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# $\delta$-$n$-IDEALS OF COMMUTATIVE RINGS

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Abstract. Let $R$ be a commutative ring with non-zero identity, and $\delta : \mathcal{I(R)} \to \mathcal{I(R)}$ be an ideal expansion where $\mathcal{I(R)}$ is the set of all ideals of $R$. In this paper, we introduce the concept of $\delta$-$n$-ideals which is an extension of $n$-ideals in commutative rings. We call a proper ideal $I$ of $R$ a $\delta$-$n$-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in \delta(I)$. For example, an ideal expansion $\delta_1$ is defined by $\delta_1(I) = \sqrt{I}$. In this case, a $\delta_1$-$n$-ideal $I$ is said to be a quasi $n$-ideal or equivalently, $I$ is quasi $n$-ideal if $\sqrt{I}$ is an $n$-ideal. A number of characterizations and results with many supporting examples concerning this new class of ideals are given. In particular, we present some results regarding quasi $n$-ideals. Furthermore, we use $\delta$-$n$-ideals to characterize fields and UN-rings.

## 1. Introduction

Throughout this paper, we assume that all rings are commutative with non-zero identity. Since prime ideals have a vital place in commutative algebra, various generalizations of prime ideals have studied by many authors for years. (see for example, [2]-[8], [9], [13], [18]-[21], [15]-[20]). In 1947, the concept of quasi-primary ideals were defined by Fuchs in [9]. Recall that an ideal $I$ of a ring $R$ is called quasi-primary if $\sqrt{I}$ is a prime ideal. Many years later, in 2007, a different generalization of prime ideals were introduced. According to Badawi's celebrated paper [3], a proper ideal $I$ of a ring $R$ is called a 2-absorbing ideal if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or

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$bc \in I$. Afterwards, in [6], Badawi, Tekir and Yetkin Celikel generalized 2-absorbing ideals to 2-absorbing primary ideals by the following definition. A proper ideal $I$ of a ring $R$ is called a 2-absorbing primary if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Later, in [16], Tekir et. al. defined a proper ideal $I$ of a ring $R$ to be 2-absorbing quasi primary if $\sqrt{I}$ is a 2-absorbing ideal of $R$. On the other hand, in 2001, D. Zhao [21] introduced the concept of expansions of ideals and $\delta$-primary ideals of commutative rings. Let $R$ be a ring. By $\mathcal{I}(\mathcal{R})$, we denote the set of all ideals of $R$. A function $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ is an ideal expansion if it assigns to each ideal $I$ of $R$ to another ideal $\delta(I)$ of the same ring with the following properties: (i) $I \subseteq \delta(I)$ and (ii) if $I \subseteq J$ for some ideals $I, J$ of $R$, then $\delta(I) \subseteq \delta(J)$. For example, $\delta_0$ is the identity function where $\delta_0(I) = I$, and $\delta_1$ is defined by $\delta_1(I) = \sqrt{I}$ for all ideal $I$ of $R$, also $\delta_+(I) = I + J$ for some $J \in \mathcal{I}(\mathcal{R})$ and $\delta_*(I) = (I : P)$ for some $P \in \mathcal{I}(\mathcal{R})$ for all $I \in \mathcal{I}(\mathcal{R})$. For more examples of expansion functions, the reader may refer to [5]. Recall also from [21] that a proper ideal $I$ of $R$ is said to be a $\delta$-primary if $ab \in I$ and $a \notin I$ for some $a, b \in R$, then $b \in \delta(I)$. The notion of $\delta$-primary ideals and its generalizations have drawn considerable interest and have been studied in many studies. See, for example, [4], [5], [8], [13], [18], [19]. Afterwards, in 2015, R. Mohamadian [14] defined the concept of $r$-ideals in commutative rings. He called a proper ideal $I$ of a ring $R$ an $r$-ideal if whenever $ab \in I$ and $Ann(a) = 0$ for some $a, b \in R$, then $b \in I$. As a recent study, [17], Tekir, Koc and Oral introduced $n$-ideals which is a subclass of $r$-ideals as follows: A proper ideal $I$ of $R$ is called $n$-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$.

Motivated from these studies mentioned above, in this paper, we call a proper ideal $I$ of a ring $R$ a quasi $n$-ideal if $\sqrt{I}$ is an $n$-ideal, or equivalently if whenever $a, b \in R$ with $ab \in I$, then either $a \in \sqrt{0}$ or $b \in \sqrt{I}$. Generalizing this idea by using an ideal expansion $\delta$, we call a proper ideal $I$ of a ring $R$ a $\delta$-$n$-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in \delta(I)$. If $\delta(I)$ is an $n$-ideal of $R$, then $I$ is a $\delta$-$n$-ideal of $R$. Unlike quasi $n$-ideals which is a particular case $\delta = \delta_1$, the converse of this inclusion may not hold in general. (see Example 2.6).

The aim of this article is to introduce and characterize $\delta$-$n$-ideals which is an extension of $n$-ideals of commutative rings and to investigate relationships among some classical ideals in the literature such as prime, $\delta$-primary, $n$-ideal and this new class of ideals. It is clear from the definition that every $n$-ideal is a $\delta$-$n$-ideal for all ideal expansions $\delta$. We start with Example 2.2 to show that this generalization is proper.

Also, Example 2.6 is given to show that a prime ideal needs not to be a $\delta$-$n$-ideal. Among many results in this paper, in Proposition 2.4, we obtain some certain conditions for a prime ideal is to be a $\delta$-$n$-ideal. In Theorem 2.8, we conclude some equivalent characterizations for $\delta$-$n$-ideals. In Theorem 2.9, we determine all rings of which every proper ideal is a $\delta$-$n$-ideal. We show in Proposition 2.15 that a maximal quasi $n$-ideal of $R$ is a prime ideal of $R$. In Proposition 2.10, we show that an integral domain has no non-zero $\delta$-$n$-ideal for expansion of ideals $\delta$ of $R$ provided $\delta(I)$ is proper for all $I \in \mathcal{I}(\mathcal{R})$. Also, it is shown in Theorem 2.11 that if $\delta(0) = 0$, then $R$ is a field if and only if $R$ is a von Neumann regular ring and $\{0\}$ is a $\delta$-$n$-ideal. Furthermore, we investigate $\delta$-$n$-ideals under various contexts of constructions such as homomorphic images, direct products, localizations and in idealization rings. (See Propositions 2.20, 2.26, 2.28 and Remark 2.27).

For the sake of completeness, we state some notations and definitions which we will need throughout. For a proper ideal $I$ a ring $R$, $\sqrt{I}$ denotes the radical of $I$ defined by $\{r \in R :$ there exists $n \in \mathbb{N}$ with $r^n \in I\}$ and for $x \in R$, by $(I : x)$, we denote the set of $\{r \in R : rx \in I\}$. We used standard definitions and terminologies in this paper. For the other notations, terminologies and applications not mentioned in the paper, the readers are referred to [11].

## 2. Properties of $\delta$-$n$-ideals

The aim of this section is to study the $\delta$-$n$-ideals in commutative rings. We begin with our main definition.

**Definition 2.1.** Let $R$ be a ring and $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ an ideal expansion. A proper ideal $I$ of $R$ is called a $\delta$-$n$-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in \delta(I)$. In particular, if $\delta = \delta_1$ defined by $\delta_1(I) = \sqrt{I}$ for all $I \in \mathcal{I}(\mathcal{R})$, then $I$ is called a quasi $n$-ideal.

It is clear that any $n$-ideal is a $\delta$-$n$-ideal (in particular, a quasi $n$-ideal). However, the following examples show that the converse of this implication is not true in general.

**Example 2.2. (A $\delta$-$n$-ideal which is not an $n$-ideal)** Consider the ideal $I = (x^3)R_1$ of $R_1 = \mathbb{Z}_4[X]$. Let $R = R_1/I$. Define the expansion function of $\mathcal{I}(\mathcal{R})$ with $\delta(K) = K + \frac{(2,x)R_1}{I}$ and let $J = (x+1)R_1/I$. We show that $J$ is a $\delta$-$n$-ideal that is not a $n$-ideal of $R$. Since $((x+1)+I)(1+I) \in J$ but $((x+1)+I) \notin \sqrt{0_R} = \frac{(2,x)R_1}{I}$ and $(1+I) \notin J$, $J$ is not an $n$-ideal of $R$. Note that $\delta(J) = \frac{(x+1)R_1}{I} + \frac{(2,x)R_1}{I}$. Thus $1 + I \in \delta(J)$, that is, $\delta(J) = R$. Thus $J$ is a $\delta$-$n$-ideal.

**Example 2.3. (A quasi $n$-ideal which is not an $n$-ideal)** Let $R$ be the idealization ring $\mathbb{Z}(+)\mathbb{Z}$ and $I = 0(+)p\mathbb{Z}$ where $p$ is a prime integer. Note that $\sqrt{I} = \sqrt{0}(+)\mathbb{Z} = 0(+)\mathbb{Z}$. Since 0 is an $n$-ideal of $\mathbb{Z}$, from [17, Proposition 2.27], $\delta_1(I) = \sqrt{I} = 0(+)\mathbb{Z}$ is an $n$-ideal of $R$. Hence $I = 0(+)p\mathbb{Z}$ is a quasi $n$-ideal of $R$ by Proposition 2.4 (5). However, $I$ is not an $n$-ideal of $R$ as $(0,1),(p,0) \in R$ with $(p,0)\cdot(0,1) = (0,p) \in 0(+)p\mathbb{Z}$ but neither $(p,0) \in \sqrt{0_R} = \sqrt{0_{\mathbb{Z}}}(+)\mathbb{Z} = 0(+)\mathbb{Z}$ nor $(0,1) \in 0(+)p\mathbb{Z}$.

We justify some relationships among $\delta$-$n$-ideal and the other class of ideals in the literature such as prime, $\delta$-primary, $n$-ideal.

**Proposition 2.4.** *Let $\delta$ be an expansion of ideals of $R$ and $I$ a proper ideal of $R$ with $\delta(I) \neq R$.*

(1) If $I$ is a $\delta$-$n$-ideal of $R$, then $I \subseteq \sqrt{0}$.
(2) A prime ideal $I$ is a $\delta$-$n$-ideal if and only if $I = \sqrt{0}$.
(3) If $\delta(I)$ is an $n$-ideal of $R$, then $I$ is a $\delta$-$n$-ideal of $R$.
(4) If $I$ is a $\delta$-$n$-ideal of $R$ provided $\sqrt{\delta(I)} = \delta(\sqrt{I})$, then so is $\sqrt{I}$.
(5) $I$ is a quasi $n$-ideal of $R$ if and only if $\sqrt{I}$ is a $n$-ideal of $R$.
(6) Let $I \subseteq \sqrt{0}$. If $I$ is a $\delta$-primary ideal of $R$, then $I$ is a $\delta$-$n$-ideal of $R$. The converse is also true if $I = \sqrt{0}$.

*Proof.* (1) Assume that $I \nsubseteq \sqrt{0}$. Then there is an element $a \in R$ with $a \in I \backslash \sqrt{0}$. Since $a = a \cdot 1 \in I$ and $a \notin \sqrt{0}$, we conclude $1 \in \delta(I)$, a contradiction. Thus $I \subseteq \sqrt{0}$.

(2) If $I$ is a prime ideal of $R$, then $\sqrt{0} \subseteq I$. Since the converse inclusion hold by (1), we have the required equality. Conversely, let $I = \sqrt{0}$. Then $I$ is an $n$-ideal of $R$ by [17, Proposition 2.8], and thus, $I$ is a $\delta$-$n$-ideal of $R$.

(3) Suppose that $ab \in I$ and $a \notin \sqrt{0}$ for some $a, b \in R$. Since $I \subseteq \delta(I)$ and $\delta(I)$ is an $n$-ideal, we conclude $b \in \delta(I)$. Thus $I$ is a $\delta$-$n$-ideal of $R$.

(4) Let $a, b \in R$ with $ab \in \sqrt{I}$ and $a \notin \sqrt{0}$. Then $(ab)^n = a^n b^n \in I$ for some positive integer $n$. Since $I$ is $\delta$-$n$-ideal and $a^n \notin \sqrt{0}$, we have $b^n \in \delta(I)$. Hence $b \in \sqrt{\delta(I)} = \delta(\sqrt{I})$. Thus $\sqrt{I}$ is a $\delta$-$n$-ideal of $R$.

(5) Follows from (4) as $\sqrt{\sqrt{I}} = \sqrt{I}$.

(6) Suppose $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$. Since $I$ is a $\delta$-primary and clearly $a \notin I$, we have $a \in \delta(I)$, as needed. In particular, it is clear from the definitions that $\sqrt{0}$ is a $\delta$-primary ideal if and only if $\sqrt{0}$ is a $\delta$-$n$-ideal. □

Note that the converse of Proposition 2.4 (1) is not satisfied in general. See the following example.

**Example 2.5.** Let $R = \mathbb{Z}_{15}[X]$ and $M(I)$ be the intersection of all maximal ideals containing $I$ of $R$. Note that $M$ is an expansion of ideals (See [21]). Consider the ideal $I = \{0\}$ of $R$. Since $3 \cdot 5 \in I$ but neither $3 \in \sqrt{0}$ nor $5 \in M(I)$, $I$ is not an $M$-$n$-ideal of $R$.

We show in the next example that a prime ideal needs not to be a $\delta$-$n$-ideal of $R$ in general which means that the condition $I = \sqrt{0}$ in Proposition 2.4 (2) is crucial.

**Example 2.6.** Let $\delta_+ : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ be an expansion of ideals of $R = \mathbb{Z}$ defined by $\delta_+(J) = J + q\mathbb{Z}$ where $q$ is prime integer with $(p, q) = 1$. Consider the ideal $I = p\mathbb{Z}$ where $p$ is a prime integer of $R$. Then $I$ is a $\delta_+$-$n$-ideal of $R$ that is neither $\delta_0$-$n$-ideal ($n$-ideal) nor $\delta_1$-$n$-ideal (quasi $n$-ideal) of $R$. Indeed, $p \cdot 1 \in I$ but $p \notin \sqrt{0}$ and $1 \notin \delta_0(I) = I$ and also $1 \notin \delta_1(I) = \sqrt{I}$.

Also, observe that the converse of Proposition 2.4 (3) may not be true. Indeed, in Example 2.6, we show that $I = p\mathbb{Z}$ is a $\delta_+$-$n$-ideal of $\mathbb{Z}$ where $p$ is a prime integer. But $\delta_+(I)$ is not an $n$-ideal of $\mathbb{Z}$ since it is not proper.

In view of Proposition 2.4 and [17, Corollary 2.9], we have the following equivalent statements.

**Corollary 2.7.** *For any ring $R$ and an expansion of fuction $\delta$, the following are equivalent.*

(1) $\sqrt{0}$ is a prime ideal of $R$.
(2) $\sqrt{0}$ is an $n$-ideal of $R$.
(3) $\sqrt{0}$ is an $\delta$-$n$-ideal of $R$.
(4) $\sqrt{0}$ is an $\delta$-primary ideal of $R$.

The next theorem gives a characterization for $\delta$-$n$-ideal of $R$ in terms of the ideals of $R$.

**Theorem 2.8.** *For a proper ideal $I$ of $R$ and an expansion of function $\delta$, the following statements are equivalent.*

(1) $I$ is a $\delta$-$n$-ideal of $R$.
(2) $(I : x) \subseteq \sqrt{0}$ for all $x \in R - \delta(I)$.
(3) Whenever $a \in R$ and an ideal $K$ of $R$ with $aK \subseteq I$, then $a \in \sqrt{0}$ or $K \subseteq \delta(I)$.
(4) Whenever $J$ and $K$ are ideals of $R$ with $JK \subseteq I$, then $J \nsubseteq \sqrt{0}$ or $K \subseteq \delta(I)$.

*Proof.* (1) $\Rightarrow$(2) Suppose that $I$ is a $\delta$-$n$-ideal of $R$. Let $b \in (I : x)$. Since $I$ is $\delta$-$n$-ideal, $xb \in I$ and $x \notin \delta(I)$, we have $b \in \sqrt{0}$. Thus $(I : x) \subseteq \sqrt{0}$.

(2)⇒(3) Suppose that $aK \subseteq I$ but $K \nsubseteq \delta(I)$. Then there exists an element $x$ of $K$ with $x \notin \delta(I)$. Hence $a \in (I : x)$ which implies that $a \in \sqrt{0}$ by (2).

(3)⇒(4) Suppose that $JK \subseteq I$ and $J \nsubseteq \sqrt{0}$. Then there is some $a \in R$ such that $a \in J \cap (R - \sqrt{0})$. Since $aK \subseteq I$ and $a \notin \sqrt{0}$, we conclude $K \subseteq \delta(I)$ by (3).

(4) ⇒(1) Let $a, b \in R$ with $ab \in I$ and for some $a \notin \sqrt{0}$. Put $J = (a)$ and $K = (b)$ in (4). So we have the result by our assumption. □

Next, we justify some equivalent characterizations for rings of which every proper ideal is $\delta$-$n$-ideal.

**Theorem 2.9.** *For every expansion function $\delta$ of ideals of $R$, the following are equivalent.*

(1) Every proper principal ideal is a $\delta$-$n$-ideal of $R$.
(2) Every proper ideal is a $\delta$-$n$-ideal of $R$.
(3) $\sqrt{0}$ is the unique prime ideal of $R$.
(4) $R$ is a quasi local ring with maximal element $M = \sqrt{0}$. (i.e., $R$ is a UN-ring)

*Proof.* (1)⇒(2) Let $I$ be a proper ideal of $R$ and $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$. Put $J = (ab)$. Since $J$ is a $\delta$-$n$ -ideal, we conclude that $b \in \delta(J) \subseteq \delta(I)$, as needed.

(2)⇒(3) Suppose that $I$ is a prime ideal of $R$. Then it is $\delta$-$n$-ideal by our assumption, and thus $I = \sqrt{0}$ by Proposition 2.4 (2).

(3)⇒(4) Follows from [7, Proposition 2 (3)].

(4)⇒(1) Suppose that $(R, \sqrt{0})$ is a quasi local ring. Then every element of $R$ is either unit or nilpotent. Let $I = (x)$ be a principal ideal and let $a, b \in R$, $ab \in (x)$ and $a \notin \sqrt{0}$. Then $a$ is unit and so $b \in (x) = I \subseteq \delta(I)$. Thus $I$ is a $\delta$-$n$-ideal. □

**Proposition 2.10.** *Let $\delta$ be an expansion of $\mathcal{I}(\mathcal{R})$ such that $\delta(I) \neq R$ for every $I \in \mathcal{I}(\mathcal{R})$.*

(1) If $R$ is an integral domain, then $\{0\}$ is the only $\delta$-$n$-ideal of $R$.
(2) Let $R$ be a reduced ring which is not an integral domain. Then $R$ has no $\delta$-$n$ -ideal.

*Proof.* (1) Suppose that $R$ is an integral domain. Then $\sqrt{0} = \{0\}$ is prime, so it is a $\delta$-$n$-ideal of $R$ by Proposition 2.4 (2). Now, assume that $I$ is a nonzero $\delta$-$n$-ideal of $R$. Then $I \subseteq \sqrt{0} = 0$ by Proposition 2.4 (1) which is a contradiction.

(2) Assume that $I$ is a $\delta$-$n$ -ideal of $R$. Hence, $I \subseteq \delta(I) \subseteq \sqrt{0} = \{0\}$ by Proposition 2.4 (1). Since $\{0\}$ is not prime ideal of $R$, there exists $a, b \in R \backslash \{0\}$ with $ab \in \{0\}$, a contradiction. □

Recall from [12] that a von Neumann regular ring is a ring such that for all $a \in R$, there exists an $x \in R$ satisfying $a = a^2 x$. In particular, $R$ is a Boolean ring if for all $a \in R$, $a = a^2$. We have the following result.

**Theorem 2.11.** *Let $\delta$ be an ideal expansion of ideals of $R$ with $\delta(0) = \{0\}$. Then $R$ is a field if and only if $R$ is a von Neumann regular ring and $\{0\}$ is a $\delta$-$n$-ideal.*

*Proof.* Suppose that $R$ is a von Neumann regular ring and $\{0\}$ is a $\delta$-$n$-ideal. Then clearly $\sqrt{0} = \{0\}$. We show that every nonzero element $a$ of $R$ is unit. Since $R$ is von Neumann regular, there exists $x \in R$ such that $a = a^2 x$. Hence $a(1 - ax) = 0$. Since $a \notin \sqrt{0}$, we conclude that $1 - ax \in \delta(0) = 0$. Thus $ax = 1$, as needed. Therefore $R$ is a field. The converse part is clear by [17, Theorem 2.15]. □

**Proposition 2.12.** *Let $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$ and $I$ be proper ideal of $R$ with $\delta(\delta(I)) = \delta(I)$. Then the following hold.*

(1) If $I$ is $\delta$-$n$-ideal and $a \in R \backslash \sqrt{0}$, then $\delta(I : a) = \delta(I)$.
(2) Let $I, J$ are $\delta$-$n$-ideals of $R$ with $\delta(\delta(I)) = \delta(I)$ and $\delta(\delta(J)) = \delta(J)$. If $IK = JK$ where and $K \nsubseteq \sqrt{0}$ for some ideal $K$ of $R$, then $\delta(I) = \delta(J)$.
(3) Let $K$ be an ideal of $R$ with $K \nsubseteq \sqrt{0}$. If $IK$ is a $\delta$-$n$-ideal with $\delta(\delta(IK)) = \delta(IK)$, then $\delta(IK) = \delta(I)$.

*Proof.* (1) Suppose that $I$ is a $\delta$-$n$-ideal and $a \notin \sqrt{0}$. Since $I \subseteq (I : a)$, clearly we have $\delta(I) \subseteq \delta(I : a)$. Let $x \in (I : a)$. Since $xa \in I$ and $a \notin \sqrt{0}$, we conclude $x \in \delta(I)$. Thus $(I : a) \subseteq \delta(I)$ and so $\delta(I : a) \subseteq \delta(\delta(I)) = \delta(I)$. As the converse inclusion always holds, we conclude the equality.

(2) Note that $IK = JK \subseteq I, J$. Since $IK \subseteq I$ and $K \nsubseteq \sqrt{0}$, we have $J \subseteq \delta(I)$ by Theorem 2.8. Since $JK \subseteq J$ and $K \nsubseteq \sqrt{0}$, we have $I \subseteq \delta(J)$ again by Theorem 2.8. Thus $\delta(I) = \delta(J)$ as $\delta(\delta(I)) = \delta(I)$ and $\delta(\delta(J)) = \delta(J)$.

(3) Since $IK \subseteq I$, we have $\delta(IK) \subseteq \delta(I)$. Since $IK \subseteq IK$ and $K \nsubseteq \sqrt{0}$, we conclude $I \subseteq \delta(IK)$ by Theorem 2.8. It follows $\delta(I) \subseteq \delta(\delta(IK)) = \delta(IK) \subseteq \delta(I)$, and thus $\delta(IK) = \delta(I)$. □

In view of Proposition 2.12, we conclude the following result for quasi $n$-ideals.

**Proposition 2.13.** *Let $I$ be proper ideal of $R$. Then*

(1) If $I$ is a quasi $n$-ideal, then $\sqrt{(I:x)} = (\sqrt{I}:x)$ for all $x \in R \backslash \sqrt{I}$.
(2) If $I$ is a quasi $n$-ideal, then $\sqrt{(I:a)} = (\sqrt{I}:a) = \sqrt{I}$ for all $a \notin \sqrt{0}$.
(3) If $I$, $J$ are quasi $n$-ideals of $R$ and $K$ is an ideal with $K \nsubseteq \sqrt{0}$ with $IK = JK$, then $\sqrt{I} = \sqrt{J}$.
(4) Let $K$ be an ideal of $R$ with $K \nsubseteq \sqrt{0}$. If $IK$ is quasi $n$-ideal, then $\sqrt{IK} = \sqrt{I}$.

*Proof.* (1) Let $a \in (\sqrt{I}:x)$. Then $ax \in \sqrt{I}$. Since clearly $a^n x^n \in I$ for some positive integer $n$, $I$ is a quasi-$n$-ideal and $x^n \notin I$, we conclude $a^n \in \sqrt{0}$, that is, $a \in \sqrt{0} \subseteq \sqrt{(I:x)}$. Since the inverse inclusion is always satisfied, we get the equality.

(2) From Proposition 2.12 (1), we have $\sqrt{(I:a)} = \sqrt{I}$. Let $b \in (\sqrt{I}:a)$. Hence $ab \in \sqrt{I}$ implies $a^n b^n \in I$ for some positive integer $n$. Since $a^n \notin \sqrt{0}$, we conclude $b \in \sqrt{I}$. Thus we have $(\sqrt{I}:a) \subseteq \sqrt{I}$ and we conclude the required equality.

(3) and (4) are clear from Proposition 2.12 as $\sqrt{\sqrt{I}} = \sqrt{I}$. □

**Lemma 2.14.** *Let $\delta$ be an expansion of $\mathcal{I}(\mathcal{R})$ and $I$ be an ideal of $R$. If $I$ is a $\delta$-$n$-ideal of $R$ such that $(\delta(I):x) \subseteq \delta(I:x) \neq R$ for all $x \in R \backslash \delta(I)$, then $(I:x)$ is a $\delta$-$n$-ideal of $R$. In particular, if $I$ is a quasi $n$-ideal of $R$, then $(I:x)$ is a quasi $n$-ideal of $R$ for all $x \in R \backslash \sqrt{I}$.*

*Proof.* Suppose that $ab \in (I:x)$ and $a \notin \sqrt{0}$. Since $abx \in I$ and $I$ is a $\delta$-$n$-ideal, we conclude that $bx \in \delta(I)$. Thus $b \in (\delta(I):x) \subseteq \delta(I:x)$, so we are done. For the "in particular" case, observe that the inclusion $(\delta_1(I):x) \subseteq \delta_1(I:x)$ is satisfied for all $x \in R \backslash \delta_1(I)$ by Proposition 2.13 (1). Therefore the claim follows from the general case. □

**Theorem 2.15.** *Let $I$ be a proper ideal of $R$.*

(1) Let $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$ with $(\delta(I):x) \subseteq \delta(I:x) \neq R$ for all $x \in R \backslash \delta(I)$. If $I$ is a maximal $\delta$-$n$-ideal of $R$ where $x \in R \backslash \delta(I)$, then $I = \sqrt{0}$ is a prime ideal of $R$.
(2) If $I$ is a maximal quasi $n$-ideal of $R$, then $I = \sqrt{0}$ is a prime ideal of $R$.

*Proof.* (1) Suppose that $I$ is a maximal $\delta$-$n$-ideal of $R$. We show that $I$ is prime. Let $ab \in I$ and $a \notin I$. Hence $(I:a)$ is a $\delta$-$n$-ideal of $R$ by Lemma 2.14. Thus $(I:a) = I$ from the maximality of $I$. It means $b \in I$, and thus $I$ is a prime ideal of $R$. From Proposition 2.4 (2), we conclude that $I = \sqrt{0}$.

(2) Let $I$ be a a maximal quasi $n$-ideal of $R$. Then $\sqrt{I:x} = (\sqrt{I}:x)$ for all $x \in R\backslash\sqrt{I}$ by Proposition 2.13 (1). By Lemma 2.14, $(I:a)$ is a quasi $n$-ideal of $R$ and the maximality of $I$ implies that $(I:a) = I$ and similar to the general case, $I$ is prime and $I = \sqrt{0}$ by Proposition 2.4 (2). □

Now, we are ready for the following result.

**Theorem 2.16.** *For a ring $R$, the following statements hold.*

1. Let $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$ with $(\delta(J):x) \subseteq \delta(J:x) \neq R$ for all ideal $J$ of $R$ and $x \in R\backslash\delta(J)$. Then $R$ has a $\delta$-$n$-ideal if and only if $\sqrt{0}$ is a prime ideal of $R$.
2. $R$ has a quasi $n$-ideal of $R$ if and only if $\sqrt{0}$ is a prime ideal of $R$.

*Proof.* (1) Let $I$ is a $\delta$-$n$-ideal of $R$ and $W = \{J : J \text{ is an } n\text{-ideal of } R\}$. Then $W$ is a nonempty partially ordered set by the set inclusion. Take a chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$ of $W$. We show that $I = \cup_{i\in\Lambda} I_i$ is a $\delta$-$n$-ideal of $R$. Suppose that $ab \in I$ and $a \notin I$ for some $a, b \in R$. Then $ab \in I_k$ for some $k \in \Lambda$. Since $a \notin I_k$ and $I_k$ is $\delta$-$n$-ideal, we conclude that $b \in \sqrt{0}$. Thus $I = \cup_{i\in\Lambda} I_i$ is an upper bound of the chain. So, there exists a maximal element $M$ of $W$ by the Zorn's Lemma. It follows that $M = \sqrt{0}$ is a prime ideal by Theorem 2.15. The converse part follows from Proposition 2.4 (2).

(2) From Proposition 2.13 (1), we have $(\delta_1(J):x) = \delta_1(J:x)$. Thus, the result is clear from (1). □

Generalizing nilpotent elements in a ring, we call an element $a \in R$ a $\delta$-nilpotent if $a \in \delta(0)$. Hence, we have the following result.

**Proposition 2.17.** *Let $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$. Then $\sqrt{0}$ is a $\delta$-$n$-ideal of $R$ if and only if every zero-divisor of the quotient ring $R/\sqrt{0}$ is $\delta_q$-nilpotent.*

*Proof.* Suppose that $\overline{a} = a + \sqrt{0}$ is a zero-divisor of $R/\sqrt{0}$. Then $\overline{a}\overline{b} = (a+\sqrt{0})(b+\sqrt{0}) = \sqrt{0}$ for some $\sqrt{0} \neq \overline{b} \in R/\sqrt{0}$. It means $ab \in \sqrt{0}$ but $b \notin \sqrt{0}$. Since $\sqrt{0}$ is a $\delta$-$n$-ideal, we conclude $a \in \delta(\sqrt{0})$. Hence $\overline{a} = a + \sqrt{0} \in \delta(\sqrt{0})/\sqrt{0}$. Now consider the natural epimorphism $\Pi : R \to R/\sqrt{0}$. Note that $\Pi$ is a $\delta\delta_q$-epimorphism. We have $\delta(\sqrt{0})/\sqrt{0} = \delta(\Pi^{-1}(0_{R/\sqrt{0}})) = \Pi^{-1}(\delta_q(0_{R/\sqrt{0}}))$. Since $\Pi$ is epimorphism, then $\delta(\sqrt{0})/\sqrt{0} = \Pi(\delta(\sqrt{0})) = \delta(0_{R/\sqrt{0}})$. Thus $\overline{a} \in \delta_q(0_{R/\sqrt{0}})$; so $\overline{a}$ is $\delta_q$-nilpotent. Conversely, Suppose that $ab \in \sqrt{0}$ and $a \notin \sqrt{0}$ for some $a, b \in R$. Then $\overline{a}\overline{b} = \sqrt{0} = 0_{R/\sqrt{0}}$ but $\overline{a} \neq 0_{R/\sqrt{0}}$. It means that $\overline{b}$

is a zero divisor of $R/\sqrt{0}$. Then $\overline{b}$ is a $\delta_q$-nilpotent from our assumption. Hence $\overline{b} \in \delta_q(0_{R/\sqrt{0}}) = \delta(\sqrt{0})/\sqrt{0}$. So $b + \sqrt{0} = c + \sqrt{0}$ for some $c \in \delta(\sqrt{0})$. It follows $b - c \in \sqrt{0} \subseteq \delta(\sqrt{0})$. Thus $b = (b-c)+c \in \delta(\sqrt{0})$; so $\sqrt{0}$ is a $\delta$-$n$-ideal of $R$. □

An ideal expansion $\delta$ is intersection preserving if it satisfies $\delta(I \cap J) = \delta(I) \cap \delta(J)$ for any $I, J \in \mathcal{I}(\mathcal{R})$, [21].

**Proposition 2.18.** *Let $\delta$ be an ideal expansion which preserves intersection. Then the following statements hold.*

(1) If $I_1, I_2, ..., I_n$ are $\delta$-$n$-ideals of $R$, then $I = \bigcap_{i=1}^{n} I_i$ is a $\delta$-$n$-ideal of $R$.
(2) Let $I_1, I_2, ..., I_n$ be proper ideals of $R$ such that $\delta(I_i)$'s are non-comparable prime ideals of $R$. If $\bigcap_{i=1}^{n} I_i$ is a $\delta$-$n$-ideal of $R$, then $I_i$ is a $\delta$-$n$-ideal of $R$ for all $i = 1, 2, ..., n$.

*Proof.* (1) Let $ab \in I$ and $b \notin \delta(I)$ for some $a, b \in R$. Since $\delta(I) = \cap_{i=1}^{n} \delta(I_i)$, $b \notin \delta(I_k)$ for some $k \in \{1, ..., n\}$. It follows $a \in \sqrt{0}$, and thus $I$ is a $\delta$-$n$-ideal of $R$.

(2) Suppose that $ab \in I_k$ and $a \notin \sqrt{0}$ for some $k \in \{1, 2, ..., n\}$. Choose an element $x \in \left( \prod_{\substack{i=1 \\ i \neq k}}^{n} I_i \right) \setminus \delta(I_k)$. Hence, $abx \in \bigcap_{i=1}^{n} I_i$. Since $\bigcap_{i=1}^{n} I_i$ is a $\delta$-$n$-ideal, we have $bx \in \delta \left( \bigcap_{i=1}^{n} I_i \right) = \bigcap_{i=1}^{n} \delta(I_i) \subseteq \delta(I_k)$ which implies $b \in \delta(I_k)$ as $\delta(I_k)$ is prime, so we are done. □

Consequently, we have the following result for quasi $n$-ideals.

**Corollary 2.19.** *Let $R$ be a ring and $I_1, I_2, ..., I_n$ be proper ideals of $R$. Then we have:*

(1) If $I_i$ 's are quasi $n$-ideals of $R$ for all $i = 1, ..., n$, then so are $I = \bigcap_{i=1}^{n} I_i$ and $\prod_{i=1}^{n} I_i$.
(2) Let $I_i$ be quasi primary ideals of $R$ for all $i = 1, ..., n$ in which their radicals are not comparable.
  **(i):** If $\bigcap_{i=1}^{n} I_i$ is a quasi $n$-ideal of $R$, then so is $I_i$ for each $i = 1, 2, ..., n$.

**(ii):** If $\prod_{i=1}^{n} I_i$ is a quasi $J$-ideal of $R$, then so is $I_i$ for each $i = 1, 2, ..., n$.

*Proof.* Similar to the proof of Proposition 2.18. □

Let $R$ and $S$ be commutative rings and $\delta, \gamma$ be expansion functions of $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}(\mathcal{S})$, respectively. Then a ring homomorphism $f : R \to S$ is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$ for all ideal $J$ of $S$. Let $\gamma_1$ be a radical operation on ideals of $S$ and $\delta_1$ be a radical operation on ideals of $R$. A homomorphism from $R$ to $S$ is an example of $\delta_1\gamma_1$-homomorphism. Additionaly, if $f$ is a $\delta\gamma$-epimorphism and $I$ is an ideal of $R$ containing $\ker(f)$, then $\gamma(f(I)) = f(\delta(I))$.

**Proposition 2.20.** *Let $f : R \to S$ be a $\delta\gamma$-homomorphism, where $\delta$ and $\gamma$ are expansion functions of $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}(\mathcal{S})$, respectively. Then the following hold:*

(1) Let $f$ be a monomorphism. If $J$ is a $\gamma$-$n$-ideal of $S$, then $f^{-1}(J)$ is a $\delta$-$n$-ideal of $R$.
(2) Suppose that $f$ is an epimorphism and $I$ is a proper ideal of $R$ with $\ker(f) \subseteq I$. If $I$ is a $\delta$-$n$-ideal of $R$, then $f(I)$ is a $\gamma$-$n$-ideal of $S$.

*Proof.* (1) Let $ab \in f^{-1}(J)$ for $a, b \in R$. Then $f(ab) = f(a)f(b) \in J$, which implies $f(a) \in \sqrt{0_S}$ or $f(b) \in \gamma(J)$. If $f(a) \in \sqrt{0_S}$, then $a \in \sqrt{0_R}$ as $\ker(f) = \{0\}$. If $f(b) \in \gamma(J)$, then we have $b \in f^{-1}(\gamma(J)) = \delta(f^{-1}(J))$ since $f$ is $\delta\gamma$-homomorphism. Thus $f^{-1}(J)$ is a $\delta$-$n$-ideal of $R$.
(2) Suppose that $a, b \in S$ with $ab \in f(I)$ and $a \notin \sqrt{0_S}$. Since $f$ is an epimorphism, there exist $x, y \in R$ such that $a = f(x)$ and $b = f(y)$. Then clearly we have $x \notin \sqrt{0_R}$ as $a \notin \sqrt{0_S}$. Since $\ker(f) \subseteq I$, $ab = f(xy) \in f(I)$ implies that $xy \in I$. Thus $y \in \delta(I)$; and so $b = f(y) \in f(\delta(I))$. On the other hand, since $\gamma(f(I)) = f(\delta(I))$, we have $b \in \gamma(f(I))$. Thus $f(I)$ is a $\gamma$-$n$-ideal of $S$.

□

Let $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$ and $I$ be an ideal of $R$. Then the function $\delta_q : R/I \to R/I$ is defined by $\delta_q(J/I) = \delta(J)/I$ for all ideals $I \subseteq J$, becomes an expansion function of $\mathcal{I}(R/I)$.

**Corollary 2.21.** *Let $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$ and $J \subseteq I$ proper ideals of $R$. Then the followings hold.*

(1) *If $I$ is a $\delta$-$n$-ideal of $R$, then $I/J$ is a $\delta_q$-$n$-ideal of $R/J$.*

(2) *$I/J$ is a $\delta_q$-n-ideal of $R/J$ and $J \subseteq \sqrt{0_R}$, then $I$ is a $\delta$-n-ideal of $R$.*
(3) *$I/J$ is a $\delta_q$-n-ideal of $R/J$ and $J$ is a $\delta$-n-ideal of $R$ where $\delta(J) \neq R$, then $I$ is a $\delta$-n-ideal of $R$.*
(4) *Let $K$ be a subring of $R$ with $S \nsubseteq I$. If $I$ is a $\delta$-n-ideal of $R$, then $S \cap I$ is a $\delta$-n-ideal of $R$.*

*Proof.* (1) Consider the natural homomorphism $\pi : R \to R/J$. By Proposition 2.20 (2), we have $I/J$ is a $\delta_q$-$n$-ideal of $R/J$ since $\ker(\pi) \subseteq I$.
(2) Let $I/J$ be a $\delta_q$-$n$-ideal of $R/J$ and $J \subseteq \sqrt{0_R}$. Assume that $ab \in I$ and $a \notin \sqrt{0}$ for some $a, b \in R$. Then $ab + J = (a + J)(b + J) \in I/J$ and $a + J \notin \sqrt{0_{R/J}}$. By our assumption, $b + J \in \delta_q(I/J) = \delta(I)/J$, that is, $b \in \delta(I)$.
(3) It is clear by (2) and Proposition 2.4.
(4) Let the injection $i : S \to R$ be defined with $i(a) = a$ for every $a \in S$. Then the proof is clear by Proposition 2.20(1).

□

**Proposition 2.22.** *Let $f : R \to S$ be an epimorphism.*

(1) If $I$ is a quasi $n$-ideal of $R$ with $Kerf \subseteq I$, then $f(I)$ is a quasi $n$-ideal of $S$.
(2) If $J$ is a quasi $n$-ideal of $S$ and $Kerf \subseteq \sqrt{0_R}$, then $f^{-1}(J)$ is a quasi $n$-ideal of $R$.

*Proof.* (1) Suppose that $I$ is a quasi $n$-ideal of $R$. Since $\sqrt{I}$ is a $n$-ideal of $R$ and $Kerf \subseteq I \subseteq \sqrt{I}$, then $f(\sqrt{I})$ is a $n$-ideal of $R$ by [17, Theorem 2.17. (1)]. Let $a, b \in R$ such that $ab \in \sqrt{f(I)}$ and $a \notin \sqrt{0_S}$, then $a^n b^n \in f(I) \subseteq f(\sqrt{I})$ for some positive integer $n$. Since clearly $a^n \notin \sqrt{0_S}$, then $b^n \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Thus $b \in \sqrt{f(I)}$ and $\sqrt{f(I)}$ is an $n$-ideal of $S$. It follows that $f(I)$ is a quasi $n$-ideal of $S$.

(2) Let $J$ be a quasi $n$-ideal of $S$. Then $\sqrt{J}$ is an $n$-ideal of $S$. First, we show that $f^{-1}(\sqrt{J})$ is an $n$-ideal of $R$. Let $a, b \in R$ with $ab \in f^{-1}(\sqrt{J})$ and $a \notin \sqrt{0_R}$. Then $f(a^n)f(b^n) \in J$ for some positive integer $n$. If $f(a^n) \in \sqrt{0_S}$, then $a^n \in Kerf \subseteq \sqrt{0_R}$, a contradiction. Thus $f(a^n) \notin \sqrt{0_S}$. Hence, it implies that $f(b)^n = f(b^n) \in \sqrt{J}$, which follows $b \in f^{-1}(\sqrt{J})$ and $f^{-1}(\sqrt{J})$ is a $n$-ideal of $R$. Now, let $c, d \in R$ such that $c, d \in \sqrt{f^{-1}(J)}$ and $c \notin \sqrt{0_R}$. Then $c^m d^m \in f^{-1}(J) \subseteq f^{-1}(\sqrt{J})$ for some positive integer $m$. Since $c^m \notin \sqrt{0_R}$ and $f^{-1}(\sqrt{J})$ is $n$-ideal, we conclude $d^m \in f^{-1}(\sqrt{J}) \subseteq \sqrt{f^{-1}(J)}$. This yields $d \in \sqrt{f^{-1}(J)}$ and $\sqrt{f^{-1}(J)}$ is an $n$-ideal of $R$. □

**Corollary 2.23.** *Let $I$ and $J$ be proper ideals of $R$ with $J \subseteq I$.*

(1) *If $I$ is a quasi $n$-ideal of $R$, then $I/J$ is a quasi $n$-ideal of $R/J$.*
(2) *$I/J$ is a quasi $n$-ideal of $R/J$ and $J$ is a quasi $n$-ideal of $R$, then $I$ is a quasi $n$-ideal of $R$.*
(3) *Let $K$ be a subring of $R$ with $S \nsubseteq I$. If $I$ is a quasi $n$-ideal of $R$, then $S \cap I$ is a quasi $n$-ideal of $R$.*

*Proof.* (1) Take the natural epimorphism $\pi : R \to R/J$ with $Ker(\pi) = J \subseteq I$ .From Proposition 2.22 (1), $\pi(I) = I/J$ is a quasi $n$-ideal of $R/J$.

(2) Again, consider the natural epimorphism $\pi : R \to R/J$. Since $J$ is a $\delta$-$n$-ideal of $R$, we have $Ker(\pi) = J \subseteq \sqrt{0}$ by Proposition 2.4 (1). Thus $\pi^{-1}(I/J) = I$ is a quasi $n$-ideal of $R$ by Proposition 2.22 (2).

(3) Consider the injection $i : S \to R$ be defined with $i(a) = a$ for every $a \in S$. Then $i^{-1}(I) = I \cap S$ is a quasi $n$-ideal of $R$ by Proposition 2.22 (2). □

Let $I$ be a proper ideal of a ring $R$. Recall that $I$ is said to be superfluous if there is no proper ideal $J$ of $R$ such that $I + J = R$. In the following, by $J(R)$, we denote the Jacobson radical of $R$.

**Lemma 2.24.** *Let $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$. Any $\delta$-$n$-ideal a ring $R$ with $\delta(I) \neq R$ is superfluous.*

*Proof.* Let $I$ be a $\delta$-$n$-ideal of $R$ with $\delta(I) \neq R$. Assume that there exists a proper ideal $J$ of $R$ with $I + J = R$. Then $1 = a + b$ for some $a \in I$ and $b \in J$ and so $1 - b \in I \subseteq \sqrt{0} \subseteq J(R)$ by Proposition 2.4. Thus $b \in J$ is a unit and so, we get $J = R$, a contradiction. □

**Proposition 2.25.** *Let $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$, $I$ and $J$ be $\delta$-$n$-ideals of a ring $R$ where $\delta(I)$ and $\delta(J)$ are proper. Then $I + J$ is a $\delta$-$n$-ideal of $R$.*

*Proof.* Let $I$ and $J$ be $\delta$-$n$-ideals of a ring $R$ such that $\delta(I) \neq R$ and $\delta(J) \neq R$. Since they are superfluous by Lemma 2.24, $I + J \neq R$. Hence, $I \cap J$ is a $\delta$-$n$-ideal by Proposition 2.18. Also, $I/(I \cap J)$ is a $\delta_q$-$n$-ideal of $R/(I \cap J)$ by Corollary 2.21 (1). Now, by the isomorphism $I/(I \cap J) \cong (I + J)/J$, $(I + J)/J$ is a $\delta_q$-$n$-ideal of $R/J$. Therefore, Corollary 2.21 (3) implies that $I + J$ is a $\delta$-$n$-ideal of $R$. □

Let $S$ be a multiplicatively closed subset of $R$. Note that $\delta_S$ is an expansion function of $\mathcal{I}(\mathcal{S}^{-1}\mathcal{R})$ such that $\delta_S(S^{-1}I) = S^{-1}(\delta(I))$ where $\delta$ is an expansion function of $R$. By $Z_I(R)$, we denote the set of $\{r \in R | rs \in I \text{ for some } s \in R\backslash I\}$ where $I$ is a proper ideal of $R$.

**Proposition 2.26.** *Let $S$ be a multiplicatively closed subset of $R$ and $\delta$ be an expansion function of $\mathcal{I}(\mathcal{R})$.*

1. If $I$ is a $\delta$-$n$-ideal of $R$ with $I \cap S = \emptyset$, then $S^{-1}I$ is a $\delta_S$-$n$-ideal of $S^{-1}R$.
2. Let $S \cap Z(R) = S \cap Z_{\delta(I)}(R) = \emptyset$. If $S^{-1}I$ is a $\delta_S$-$n$-ideal of $S^{-1}R$, then $I$ is a $\delta$-$n$-ideal of $R$.

*Proof.* (1) Suppose that $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ and $\frac{a}{s} \notin \sqrt{0_{S^{-1}R}}$ for some $a, b \in R$ and $s, t \in S$. Then there is $u \in S$ with $abu \in I$. Thus $bu \in \delta(I)$ since $a \notin \sqrt{0}$. Hence $\frac{b}{t} = \frac{bu}{tu} \in S^{-1}(\delta(I)) = \delta_S(S^{-1}I)$. Consequently, $S^{-1}I$ is a $\delta_S$-$n$-ideal of $S^{-1}R$.

(2) Let $a, b \in R$ with $ab \in I$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ implies that either $\frac{a}{1} \in \sqrt{0_{S^{-1}R}}$ or $\frac{b}{1} \in \delta_S(S^{-1}I)$. If $\frac{a}{1} \in \sqrt{0_{S^{-1}R}}$, then $ua^n = 0$ for some $u \in S$ and a positive integer $n$. Since $S \cap Z(R) = \emptyset$, we conclude $a^n = 0$ and $a \in \sqrt{0}$. If $\frac{b}{1} \in \delta_S(S^{-1}I) = S^{-1}(\delta(I))$, then $vb \in \delta(I)$ for some $v \in S$. Our assumption $S \cap Z_{\delta(I)}(R) = \emptyset$ implies that $b \in \delta(I)$, as needed. □

An element $a \in R$ is called regular if $ann(a) = 0$. Let $r(R)$ be the set of all regular elements of $R$. Note that $r(R)$ is a multiplicatively closed subset of $R$. From [17, Proposition 2.20], we obtain that if $I$ is a $\delta_{r(R)}$-$n$-ideal of $R_{r(R)}$, then $I^c$ is $\delta$-$n$-ideal of $R$.

*Remark* 2.27. Let $R = R_1 \times R_2$ be a commutative ring where $R_i$ is a commutative ring with nonzero identity for each $i \in \{1, 2\}$. Every ideal $I$ of $R$ is of the form of $I = I_1 \times I_2$ where $I_i$ is an ideal of $R_i$ for all $i \in \{1, 2\}$. Let $\delta_i$ be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for each $i \in \{1, 2\}$. Let $\delta_\times$ be a function of $\mathcal{I}(\mathcal{R})$, which is defined by $\delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ . Then $\delta_\times$ is an expansion function of $\mathcal{I}(\mathcal{R})$. If $\delta_i(I_i) \neq R_i$ for some $i \in \{1, 2\}$, then $R$ has no a $\delta_\times$-$n$-ideal.Indeed, suppose that $I = I_1 \times I_2$ is a $\delta_\times$-$n$-ideal of $R$ where $I_i$ is an ideal of $R_i$ for $i \in \{1, 2\}$. As $(1, 0)(0, 1) \in I$ and $(1, 0), (0, 1) \notin \sqrt{0_R}$, then we have $(1, 0), (0, 1) \in \delta_\times(I)$. Thus $\delta_\times(I) = \delta_1(I_1) \times \delta_2(I_2) = R_1 \times R_2$, a contradiction.

Let $R$ be a ring and $M$ be a unitary $R$-module. Recall that the idealization $R(+)M = \{(r, m) : r \in R,\ m \in M\}$ is a commutative ring with the addition $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $r_1, r_2 \in R$; $m_1, m_2 \in M$. For an ideal $I$ of $R$ and a submodule $N$ of $M$, it is well-known that $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$, [1] and

[11]. We recall also from [1] that $\sqrt{I(+)N} = \sqrt{I}(+)M$. Let $R(+)M$ be the idealization where $M$ is an $R$-module. For an expansion function $\delta$ of $R$, define $\delta_{(+)}$ as $\delta_{(+)}(I(+)N) = \delta(I)(+)M$ for some ideal $I(+)N$ of $R(+)M$. It is clear that $\delta_{(+)}$ is an expansion function of $R(+)M$. Next, we characterize $\delta$-$n$-ideals in any idealization ring $R(+)M$.

**Proposition 2.28.** *Let $I$ be an ideal of a ring $R$ and $N$ be a submodule of an $R$-module $M$. Then $I$ is a $\delta$-$n$-ideal of $R$ if and only if $I(+)N$ is a $\delta_{(+)}$-$n$-ideal of $R(+)M$.*

*Proof.* Let $I$ be a $\delta$-$n$-ideal of $R$. Assume that $(r, m)(s, m') \in I(+)N$ and $(s, m') \notin \sqrt{0}(+)M$ for some $(r, m)(s, m') \in R(+)M$. Then $s \in \delta(I)$ since $rs \in I$ and $s \notin \sqrt{0}$. Thus $(s, m') \in \delta(I)(+)M = \delta_{(+)}(I(+)M)$. Conversely, suppose that $I(+)N$ is a $\delta_{(+)}$-$n$-ideal of $R(+)M$. Let $r, s \in R$ with $rs \in I$ and $s \notin \sqrt{0}$. Hence, we get $(r, m)(s, m') \in I(+)N$ and clearly $(s, m') \notin \sqrt{0}(+)M$ which follows $(r, m) \in \delta_{(+)}(I(+)M)$ and $r \in \delta(I)$. Thus $I$ is a $\delta$-$n$-ideal of $R$. □

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