 | 0 |
| i$\varepsilon$ | i$\varepsilon$ | -$\varepsilon$ | 0 | 0 |

Five different conjugation operates on dual complex numbers [11] in the following manner:
Let $w = z_1 + \varepsilon z_2 = x_1 + iy_1 + \varepsilon x_2 + i\varepsilon y_2$, $z_2 \neq 0$,
$w^{\star 1} = (x_1 - iy_1) + \varepsilon(x_2 - iy_2) = z_1^\star + \varepsilon z_2^\star$, Complex-conjugation,
$w^{\star 2} = (x_1 + iy_1) - \varepsilon(x_2 + iy_2) = z_1 - \varepsilon z_2$, Dual-conjugation,
$w^{\star 3} = (x_1 - iy_1) - \varepsilon(x_2 - iy_2) = z_1^\star - \varepsilon z_2^\star$, Coupled-conjugation,
$w^{\star 4} = (x_1 - iy_1)(1 - \frac{\varepsilon(x_2+iy_2)}{(x_1+iy_1)}) = z_1^\star(1 - \frac{\varepsilon z_2}{z_1})$, Dual-complex-conjugation,
$w^{\star 5} = (x_2 + iy_2) - \varepsilon(x_1 + iy_1) = z_2 - \varepsilon z_1$, Anti-dual-conjugation.
The norm of dual complex numbers defined in the following manner:
$N_{w^{\star 1}} = \|w \times w^{\star 1}\| = \sqrt{|z_1^2| + 2\varepsilon Re(z_1 z_2^\star)}$,
$N_{w^{\star 2}} = \|w \times w^{\star 2}\| = \sqrt{z_1^2}$,
$N_{w^{\star 3}} = \|w \times w^{\star 3}\| = \sqrt{|z_1^2| - 2i\varepsilon Im(z_1 z_2^\star)}$,
$N_{w^{\star 4}} = \|w \times w^{\star 4}\| = \sqrt{|z_1^2|}$,
$N_{w^{\star 5}} = \|w \times w^{\star 5}\| = \sqrt{z_1 z_2 + \varepsilon(z_2^2 - z_1^2)}$.
In 2017, Gungor and Azak [8] defined the dual complex Fibonacci numbers as follow: $DF_n^\star = F_n + iF_{n+1} + \varepsilon(F_{n+2} + iF_{n+3})$ where the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfies the condition
$i^2 = -1$, $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $(i\varepsilon)^2 = 0$.
In 2018, Aydin Torunbalci [1] defined the dual complex $k$-Fibonacci

numbers as follow:

$$DCF_{k,n} = F_{k,n} + iF_{k,n+1} + \varepsilon(F_{k,n+2} + iF_{k,n+3})$$

where the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfies the condition
$i^2 = -1$, $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $(i\varepsilon)^2 = 0$.
In 2019, Prasad [12] introduced complex Fibonacci $p$-numbers by the following recurrence relation:

$$F_p^\star(n) = F_p^\star(n-1) + F_p^\star(n-p-1)$$

with $n > p+1$ and initial terms

$$F_p^\star(0) = i, F_p^\star(1) = F_p^\star(2) = \cdots = F_p^\star(p) = 1 + i.$$

In 2021, Prasad [14] introduced dual complex Fibonacci $p$-numbers by the following recurrence relation:

$$DF_p^\star(n) = F_p^\star(n-1) + \varepsilon F_p^\star(n-p-1) = F_p(n-1) + iF_p(n) + \varepsilon(F_p(n-p-1) + iF_p(n-p))$$

with $n > p+1$ and initial terms

$$F_p^\star(0) = i, F_p^\star(1) = F_p^\star(2) = \cdots = F_p^\star(p) = 1 + i,$$

and the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfies the condition

$$i^2 = -1,\ \varepsilon \neq 0,\ \varepsilon^2 = 0,\ (i\varepsilon)^2 = 0$$

.
In this paper, we introduce $m$-extension dual complex Fibonacci $p$-numbers by the following recurrence relation:

$$DF_{p,m}^\star(n) = mF_{p,m}^\star(n-1) + \varepsilon F_{p,m}^\star(n-p-1) \quad (1.1)$$

$$= m(F_{p,m}(n-1) + iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p))$$

with $n > p+1$ and initial terms

$$F_{p,m}^\star(n) = m^{n-1}, n = 1, 2, 3, 4, \cdots, p+1$$

and the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfies the condition

$$i^2 = -1,\ \varepsilon \neq 0,\ \varepsilon^2 = 0,\ (i\varepsilon)^2 = 0.$$

## 2. On the $m$-extension dual complex Fibonacci $p$-numbers

**Definition 2.1.** On the $m$-extension dual complex Fibonacci $p$-numbers are given by the following recurrence relation:

$$DF_{p,m}^\star(n) = mF_{p,m}^\star(n-1) + \varepsilon F_{p,m}^\star(n-p-1)$$

$$= m(F_{p,m}(n-1) + iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p))$$

$$= mF_{p,m}(n-1) + imF_{p,m}(n) + \varepsilon F_{p,m}(n-p-1) + i\varepsilon F_{p,m}(n-p) \quad (2.1)$$

with $n > p+1$ and initial terms

$$F^{\star}_{p,m}(n) = m^{n-1}, n = 1, 2, 3, 4, \cdots, p+1$$

and the basis $\{1, i, \varepsilon, i\varepsilon\}$ satisfies the condition
$i^2 = -1$, $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $(i\varepsilon)^2 = 0$.
With the addition and multiplication by real scalar of two $m$-extension dual complex Fibonacci $p$-numbers gives the $m$-extension dual complex Fibonacci $p$-numbers.

**Definition 2.2.** The addition and subtraction of the $m$-extension dual complex Fibonacci $p$-numbers are defined by
$DF^{\star}_{p,m}(n_1) \pm DF^{\star}_{p,m}(n_2) = mF^{\star}_{p,m}(n_1-1) + \varepsilon F^{\star}_{p,m}(n_1-p-1) \pm mF^{\star}_{p,m}(n_2-1) + \varepsilon F^{\star}_{p,m}(n_2-p-1)$
$= m(F_{p,m}(n_1-1) + iF_{p,m}(n_1)) + \varepsilon(F_{p,m}(n_1-p-1) + iF_{p,m}(n_1-p)) \pm m(F_{p,m}(n_2-1) + iF_{p,m}(n_2)) + \varepsilon(F_{p,m}(n_2-p-1) + iF_{p,m}(n_2-p))$
$= m(F_{p,m}(n_1-1) \pm F_{p,m}(n_2-1)) + im(F_{p,m}(n_1) \pm F_{p,m}(n_2)) + \varepsilon(F_{p,m}(n_1-p-1) \pm F_{p,m}(n_2-p-1)) + i\varepsilon(F_{p,m}(n_1-p) \pm F_{p,m}(n_2-p))$.

**Definition 2.3.** The multiplication of a $m$-extension dual complex Fibonacci $p$-number by real scalar $\lambda$ is defined by

$$\lambda DF^{\star}_{p,m}(n) = \lambda mF_{p,m}(n-1) + i\lambda mF_{p,m}(n) + \varepsilon\lambda F_{p,m}(n-p-1) + i\varepsilon\lambda F_{p,m}(n-p).$$

**Definition 2.4.** The multiplication of two $m$-extension dual complex Fibonacci $p$-numbers is defined by

$DF^{\star}_{p,m}(n_1)DF^{\star}_{p,m}(n_2) = [mF_{p,m}(n_1-1) + imF_{p,m}(n_1) + \varepsilon F_{p,m}(n_1-p-1) + i\varepsilon F_{p,m}(n_1-p)][mF_{p,m}(n_2-1) + imF_{p,m}(n_2) + \varepsilon F_{p,m}(n_2-p-1) + i\varepsilon F_{p,m}(n_2-p)]$
$= m^2[F_{p,m}(n_1-1)F_{p,m}(n_2-1) - F_{p,m}(n_1)F_{p,m}(n_2)] + im[F_{p,m}(n_1-1)F_{p,m}(n_2) + F_{p,m}(n_1)F_{p,m}(n_2-1)] + m\varepsilon[F_{p,m}(n_1-1)F_{p,m}(n_2-p-1) + F_{p,m}(n_1-p-1)F_{p,m}(n_2-1) - F_{p,m}(n_1)F_{p,m}(n_2-p) - F_{p,m}(n_2)F_{p,m}(n_1-p)]$
$+ im\varepsilon[F_{p,m}(n_1-1)F_{p,m}(n_2-p) + F_{p,m}(n_1)F_{p,m}(n_2-p-1) + F_{p,m}(n_2)F_{p,m}(n_1-p-1) + F_{p,m}(n_1-p)F_{p,m}(n_2-1)] = DF^{\star}_{p,m}(n_2)DF^{\star}_{p,m}(n_1)$.

On the $m$-extension dual complex Fibonacci $p$-numbers also provide the properties of norm and conjugation.

**Proposition 2.5.** *Five kinds of conjugation can be defined for dual complex numbers* [11] *. Therefore, conjugation of the $m$-extension dual complex Fibonacci $p$-number is defined in five different ways in the* *following manner:*

$$DF^{\star 1}_{p,m}(n) = m(F_{p,m}(n-1) - iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) - iF_{p,m}(n-p)), Complex - conjugation \quad (2.2)$$

$$DF^{\star 2}_{p,m}(n) = m(F_{p,m}(n-1) + iF_{p,m}(n)) - \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p)), Dual - conjugation \quad (2.3)$$

$$DF^{\star 3}_{p,m}(n) = m(F_{p,m}(n-1) - iF_{p,m}(n)) - \varepsilon(F_{p,m}(n-p-1) - iF_{p,m}(n-p)), Coupled - conjugation \quad (2.4)$$

$$DF^{\star 4}_{p,m}(n) = m(F_{p,m}(n-1) - iF_{p,m}(n))(1 - \varepsilon\frac{F_{p,m}(n-p-1) + iF_{p,m}(n-p)}{m(F_{p,m}(n-1) + iF_{p,m}(n))}), Dual - complex - conjugation \quad (2.5)$$

$$DF^{\star 5}_{p,m}(n) = m(F_{p,m}(n-p-1) + iF_{p,m}(n-p)) - \varepsilon(F_{p,m}(n-1) + iF_{p,m}(n)), Anti - dual - conjugation \quad (2.6)$$

*Some properties of conjugations for $m$-extension dual complex Fibonacci $p$-numbers are given in the following theorems.*

**Theorem 2.6.** Let $DF^{\star 1}_{p,m}(n)$, $DF^{\star 2}_{p,m}(n)$, $DF^{\star 3}_{p,m}(n)$, $DF^{\star 4}_{p,m}(n)$, $DF^{\star 5}_{p,m}(n)$ be five kinds of conjugation of $m$-extension dual complex Fibonacci $p$-number. Thereby, we give the following relations.

$$DF^{\star}_{p,m}(n)DF^{\star 1}_{p,m}(n) = m^2[F^2_{p,m}(n-1) + F^2_{p,m}(n)] + 2m\varepsilon[F_{p,m}(n-1)F_{p,m}(n-p-1) + F_{p,m}(n)F_{p,m}(n-p)] \quad (2.7)$$

$$DF^{\star}_{p,m}(n)DF^{\star 2}_{p,m}(n) = m^2[F^2_{p,m}(n-1) - F^2_{p,m}(n)] + i2m^2F_{p,m}(n-1)F_{p,m}(n) \quad (2.8)$$

$$DF^{\star}_{p,m}(n)DF^{\star 3}_{p,m}(n) = m^2[F^2_{p,m}(n-1) + F^2_{p,m}(n)] + i2m\varepsilon[F_{p,m}(n-p)F_{p,m}(n-1) - F_{p,m}(n)F_{p,m}(n-p-1)] \quad (2.9)$$

$$DF^{\star}_{p,m}(n)DF^{\star 4}_{p,m}(n) = m^2[F^2_{p,m}(n-1) + F^2_{p,m}(n)] \quad (2.10)$$

$$DF^{\star}_{p,m}(n)DF^{\star 5}_{p,m}(n) = m^2[F_{p,m}(n-1)F_{p,m}(n-p-1) - F_{p,m}(n)F_{p,m}(n-p)]$$
$$+im^2[F_{p,m}(n-1)F_{p,m}(n-p) + F_{p,m}(n)F_{p,m}(n-p-1)]$$
$$+m\varepsilon[F^2_{p,m}(n-p-1) + F^2_{p,m}(n) - F^2_{p,m}(n-1) - F^2_{p,m}(n-p)]$$

$$+i2m\varepsilon[F_{p,m}(n-p-1)F_{p,m}(n-p) - F_{p,m}(n-1)F_{p,m}(n)] \tag{2.11}$$

$$DF^{\star}_{p,m}(n) + DF^{\star 1}_{p,m}(n) = 2[mF_{p,m}(n-1) + \varepsilon F_{p,m}(n-p-1)] \tag{2.12}$$

$$DF^{\star}_{p,m}(n) + DF^{\star 2}_{p,m}(n) = 2m[F_{p,m}(n-1) + iF_{p,m}(n)] \tag{2.13}$$

$$DF^{\star}_{p,m}(n) + DF^{\star 3}_{p,m}(n) = 2[mF_{p,m}(n-1) + i\varepsilon F_{p,m}(n-p)] \tag{2.14}$$

$$(F_{p,m}(n-1) + iF_{p,m}(n))DF^{\star 4}_{p,m}(n) = [F_{p,m}(n-1) - iF_{p,m}(n)]DF^{\star 2}_{p,m}(n) \tag{2.15}$$

$$\varepsilon DF^{\star}_{p,m}(n) + mDF^{\star 5}_{p,m}(n) = m^2[F_{p,m}(n-p-1) + iF_{p,m}(n-p)] \tag{2.16}$$

$$mDF^{\star}_{p,m}(n) - \varepsilon DF^{\star 5}_{p,m}(n) = m^2[F_{p,m}(n-1) + iF_{p,m}(n)] \tag{2.17}$$

$$DF^{\star}_{p,m}(n) + DF^{\star}_{p,m}(n+p) = DF^{\star}_{p,m}(n+p+1) \tag{2.18}$$

*Proof.* $DF^{\star}_{p,m}(n)DF^{\star 1}_{p,m}(n) = [m(F_{p,m}(n-1) + iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p))][m(F_{p,m}(n-1) - iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) - iF_{p,m}(n-p))] = m^2[F^2_{p,m}(n-1) + F^2_{p,m}(n)] + 2m\varepsilon[F_{p,m}(n-1)F_{p,m}(n-p-1) + F_{p,m}(n)F_{p,m}(n-p)]$,by equations (2) and (3).

$DF^{\star}_{p,m}(n)DF^{\star 2}_{p,m}(n) = [m(F_{p,m}(n-1) + iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p))][m(F_{p,m}(n-1) + iF_{p,m}(n)) - \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p))] = m^2[F^2_{p,m}(n-1) - F^2_{p,m}(n)] + i2m^2F_{p,m}(n-1)F_{p,m}(n)$, by equations (2) and (4).

$DF^{\star}_{p,m}(n)DF^{\star 3}_{p,m}(n) = [m(F_{p,m}(n-1) + iF_{p,m}(n)) + \varepsilon(F_{p,m}(n-p-1) + iF_{p,m}(n-p))][m(F_{p,m}(n-1) - iF_{p,m}(n)) - \varepsilon(F_{p,m}(n-p-1) - iF_{p,m}(n-p))] = m^2[F^2_{p,m}(n-1) + F^2_{p,m}(n)] + i2m\varepsilon[F_{p,m}(n-p)F_{p,m}(n-1) - F_{p,m}(n)F_{p,m}(n-p-1)]$, by equations (2) and (5).

$DF^{\star}_{p,m}(n)DF^{\star 4}_{p,m}(n) = [m(F_{p,m}(n-1)+iF_{p,m}(n))+\varepsilon(F_{p,m}(n-p-1)+ iF_{p,m}(n-p))][m(F_{p,m}(n-1)-iF_{p,m}(n))(1-\varepsilon\frac{F_{p,m}(n-p-1)+iF_{p,m}(n-p)}{m(F_{p,m}(n-1)+iF_{p,m}(n))})] = m^2[F^2_{p,m}(n-1)+F^2_{p,m}(n)]$, by equation (2) and (6).

$DF^{\star}_{p,m}(n)DF^{\star 5}_{p,m}(n) = [m(F_{p,m}(n-1)+iF_{p,m}(n))+\varepsilon(F_{p,m}(n-p-1)+iF_{p,m}(n-p))][m(F_{p,m}(n-p-1)+iF_{p,m}(n-p))-\varepsilon(F_{p,m}(n-1)+iF_{p,m}(n))] = m^2[F_{p,m}(n-1)F_{p,m}(n-p-1)-F_{p,m}(n)F_{p,m}(n-p)] + im^2[F_{p,m}(n-1)F_{p,m}(n-p)+F_{p,m}(n)F_{p,m}(n-p-1)]+m\varepsilon[F^2_{p,m}(n-p-1)+F^2_{p,m}(n)-F^2_{p,m}(n-1)-F^2_{p,m}(n-p)]+i2m\varepsilon[F_{p,m}(n-p-1)F_{p,m}(n-p)-F_{p,m}(n-1)F_{p,m}(n)]$, by equations (2) and (7).

$DF^{\star}_{p,m}(n)+DF^{\star 1}_{p,m}(n) = [m(F_{p,m}(n-1)+iF_{p,m}(n))+\varepsilon(F_{p,m}(n-p-1)+iF_{p,m}(n-p)]+[m(F_{p,m}(n-1)-iF_{p,m}(n))+\varepsilon(F_{p,m}(n-p-1)-iF_{p,m}(n-p)] = 2[mF_{p,m}(n-1)+\varepsilon F_{p,m}(n-p-1)]$, by equations (2) and (3).

$DF^{\star}_{p,m}(n)+DF^{\star 2}_{p,m}(n) = [m(F_{p,m}(n-1)+iF_{p,m}(n)+\varepsilon(F_{p,m}(n-p-1)+iF_{p,m}(n-p)]+[m(F_{p,m}(n-1)+iF_{p,m}(n))-\varepsilon(F_{p,m}(n-p-1)+iF_{p,m}(n-p))] = 2m[F_{p,m}(n-1)+iF_{p,m}(n)]$, by equations (2) and (4).

$DF^{\star}_{p,m}(n)+DF^{\star 3}_{p,m}(n) = [m(F_{p,m}(n-1)+iF_{p,m}(n))+\varepsilon(F_{p,m}(n-p-1)+iF_{p,m}(n-p))]+[m(F_{p,m}(n-1)-iF_{p,m}(n))-\varepsilon(F_{p,m}(n-p-1)-iF_{p,m}(n-p))] = 2[mF_{p,m}(n-1)+i\varepsilon F_{p,m}(n-p)]$, by equations (2) and (5).

$(F_{p,m}(n-1)+iF_{p,m}(n))DF^{\star 4}_{p,m}(n) = (F_{p,m}(n-1)+iF_{p,m}(n))[m(F_{p,m}(n-1)-iF_{p,m}(n))(1-\varepsilon\frac{F_{p,m}(n-p-1)+iF_{p,m}(n-p)}{m(F_{p,m}(n-1)+iF_{p,m}(n))})] =$

$\mathrm{F}_{p,m}(n-1)-iF_{p,m}(n)$

$[m(F_{p,m}(n-1)+iF_{p,m}(n))-\varepsilon(F_{p,m}(n-p-1)+iF_{p,m}(n-p))] = [F_{p,m}(n-1)-iF_{p,m}(n)]DF^{\star 2}_{p,m}(n)$, by equations (4) and (6).

$\varepsilon DF^{\star}_{p,m}(n)+mDF^{\star 5}_{p,m}(n) = m\varepsilon(F_{p,m}(n-1)+iF_{p,m}(n))+m^2(F_{p,m}(n-p-1)+iF_{p,m}(n-p))-m\varepsilon(F_{p,m}(n-1)+iF_{p,m}(n)) = m^2[F_{p,m}(n-p-1)+iF_{p,m}(n-p)]$, by equations (2) and (7).

$mDF^{\star}_{p,m}(n)-\varepsilon DF^{\star 5}_{p,m}(n) = m[m(F_{p,m}(n-1)+iF_{p,m}(n))+\varepsilon(F_{p,m}(n-p-1)+iF_{p,m}(n-p))]-\varepsilon[m(F_{p,m}(n-p-1)+iF_{p,m}(n-p))-\varepsilon(F_{p,m}(n-1)+iF_{p,m}(n))] = m^2[F_{p,m}(n-1)+iF_{p,m}(n)]$, by equations (2) and (7).
We know that $DF^{\star}_{p,m}(n) = DF^{\star}_{p,m}(n-1)+DF^{\star}_{p,m}(n-p-1)$
We put $n = n+p+1$, we get

$DF^{\star}_{p,m}(n+p+1) = DF^{\star}_{p,m}(n+p)+DF^{\star}_{p,m}(n) = DF^{\star}_{p,m}(n)+DF^{\star}_{p,m}(n+p)$.

Therefore, the norm of $m$-extension dual complex Fibonacci $p$-number, $NDF^{\star}_{p,m}(n)$ is defined in five different ways as follows:

$$NDF^{\star 1}_{p,m}(n) = \|DF^{\star}_{p,m}(n)DF^{\star 1}_{p,m}(n)\|$$

$$= [m^2(F^2_{p,m}(n-1) + F^2_{p,m}(n)) + 2m\varepsilon(F_{p,m}(n-1)F_{p,m}(n-p-1) + F_{p,m}(n)F_{p,m}(n-p))]^{\frac{1}{2}}$$

$$NDF^{\star 2}_{p,m}(n) = \|DF^{\star}_{p,m}(n)DF^{\star 2}_{p,m}(n)\|$$

$$= [(m^2(F^2_{p,m}(n-1) - F^2_{p,m}(n)))^2 + 4m^4F^2_{p,m}(n)F^2_{p,m}(n-1)]^{\frac{1}{2}} = m^2(F^2_{p,m}(n-1) + F^2_{p,m}(n))$$

$$NDF^{\star 3}_{p,m}(n) = \|DF^{\star}_{p,m}(n)DF^{\star 3}_{p,m}(n)\| = m^2(F^2_{p,m}(n-1) + F^2_{p,m}(n))$$

$$NDF^{\star 4}_{p,m}(n) = \|DF^{\star}_{p,m}(n)DF^{\star 4}_{p,m}(n)\| = m(F^2_{p,m}(n-1) + F^2_{p,m}(n))^{\frac{1}{2}}$$

$(NDF^{\star 5}_{p,m}(n))^2 = (\|DF^{\star}_{p,m}(n)DF^{\star 5}_{p,m}(n)\|)^2=$
$m^4[F^2_{p,m}(n-1)F^2_{p,m}(n-p-1) + F^2_{p,m}(n)F^2_{p,m}(n-p) + F^2_{p,m}(n-1)F^2_{p,m}(n-p) + F^2_{p,m}(n)F^2_{p,m}(n-p-1)]$
$+2m^3\varepsilon[F_{p,m}(n-1)F^3_{p,m}(n-p-1) - F^3_{p,m}(n-1)F_{p,m}(n-p-1) + F_{p,m}(n)F^3_{p,m}(n-p) - F^3_{p,m}(n)F_{p,m}(n-p) + F_{p,m}(n-1)F_{p,m}(n-p-1)F^2_{p,m}(n-p) - F^2_{p,m}(n)F_{p,m}(n-1)F_{p,m}(n-p-1) + F_{p,m}(n)F_{p,m}(n-p)F^2_{p,m}(n-p-1) - F^2_{p,m}(n-1)F_{p,m}(n)F_{p,m}(n-p)]$.

□

**Theorem 2.7.** *Let $n_1, n_2 \geq 0$ the Honsberger identity for $m$-extension dual complex Fibonacci $p$-numbers $DF^{\star}_{p,m}(n_1)$, $DF^{\star}_{p,m}(n_2)$ is given by* $DF^{\star}_{p,m}(n_1)DF^{\star}_{p,m}(n_2) + DF^{\star}_{p,m}(n_1+1)DF^{\star}_{p,m}(n_2+1) = m^2[F_{p,m}(n_1-1)F_{p,m}(n_2-1) - F_{p,m}(n_1+1)F_{p,m}(n_2+1)] + im^2[F_{p,m}(n_1-1)F_{p,m}(n_2) + F_{p,m}(n_1)F_{p,m}(n_2-1) + F_{p,m}(n_1+1)F_{p,m}(n_2) + F_{p,m}(n_1)F_{p,m}(n_2+1)]$
$+\varepsilon m[F_{p,m}(n_1-1)F_{p,m}(n_2-p-1) + F_{p,m}(n_1-p-1)F_{p,m}(n_2-1) - F_{p,m}(n_1+1)F_{p,m}(n_2-p+1) - F_{p,m}(n_1-p+1)F_{p,m}(n_2+1)]$
$+i\varepsilon m[F_{p,m}(n_1)F_{p,m}(n_2-p-1) + F_{p,m}(n_1-1)F_{p,m}(n_2-p) + F_{p,m}(n_1-p)F_{p,m}(n_2-1) + F_{p,m}(n_1-p-1)F_{p,m}(n_2) + F_{p,m}(n_1+1)F_{p,m}(n_2-p) + F_{p,m}(n_1)F_{p,m}(n_2-p+1) + F_{p,m}(n_1-p)F_{p,m}(n_2+1) + F_{p,m}(n_1-p+1)F_{p,m}(n_2)]$.

*Proof.* $DF^{\star}_{p,m}(n_1)DF^{\star}_{p,m}(n_2)+DF^{\star}_{p,m}(n_1+1)DF^{\star}_{p,m}(n_2+1) = [m(F_{p,m}(n_1-1) + iF_{p,m}(n_1)) + \varepsilon F_{p,m}(n_1-p-1) + i\varepsilon F_{p,m}(n_1-p)][m(F_{p,m}(n_2-1) + iF_{p,m}(n_2)) + \varepsilon F_{p,m}(n_2-p-1) + i\varepsilon F_{p,m}(n_2-p)]$
$+[m(F_{p,m}(n_1) + iF_{p,m}(n_1+1)) + \varepsilon F_{p,m}(n_1-p) + i\varepsilon F_{p,m}(n_1-p+1)]$

$[m(F_{p,m}(n_2)+iF_{p,m}(n_2+1))+\varepsilon F_{p,m}(n_2-p)+i\varepsilon F_{p,m}(n_2-p+1)]$
$=m^2[F_{p,m}(n_1-1)F_{p,m}(n_2-1)-F_{p,m}(n_1+1)F_{p,m}(n_2+1)]+im^2[F_{p,m}(n_1-1)F_{p,m}(n_2)+F_{p,m}(n_1)F_{p,m}(n_2-1)+F_{p,m}(n_1+1)F_{p,m}(n_2)+F_{p,m}(n_1)F_{p,m}(n_2+1)]$
$+\varepsilon\mathrm{m}[\mathrm{F}_{p,m}(n_1-1)F_{p,m}(n_2-p-1)+F_{p,m}(n_1-p-1)F_{p,m}(n_2-1)-F_{p,m}(n_1+1)F_{p,m}(n_2-p+1)-F_{p,m}(n_1-p+1)F_{p,m}(n_2+1)]$
$+\mathrm{i}\varepsilon\mathrm{m}[\mathrm{F}_{p,m}(n_1)F_{p,m}(n_2-p-1)+F_{p,m}(n_1-1)F_{p,m}(n_2-p)+F_{p,m}(n_1-p)F_{p,m}(n_2-1)+F_{p,m}(n_1-p-1)F_{p,m}(n_2)+F_{p,m}(n_1+1)F_{p,m}(n_2-p)+F_{p,m}(n_1)F_{p,m}(n_2-p+1)+F_{p,m}(n_1-p)F_{p,m}(n_2+1)+F_{p,m}(n_1-p+1)F_{p,m}(n_2)]$. □

**Theorem 2.8.** *Let $n_1, n_2 \geq 0$ the D'Ocagne's identity for $m$-extension dual complex Fibonacci $p$-numbers $DF^{\star}_{p,m}(n_1)$, $DF^{\star}_{p,m}(n_2)$ is given by* $DF^{\star}_{p,m}(n_1)DF^{\star}_{p,m}(n_2+1)-DF^{\star}_{p,m}(n_1+1)DF^{\star}_{p,m}(n_2)=m^2[F_{p,m}(n_1-1)F_{p,m}(n_2)-F_{p,m}(n_1)F_{p,m}(n_2+1)+F_{p,m}(n_1+1)F_{p,m}(n_2)-F_{p,m}(n_1)F_{p,m}(n_2-1)]+im^2[F_{p,m}(n_1-1)F_{p,m}(n_2+1)-F_{p,m}(n_1+1)F_{p,m}(n_2-1)]$
$+\varepsilon m[F_{p,m}(n_1-1)F_{p,m}(n_2-p)-F_{p,m}(n_1)F_{p,m}(n_2-p+1)+F_{p,m}(n_1-p-1)F_{p,m}(n_2)-F_{p,m}(n_1-p)F_{p,m}(n_2+1)-F_{p,m}(n_1)F_{p,m}(n_2-p-1)+F_{p,m}(n_1+1)F_{p,m}(n_2-p)-F_{p,m}(n_1-p)F_{p,m}(n_2-1)+F_{p,m}(n_1-p+1)F_{p,m}(n_2)]$
$+i\varepsilon m[F_{p,m}(n_1-1)F_{p,m}(n_2-p+1)+F_{p,m}(n_1-p-1)F_{p,m}(n_2+1)-F_{p,m}(n_1+1)F_{p,m}(n_2-p-1)-F_{p,m}(n_1-p+1)F_{p,m}(n_2-1)]$.

*Proof.* $DF^{\star}_{p,m}(n_1)DF^{\star}_{p,m}(n_2+1)-DF^{\star}_{p,m}(n_1+1)DF^{\star}_{p,m}(n_2)=[m(F_{p,m}(n_1-1)+iF_{p,m}(n_1))+\varepsilon F_{p,m}(n_1-p-1)+i\varepsilon F_{p,m}(n_1-p)][m(F_{p,m}(n_2)+iF_{p,m}(n_2+1))+\varepsilon F_{p,m}(n_2-p)+i\varepsilon F_{p,m}(n_2-p+1)]-[m(F_{p,m}(n_1)+iF_{p,m}(n_1+1))+\varepsilon F_{p,m}(n_1-p)+i\varepsilon F_{p,m}(n_1-p+1)][m(F_{p,m}(n_2-1)+iF_{p,m}(n_2))+\varepsilon F_{p,m}(n_2-p-1)+i\varepsilon F_{p,m}(n_2-p)]$
$=m^2[F_{p,m}(n_1-1)F_{p,m}(n_2)-F_{p,m}(n_1)F_{p,m}(n_2+1)+F_{p,m}(n_1+1)F_{p,m}(n_2)-F_{p,m}(n_1)F_{p,m}(n_2-1)]+im^2[F_{p,m}(n_1-1)F_{p,m}(n_2+1)-F_{p,m}(n_1+1)F_{p,m}(n_2-1)]$
$+\varepsilon\mathrm{m}[\mathrm{F}_{p,m}(n_1-1)F_{p,m}(n_2-p)-F_{p,m}(n_1)F_{p,m}(n_2-p+1)+F_{p,m}(n_1-p-1)F_{p,m}(n_2)-F_{p,m}(n_1-p)F_{p,m}(n_2+1)-F_{p,m}(n_1)F_{p,m}(n_2-p-1)+F_{p,m}(n_1+1)F_{p,m}(n_2-p)-F_{p,m}(n_1-p)F_{p,m}(n_2-1)+F_{p,m}(n_1-p+1)F_{p,m}(n_2)]$
$+\mathrm{i}\varepsilon\mathrm{m}[\mathrm{F}_{p,m}(n_1-1)F_{p,m}(n_2-p+1)+F_{p,m}(n_1-p-1)F_{p,m}(n_2+1)-F_{p,m}(n_1+1)F_{p,m}(n_2-p-1)-F_{p,m}(n_1-p+1)F_{p,m}(n_2-1)]$. □

**Theorem 2.9.** Let $DF^{\star}_{p,m}(n)$ be $m$-extension dual complex Fibonacci $p$-number. For $n \geq 1$, Cassini's identity for $DF^{\star}_{p,m}(n)$ is given by $DF^{\star}_{p,m}(n-1)DF^{\star}_{p,m}(n+1)-D^2F^{\star}_{p,m}(n)=m^2[F^2_{p,m}(n)-F^2_{p,m}(n-1)+F_{p,m}(n-2)F_{p,m}(n)-F_{p,m}(n+1)F_p(n-1)]+im^2[F_{p,m}(n+1)F_{p,m}(n-2)-F_{p,m}(n)F_{p,m}(n-1)]$

$+\varepsilon m[F_{p,m}(n-2)F_{p,m}(n-p)-F_{p,m}(n-1)F_{p,m}(n-p+1)+F_{p,m}(n)F_{p,m}(n-p-2)-F_{p,m}(n-p-1)F_{p,m}(n+1)-2F_{p,m}(n-1)F_{p,m}(n-p-1)+2F_{p,m}(n)F_{p,m}(n-p)]$
$+i\varepsilon m[F_{p,m}(n-2)F_{p,m}(n-p+1)-F_{p,m}(n-1)F_{p,m}(n-p)+F_{p,m}(n-p-2)F_{p,m}(n+1)-F_{p,m}(n-p-1)F_{p,m}(n)]$.

*Proof.* $DF^{\star}_{p,m}(n-1)DF^{\star}_{p,m}(n+1)-D^2F^{\star}_{p,m}(n)$
$=[m(F_{p,m}(n-2)+iF_{p,m}(n-1))+\varepsilon(F_{p,m}(n-p-2)+iF_{p,m}(n-p-1))][m(F_{p,m}(n)+iF_{p,m}(n+1))+\varepsilon(F_{p,m}(n-p)+iF_{p,m}(n-p+1))]$-$[m(F_{p,m}(n-1)+iF_{p,m}(n))+\varepsilon(F_{p,m}(n-p-1)+iF_{p,m}(n-p))]^2$
$=m^2[F^2_{p,m}(n)-F^2_{p,m}(n-1)+F_{p,m}(n-2)F_{p,m}(n)-F_{p,m}(n+1)F_p(n-1)]+\mathrm{i}m^2[F_{p,m}(n+1)F_{p,m}(n-2)-F_{p,m}(n)F_{p,m}(n-1)]$
$+\varepsilon\mathrm{m}[\mathrm{F}_{p,m}(n-2)F_{p,m}(n-p)-F_{p,m}(n-1)F_{p,m}(n-p+1)+F_{p,m}(n)F_{p,m}(n-p-2)-F_{p,m}(n-p-1)F_{p,m}(n+1)-2F_{p,m}(n-1)F_{p,m}(n-p-1)+2F_{p,m}(n)F_{p,m}(n-p)]$
$+\mathrm{i}\varepsilon\mathrm{m}[\mathrm{F}_{p,m}(n-2)F_{p,m}(n-p+1)-F_{p,m}(n-1)F_{p,m}(n-p)+F_{p,m}(n-p-2)F_{p,m}(n+1)-F_{p,m}(n-p-1)F_{p,m}(n)]$. □

**Theorem 2.10.** *Let $DF^{\star}_{p,m}(n+r)$ be the m-extension dual complex Fibonacci p-number. For $n\geq 1$, Catalan's identity for $DF^{\star}_{p,m}(n+r)$ is given by* $DF^{\star}_{p,m}(n+r-1)DF^{\star}_{p,m}(n+r+1)-D^2F^{\star}_{p,m}(n+r)=m^2[F^2_{p,m}(n+r)-F^2_{p,m}(n+r-1)+F_{p,m}(n+r-2)F_{p,m}(n+r)-F_{p,m}(n+r+1)F_p(n+r-1)]+im^2[F_{p,m}(n+r+1)F_{p,m}(n+r-2)-F_{p,m}(n+r)F_{p,m}(n+r-1)]$
$+\varepsilon m[F_{p,m}(n+r-2)F_{p,m}(n+r-p)-F_{p,m}(n+r-1)F_{p,m}(n+r-p+1)+F_{p,m}(n+r)F_{p,m}(n+r-p-2)-F_{p,m}(n+r-p-1)F_{p,m}(n+r+1)-2F_{p,m}(n+r-1)F_{p,m}(n+r-p-1)+2F_{p,m}(n+r)F_{p,m}(n+r-p)]$
$+i\varepsilon m[F_{p,m}(n+r-2)F_{p,m}(n+r-p+1)-F_{p,m}(n+r-1)F_{p,m}(n+r-p)+F_{p,m}(n+r-p-2)F_{p,m}(n+r+1)-F_{p,m}(n+r-p-1)F_{p,m}(n+r)]$.

*Proof.* $DF^{\star}_{p,m}(n+r-1)DF^{\star}_{p,m}(n+r+1)-D^2F^{\star}_{p,m}(n+r)=[m(F_{p,m}(n+r-2)+iF_{p,m}(n+r-1))+\varepsilon(F_{p,m}(n+r-p-2)+iF_{p,m}(n+r-p-1))][m(F_{p,m}(n+r)+iF_{p,m}(n+r+1))+\varepsilon(F_{p,m}(n+r-p)+iF_{p,m}(n+r-p+1))]$-$[m(F_{p,m}(n+r-1)+iF_{p,m}(n+r))+\varepsilon(F_{p,m}(n+r-p-1)+iF_{p,m}(n+r-p))]^2$
$=m^2[F^2_{p,m}(n+r)-F^2_{p,m}(n+r-1)+F_{p,m}(n+r-2)F_{p,m}(n+r)-F_{p,m}(n+r+1)F_p(n+r-1)]+\mathrm{i}m^2[F_{p,m}(n+r+1)F_{p,m}(n+r-2)-F_{p,m}(n+r)F_{p,m}(n+r-1)]$
$+\varepsilon\mathrm{m}[\mathrm{F}_{p,m}(n+r-2)F_{p,m}(n+r-p)-F_{p,m}(n+r-1)F_{p,m}(n+r-p+1)+F_{p,m}(n+r)F_{p,m}(n+r-p-2)-F_{p,m}(n+r-p-1)F_{p,m}(n+r+1)-2F_{p,m}(n+r-1)F_{p,m}(n+r-p-1)+2F_{p,m}(n+r)F_{p,m}(n+r-p)]$
$+\mathrm{i}\varepsilon\mathrm{m}[\mathrm{F}_{p,m}(n+r-2)F_{p,m}(n+r-p+1)-F_{p,m}(n+r-1)F_{p,m}(n+r-p)+F_{p,m}(n+r-p-2)F_{p,m}(n+r+1)-F_{p,m}(n+r-p-1)F_{p,m}(n+r)]$. □

## 3. Conclusion

In this paper, we introduced $m$-extension dual complex Fibonacci $p$-numbers. We established the properties of $m$-extension dual complex Fibonacci $p$-numbers. We also established the Honsberger identity, D'Ocagne's identity, Cassini's identity and Catalan's identity for $m$-extension dual complex Fibonacci $p$-numbers. I hope that these results will be useful in applied mathematics, quantum mechanics, quantum physics, Lie groups, number theory, Kinematics and differential equations like dual complex $k$-Fibonacci numbers [1], dual complex Fibonacci $p$-numbers [14], dual-complex numbers and their holomorphic functions [11] and dual-complex Jacobsthal quaternions [2] etc.

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