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CENTERS OF CENTRALIZER NEARRINGS DETERMINED BY ALL ENDOMORPHISMS OF SYMMETRIC GROUPS

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ABSTRACT. For $n = 5, 6$ and $E = \operatorname{End} S_n$, the functions in the centralizer nearring $M_E(S_n) = \{f : S_n \to S_n \mid f(1) = (1) \text{ and } f \circ s = s \circ f \text{ for all } s \in E\}$ are determined. The centers of these two nearrings are also described. Results that can be used to determine the functions in $M_E(S_n)$ and their centers for $n \geq 7$ are also presented.

1. INTRODUCTION

Let $(G, +)$ be a group written additively with identity 0, but not necessarily abelian. For a subsemigroup S of $\operatorname{End} G$, the set of all endomorphisms of G, the set $M_S(G) = \{f : G \to G \mid f(0) = 0 \text{ and } f \circ s = s \circ f \text{ for all } s \in S\}$ is a right nearring under function addition and composition called the centralizer nearring determined by G and S. Every nearring with identity is isomorphic to an $M_S(G)$ for some choice of G and S ([6], Theorem 2.8). For a fixed group G, the smallest centralizer nearring is $M_E(G)$, where $E = \operatorname{End} G$. These nearrings were considered in [2] for various groups, including the symmetric groups, and the properties of simplicity, localness, and being a ring were investigated. The nearrings $M_I(S_n)$ for $I = \operatorname{Inn} G$, the inner automorphisms of S_n, were studied in [1]. In particular, the centers of $M_I(S_n)$, $C(M_I(S_n)) = \{c \in M_I(S_n) \mid c \circ f = f \circ c \text{ for all } f \in M_I(S_n)\}$ were determined for $n = 4, 5, 6$. For more information about nearrings, consult [4], [6], and [7].

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In this paper, we find the centers of $M_E(S_n)$ for $n = 5, 6$ and develop results that may be used to find centers in the cases $n \geq 7$. In the next section, we discuss the functions in $M_I(S_n)$ for $n \geq 3$. Section 3 considers the general nearring $M_E(S_n)$ and establishes conditions for when functions commute with one another. The last two sections characterize functions in $M_E(S_n)$ and $C(M_E(S_n))$ for $n = 5$ and $n = 6$, respectively.

We let id denote the identity function from G to G. The zero function in $M_S(G)$ is denoted with 0, as is the identity element in G, but its use will be clear from the context. For $g \in G$, the cyclic subgroup of G generated by g is $\langle g \rangle$. We let G^* denote the nonzero elements of G. For $g \in G^*$ and an arbitrary group of automorphisms A of G, we let Ag denote the orbit of g determined by A. Group addition in $(S_n, +)$ refers to composition of functions in S_n. However, as in [1], we use the usual juxtaposition when referring to chosen elements in S_n, and $+$ when adding them. If a result is purely group theoretic, we use juxtaposition, whereas if a result is nearring related, we use both juxtaposition and additive notation.

2. $M_I(S_n)$

First, we consider the general case of a centralizer nearring determined by a finite group G and a group of automorphisms A of G. Let $f \in M_A(G)$ and $g \in G$. Let $z \in Ag$, say $\varphi(g) = z$ for some $\varphi \in A$. Then $f(z) = f(\varphi(g)) = \varphi(f(g))$ and $f(z)$ is completely determined by $f(g)$. In particular, $f(z)$ and $f(g)$ are in the same orbit. The next lemma appears as Lemma 2.3 in [1].

Lemma 2.1. *Let $f \in M_A(G)$, $g \in G$, and $z \in Ag$. If $f(g) = kg$ for some integer k, then $f(z) = kz$. In particular, if $f(g) = 0$, then $f(z) = 0$, and if $f(g) = g$, then $f(z) = z$.*

When describing functions in $M_A(G)$, we only need to know a function value on each orbit representative determined by A. The function can be extended on each orbit using the procedure outlined in the paragraph above. In order for the extension to be well-defined, we need Betsch's Lemmma ([6], Lemma 3.30).

Lemma 2.2. *(Betsch's Lemma) Let $(G, +)$ be a group and A be a group of automorphisms of G. For $x \in G^*$, define $\mathrm{Stab}(x) = \{a \in A \mid a(x) = x\}$, the stabilizer of x in A. Let $g \in G^*$ and $h \in G$. Then there exists $f \in M_A(G)$ such that $f(g) = h$ if and only if $\mathrm{Stab}(g) \subseteq \mathrm{Stab}(h)$.*

Thus for a fixed $g \in G^*$, if $\mathrm{Stab}(g) \subseteq \mathrm{Stab}(h)$, one can define $f(g) = h$ and extend f to other values z in the same orbit as g via $f(z) =$

$\varphi(f(g))$ as described in the first paragraph of this section. Defining $f(x) = 0$ on the remaining orbits creates the function $f \in M_A(G)$ given in the proof of Betsch's Lemma.

Lemma 2.3. *Let G be a finite group, A be a group of automorphisms of G, and $\{B_i\}$ be the collection of orbits determined by A. Let $f : G \to G$ be a function such that $f|_{B_i} = k_i \cdot id$ for some integer k_i for each orbit B_i. Then $f \in M_A(G)$.*

Proof. Let $\varphi \in A$ and $g \in G$. Then $g \in B_i$ for some orbit B_i. Hence $\varphi(g) \in B_i$. So $\varphi(f(g)) = \varphi(k_i g) = k_i\varphi(g) = f(\varphi(g))$. Since $g \in G$ is arbitrary, $\varphi \circ f = f \circ \varphi$ and $f \in M_A(G)$. □

We note that we can create the function given in the previous lemma. For $g \in B_i$ and integer k_i, we have $\mathrm{Stab}(g) \subseteq \mathrm{Stab}(k_i g)$. By Betsch's Lemma, there exists a function $f \in M_A(G)$ such that $f(g) = k_i g$. Extending f to B_i yields $f|_{B_i} = k_i \cdot id$.

We now turn our attention to the symmetric groups S_n and the group of inner automorphism $I = \mathrm{Inn}\, S_n$. In the symmetric groups, the orbits determined by I are the distinct cycle structures of S_n. To define a function in $M_I(S_n)$, we need the following lemma which appears in [2]. This lemma is a straightforward application of Betsch's Lemma to I and S_n.

Lemma 2.4. *For $g \in S_n$, let $\mathrm{Move}\, g = \{i \in \{1, 2, \ldots, n\} \mid g(i) \neq i\}$. Let $g = g_1 + g_2 + \cdots + g_r$ be a pairwise disjoint sum in S_n where each g_w is a pairwise disjoint sum of k_w cycles and $k_w \neq k_y$ for all $w \neq y$. Then there exists $f \in M_I(S_n)$ such that $f(g) = h$ if and only if*

(i) *$h \in \langle g_1\rangle + \langle g_2\rangle + \cdots + \langle g_r\rangle$ for $|\mathrm{Move}\, g| \neq n - 2$.*
(ii) *$h \in \langle g_1\rangle + \langle g_2\rangle + \cdots + \langle g_r\rangle + \langle(a\, b)\rangle$ for $|\mathrm{Move}\, g| = n - 2$ where a and b are the two distinct elements not in $\mathrm{Move}\, g$.*

Using Lemma 2.4, all functions in $M_I(S_n)$ can be determined. Table 1 describes all functions in $M_I(S_n)$ for $n = 5, 6$. For each cycle structure representative $x \in S_n$, the function $f \in M_I(S_n)$ assumes a value $f(x)$ in the set in the adjacent columns. For example, in S_5, $f(1\,2\,3) \in \langle(1\,2\,3)\rangle + \langle(4\,5)\rangle$, and in S_6, $f((1\,2\,3\,4)(5\,6)) \in \langle(1\,2\,3\,4)\rangle + \langle(5\,6)\rangle$. The remaining values for f are obtained by extending to the other elements in each orbit as described at the beginning of this section. Note that the $n-2$ case from Lemma 2.4 occurs with the 3-cycles for S_5 and with the 4-cycles and the product of two 2-cycles for S_6.

Table 1. All functions $f \in M_I(S_n)$ for $n = 5, 6$

$x \in S_n$	$f \in M_I(S_5)$	$f \in M_I(S_6)$
$(1\,2)$	$\langle(1\,2)\rangle$	$\langle(1\,2)\rangle$
$(1\,3)(2\,4)$	$\langle(1\,3)(2\,4)\rangle$	$\langle(1\,3)(2\,4)\rangle + \langle(5\,6)\rangle$
$(1\,2\,3)$	$\langle(1\,2\,3)\rangle + \langle(4\,5)\rangle$	$\langle(1\,2\,3)\rangle$
$(1\,2\,3\,4)$	$\langle(1\,2\,3\,4)\rangle$	$\langle(1\,2\,3\,4)\rangle + \langle(5\,6)\rangle$
$(1\,2\,3)(4\,5)$	$\langle(1\,2\,3)\rangle + \langle(4\,5)\rangle$	$\langle(1\,2\,3)\rangle + \langle(4\,5)\rangle$
$(1\,2\,3\,4\,5)$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$
$(1\,2)(3\,4)(5\,6)$		$\langle(1\,2)(3\,4)(5\,6)\rangle$
$(1\,2\,3)(4\,5\,6)$		$\langle(1\,2\,3)(4\,5\,6)\rangle$
$(1\,2\,3\,4)(5\,6)$		$\langle(1\,2\,3\,4)\rangle + \langle(5\,6)\rangle$
$(1\,2\,3\,4\,5\,6)$		$\langle(1\,2\,3\,4\,5\,6)\rangle$

3. $M_E(S_n)$ and Commuting Theorems

Here, we consider the nearring $M_E(S_n)$ where $E = \operatorname{End} S_n$, the set of all endomorphisms of S_n. Let Y be the set of elements in S_n of order two. For each $y \in Y$, we define $\mu_y(x) = \begin{cases} (1) & \text{if } x \in A_n \\ y & \text{if } x \in S_n \backslash A_n \end{cases}$. Then μ_y is an endomorphism of S_n.

Using [8], one can deduce that for $n \geq 3$ and $n \neq 4, 6$, the endomorphisms of S_n are the zero map, the inner automorphisms of S_n, and μ_y for each $y \in Y$. In [2], it was shown that $M_E(S_3) \cong \mathbb{Z}_6$ and $M_E(S_4) \cong \mathbb{Z}_{12}$. Thus for the rest of the paper we assume $n \geq 5$. We note that S_6 has outer automorphisms as well, but the only nonautomorphisms of S_6 are the zero map and μ_y for each $y \in Y$.

Functions in $M_E(S_n)$ must commute with μ_y for all $y \in Y$. The next result, which appears in [2], gives necessary and sufficient conditions for functions in $M_{(1)}(S_n) = \{f : S_n \to S_n \mid f(1) = (1)\}$ to commute with all μ_y.

Lemma 3.1. *Let $f \in M_{(1)}(S_n)$. Then $f\mu_y = \mu_y f$ for every $y \in Y$ if and only if $f(A_n) \subseteq A_n$, and either (a) $f(S_n \backslash A_n) \subseteq A_n$ and $f(Y) = (1)$ or (b) $f(S_n \backslash A_n) \subseteq S_n \backslash A_n$ and $f|_Y = id$.*

Next we collect some theorems showing when $f_1(f_2(g)) = f_2(f_1(g))$ for functions $f_1, f_2 \in M_E(S_n)$ and $g \in S_n$. This will facilitate showing a function f is in the center of $M_E(S_n)$, which is the focus of the next two sections. The first lemma explains that to show $f_1(f_2(x)) = f_2(f_1(x))$ for all elements $x \in G$, one only needs to verify that $f_1(f_2(g)) = f_2(f_1(g))$ for all orbit representatives $g \in G$ determined by a group of automorphisms A of G.

Lemma 3.2. *Let $g \in G$ be an orbit representative under A, and let $z \in Ag$. Then for $f_1, f_2 \in M_A(G)$, if $f_1(f_2(g)) = f_2(f_1(g))$, then $f_1(f_2(z)) = f_2(f_1(z))$.*

Proof. Suppose $f_1(f_2(g)) = f_2(f_1(g))$ for some orbit representative g under A. Since $z \in Ag$, there exists $\varphi \in A$ such that $\varphi(g) = z$. Thus $f_1(f_2(z)) = f_1(f_2(\varphi(g))) = \varphi(f_1(f_2(g))) = \varphi(f_2(f_1(g))) = f_2(f_1(\varphi(g))) = f_2(f_1(z))$. □

The following lemma handles the specific case where $f_1(g), f_2(g) \in \{(1)\} \cup Ig$.

Lemma 3.3. *Let $f_1, f_2 \in M_I(S_n)$ and $g \in S_n$ such that $f_1(g), f_2(g) \in \{(1)\} \cup Ig$. Then $f_1(f_2(g)) = f_2(f_1(g))$. In particular, $f_1(f_2(y)) = f_2(f_1(y))$ for all $y \in Y$.*

Proof. If $f_1(g) = f_2(g) = (1)$, then $f_1(f_2(g)) = f_1(1) = (1) = f_2(1) = f_2(f_1(g))$. Without a loss of generality, if $f_1(g) = (1)$ and $f_2(g) \in Ig$, then by Lemma 2.1, $f_1(f_2(g)) = (1) = f_2(1) = f_2(f_1(g))$. If $f_1(g), f_2(g) \in Ig$, the result follows from Lemma 4.4 in [1]. The last statement follows immediately since $f_1(y), f_2(y) \in \{(1), y\}$ by Lemma 3.1. □

The next lemma involves two functions in $M_E(S_n)$, with the image of an odd permutation being an element of Y for at least one of the functions.

Lemma 3.4. *Let $f_1, f_2 \in M_E(S_n)$ and $g \in S_n \backslash A_n$.*

(i) *If $f_1(g) \in \{(1), g\}$ and $f_2(g) \in Y$, then $f_1(f_2(g)) = f_2(f_1(g))$.*
(ii) *If $f_1(g) = -g$ and $f_2(g) = mg \in Y$ for some integer m, then $f_1(f_2(g)) = f_2(f_1(g))$.*
(iii) *If $f_1(g), f_2(g) \in A_n \cap Y$, then $f_1(f_2(g)) = f_2(f_1(g))$.*
(iv) *If $f_1(g) = f_2(g) \in (S_n \backslash A_n) \cap Y$, then $f_1(f_2(g)) = f_2(f_1(g))$.*

Proof. (i) First assume that $f_1(g) = (1)$ and $f_2(g) \in Y$. Since g is odd and $f_1(g) = (1) \in A_n$, it follows by Lemma 3.1 that $f_1(Y) = (1)$. Thus $f_1(f_2(g)) = (1) = f_2(1) = f_2(f_1(g))$.

Now assume that $f_1(g) = g$ and $f_2(g) \in Y$. Since g is odd and $f_1(g) = g \in S_n \backslash A_n$, it follows by Lemma 3.1 that $f_1|_Y = id$. Thus $f_1(f_2(g)) = f_2(g) = f_2(f_1(g))$.

(ii) If $f_1(g) = -g$ and $f_2(g) = mg \in Y$, then $|mg| = 2$ and $|g| = 2m$. Hence $(2m)g = mg + mg = (1)$ and $mg = m(-g)$. As above, since g is odd and $f_1(g) = -g \in S_n \backslash A_n$, it follows by Lemma 3.1 that $f_1|_Y = id$. Also, since $-g \in Ig$ and $f_2(g) = mg$, we conclude that $f_2(-g) = m(-g)$ by Lemma 2.1. Thus $f_1(f_2(g)) = f_2(g) = mg = m(-g) = f_2(-g) = f_2(f_1(g))$.

(iii) Since g is odd and $f_1(g), f_2(g) \in A_n$, it follows by Lemma 3.1 that $f_1(Y) = (1) = f_2(Y)$. As $f_1(g), f_2(g) \in Y$, we conclude that $f_1(f_2(g)) = (1) = f_2(f_1(g))$.

(iv) Since g is odd and $f_1(g) = f_2(g) \in S_n \backslash A_n$, it follows by Lemma 3.1 that $f_1|_Y = id$ and $f_2|_Y = id$. As $f_1(g), f_2(g) \in Y$, we conclude that $f_1(f_2(g)) = f_2(g) = f_1(g) = f_2(f_1(g))$. ☐

4. $M_E(S_5)$ AND $C(M_E(S_5))$

In this section, we determine the functions in $M_E(S_5)$ and $C(M_E(S_5))$. We first describe functions in $M_E(S_5)$. We use two columns in Table 2, with one column displaying $f \in M_E(S_5)$ such that $f(Y) = (1)$ and a second column for $f \in M_E(S_5)$ with $f|_Y = id$ as indicated in Lemma 3.1.

Theorem 4.1. *A function $f \in M_E(S_5)$ if and only if f is a function described in one of the columns of the table below. Furthermore, $|M_E(S_5)| = 180$.*

TABLE 2. All functions $f \in M_E(S_5)$

$x \in S_5$	$f \in M_E(S_5)$	$f \in M_E(S_5)$
$y \in Y$	(1)	y
$(1\,2\,3)$	$\langle(1\,2\,3)\rangle$	$\langle(1\,2\,3)\rangle$
$(1\,2\,3\,4)$	(1), $(1\,3)(2\,4)$	$(1\,2\,3\,4)$, $(1\,4\,3\,2)$
$(1\,2\,3)(4\,5)$	$\langle(1\,2\,3)\rangle$	$(4\,5)$, $(1\,2\,3)(4\,5)$, $(1\,3\,2)(4\,5)$
$(1\,2\,3\,4\,5)$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$

Proof. Restricting functions in $M_I(S_5)$ in Table 1 to the conditions of Lemma 3.1 gives all functions in $M_E(S_5)$, which are described in Table 2. There are $3 \cdot 2 \cdot 3 \cdot 5 = 90$ possible functions represented in each column of Table 2. Thus there are 180 total functions in $M_E(S_5)$. ☐

We now determine the functions in $C(M_E(S_5))$. We begin with some necessary conditions for functions to be in $C(M_E(S_n))$ for $n = 5$ or $n \geq 7$. Since S_6 has outer automorphisms, $M_E(S_6)$ is handled as a separate case in the next section.

Theorem 4.2. *Let $n = 5$ or $n \geq 7$ and $c \in C(M_E(S_n))$.*

(i) *If $g \in A_n \backslash Y$, then $c(g) \in \{(1)\} \cup Ig$.*

(ii) *If $g \in S_n \backslash (A_n \cup Y)$, then $c(g) \in \{(1)\} \cup Ig \cup Y$.*

Proof. (i) Let $c \in C(M_E(S_n))$ and $g \in A_n \backslash Y$. Define the function $f_1(x) = \begin{cases} x & \text{if } x \in Ig \\ (1) & \text{otherwise} \end{cases}$. Note that $f_1(1) = (1)$. By Lemma 2.3,

$f_1 \in M_I(S_n)$. Since f_1 also satisfies the conditions of Lemma 3.1, we conclude that $f_1 \in M_E(S_n)$. Thus $c(g) = c(f_1(g)) = f_1(c(g))$ and $c(g)$ is a fixed point of f_1. Therefore $c(g) \in \{(1)\} \cup Ig$.

(ii) Let $c \in C(M_E(S_n))$ and $g \in S_n \backslash (A_n \cup Y)$. Define the function $f_2(x) = \begin{cases} x & \text{if } x \in Ig \cup Y \\ 3x & \text{if } x \in S_n \backslash (A_n \cup Ig \cup Y) \\ (1) & \text{if } x \in A_n \backslash Y \end{cases}$. As above, $f_2 \in M_E(S_n)$. Thus $c(g) = c(f_2(g)) = f_2(c(g))$ and $c(g)$ is a fixed point of f_2. Note that $3x \neq x$ for $x \in S_n \backslash (A_n \cup Ig \cup Y)$, since otherwise, $2x = (1)$ and $x \in Y \cup \{(1)\}$, a contradiction. Therefore $c(g) \in \{(1)\} \cup Ig \cup Y$. □

In Table 3 below, superscripts designate corresponding function values that must be used in tandem. For example, if $c((1\,2\,3)(4\,5)) = (4\,5)$, then $c(1\,2\,3) = (1)$.

Theorem 4.3. *Let $c \in M_E(S_5)$. Then $c \in C(M_E(S_5))$ if and only if c is a function described in one of the columns of the table below. Furthermore, $|C(M_E(S_5))| = 40$.*

Table 3. All functions $c \in C(M_E(S_5))$

$x \in S_5$	$c \in C(M_E(S_5))$	$c \in C(M_E(S_5))$
$y \in Y$	(1)	y
$(1\,2\,3)$	(1)	$(1)^a$, $(1\,2\,3)^b$, $(1\,3\,2)^c$
$(1\,2\,3\,4)$	(1), $(1\,3)(2\,4)$	$(1\,2\,3\,4)$, $(1\,4\,3\,2)$
$(1\,2\,3)(4\,5)$	(1)	$(4\,5)^a$, $(1\,2\,3)(4\,5)^b$, $(1\,3\,2)(4\,5)^c$
$(1\,2\,3\,4\,5)$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$

Proof. Let $c \in C(M_E(S_5))$ and $a = (1\,2\,3)(4\,5)$. By Table 1, $c(a) \in \langle(1\,2\,3)\rangle + \langle(4\,5)\rangle$. Using Theorem 4.2, we also conclude that $c(a) \in \{(1), (4\,5), (1\,2\,3)(4\,5), (1\,3\,2)(4\,5)\}$.

By Table 2, there exists $f \in M_E(S_5)$ such that $f(a) = (1\,2\,3)$ and $f(4\,5) = (1)$. Since a and $-a$ are in the same orbit under I and $f(a) = (1\,2\,3) = 4a$, it follows from Lemma 2.1 that $f(-a) = 4(-a) = (1\,3\,2)$. Thus $c(1\,2\,3) = c(f(a)) = f(c(a))$.

If $c(a) = (1)$, then $c(1\,2\,3) = f(c(a)) = f(1) = (1)$. Also, if $c(a) = (4\,5)$, then $c(1\,2\,3) = f(c(a)) = f(4\,5) = (1)$. If $c(a) = a$, then $c(1\,2\,3) = f(c(a)) = f(a) = (1\,2\,3)$. If $c(a) = -a$, then $c(1\,2\,3) = f(c(a)) = f(-a) = (1\,3\,2)$. With these restrictions, c is one of the functions in Table 3.

Now let $\alpha \in M_E(S_5)$ be a function described in Table 3 and let $f \in M_E(S_5)$. By Lemma 3.2, we need to show that $\alpha(f(a)) = f(\alpha(a))$ for all orbit representatives a determined by I.

If $a \in Y \cup \{(1\,2\,3), (1\,2\,3\,4\,5)\}$, then $\alpha(f(a)) = f(\alpha(a))$ by Lemma 3.3. If $a = (1\,2\,3\,4)$, then $\alpha(a), f(a) \in \{(1), (1\,3)(2\,4), (1\,2\,3\,4), (1\,4\,3\,2)\}$. By considering all combinations of $\alpha(a)$ and $f(a)$ in conjunction with Lemmas 3.3 and 3.4, it follows that $\alpha(f(a)) = f(\alpha(a))$.

For $a = (1\,2\,3)(4\,5)$, we consider several cases. Assume $\alpha(a) = (1)$ and $f(a) = b \in \{(1\,2\,3), (1\,3\,2)\}$. By Table 3, $\alpha(1\,2\,3) = (1)$. It follows that $\alpha(1\,3\,2) = (1)$ by Lemma 2.1. Thus $f(\alpha(a)) = f(1) = (1) = \alpha(b) = \alpha(f(a))$.

Now assume $\alpha(a) = (4\,5)$ and $f(a) = b \in \{(1\,2\,3), (1\,3\,2)\}$. By Table 3, $\alpha(1\,2\,3) = (1)$, and by Lemma 2.1, $\alpha(1\,3\,2) = (1)$. By Table 2, $f(4\,5) = (1)$. So $f(\alpha(a)) = f(4\,5) = (1) = \alpha(b) = \alpha(f(a))$.

Next assume $\alpha(a) = a$ and $f(a) = b \in \{(1\,2\,3), (1\,3\,2)\}$. By Table 3, $\alpha(1\,2\,3) = (1\,2\,3)$. So $\alpha(b) = b$ by Lemma 2.1. Thus $f(\alpha(a)) = f(a) = b = \alpha(b) = \alpha(f(a))$.

Finally, assume $\alpha(a) = -a$ and $f(a) = b \in \{(1\,2\,3), (1\,3\,2)\}$. By Table 3 and Lemma 2.1, $\alpha(b) = -b$. By Lemma 2.1, $f(-a) = -b$. Thus $f(\alpha(a)) = f(-a) = -b = \alpha(b) = \alpha(f(a))$.

All other combinations of $\alpha(a) \in \{(1), (4\,5), (1\,2\,3)(4\,5), (1\,3\,2)(4\,5)\}$ and $f(a) \in \{(1), (4\,5), (1\,2\,3)(4\,5), (1\,3\,2)(4\,5), (1\,2\,3), (1\,3\,2)\}$ result in $f(\alpha(a)) = \alpha(f(a))$ by Lemmas 3.3 and 3.4. So $\alpha \in C(M_E(S_5))$.

There are $2 \cdot 5 = 10$ possible functions represented in the first column of Table 3 and $3 \cdot 2 \cdot 5 = 30$ possible functions represented by the second column. Thus there are 40 total functions and the proof is complete. □

For $n \geq 7$, End S_n consists of the zero map, the inner automorphisms, and μ_y for each $y \in Y$ as described at the beginning of Section 3. These are precisely all of the endomorphisms of S_5. Therefore, finding $M_E(S_n)$ and $C(M_E(S_n))$ for $n \geq 7$ follows a similar procedure to the $n = 5$ case above.

5. $M_E(S_6)$ and $C(M_E(S_6))$

To find $C(M_E(S_6))$, we follow the same line of development used to find $C(M_E(S_5))$. Since S_6 has outer automorphisms, we need an extra intermediate step. We begin by considering functions in $M_Q(S_6)$, where Q consists of the zero endomorphism, the inner automorphisms of S_6, and μ_y with $y \in Y$. Using Table 1 and Lemma 3.1, we can determine all functions in $M_Q(S_6)$ as given in each column of Table 4.

TABLE 4. All functions $f \in M_Q(S_6)$

$x \in S_6$	$f \in M_Q(S_6)$	$f \in M_Q(S_6)$
$y \in Y$	(1)	y
$(1\,2\,3)$	$\langle(1\,2\,3)\rangle$	$\langle(1\,2\,3)\rangle$
$(1\,2\,3\,4)$	$\langle(1\,2\,3\,4)(5\,6)\rangle$	$(1\,2\,3\,4)$, $(1\,4\,3\,2)$, $(5\,6)$, $(1\,3)(2\,4)(5\,6)$
$(1\,2\,3)(4\,5)$	$\langle(1\,2\,3)\rangle$	$(1\,2\,3)(4\,5)$, $(1\,3\,2)(4\,5)$, $(4\,5)$
$(1\,2\,3\,4\,5)$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$
$(1\,3\,5)(2\,4\,6)$	$\langle(1\,3\,5)(2\,4\,6)\rangle$	$\langle(1\,3\,5)(2\,4\,6)\rangle$
$(1\,2\,3\,4)(5\,6)$	$\langle(1\,2\,3\,4)(5\,6)\rangle$	$\langle(1\,2\,3\,4)(5\,6)\rangle$
$(1\,2\,3\,4\,5\,6)$	$\langle(1\,3\,5)(2\,4\,6)\rangle$	$(1\,2\,3\,4\,5\,6)$, $(1\,6\,5\,4\,3\,2)$, $(1\,4)(2\,5)(3\,6)$

In [5], an outer automorphism ϕ of S_6 is described as follows.

Theorem 5.1. *There exists an outer automorphism ϕ of S_6 such that $\phi(1\,2) = (1\,2)(3\,6)(4\,5)$, $\phi(1\,3) = (1\,6)(2\,4)(3\,5)$, $\phi(1\,4) = (1\,3)(2\,5)(4\,6)$, $\phi(1\,5) = (1\,5)(2\,6)(3\,4)$, and $\phi(1\,6) = (1\,4)(2\,3)(5\,6)$.*

Other values of ϕ can be obtained by writing any element of S_6 as the product of transpositions of the form $(1\,i)$. The next lemma provides values of ϕ that are needed in the sequel.

Lemma 5.2. *Let ϕ be the outer automorphism of Theorem 5.1 and $\lambda_{(2\,6\,3)}$ be the inner automorphism determined by $(2\,6\,3)$, i.e., $\lambda_{(2\,6\,3)}(x) = (2\,3\,6)\,x\,(2\,6\,3)$. Then*

(i) $\lambda_{(2\,6\,3)}(\phi(1\,2\,3\,4)) = (1\,2\,3\,4)$;
(ii) $\lambda_{(2\,6\,3)}(\phi(5\,6)) = (1\,3)(2\,4)(5\,6)$;
(iii) $\lambda_{(2\,6\,3)}(\phi((1\,2\,3\,4)(5\,6))) = (1\,4\,3\,2)(5\,6)$;
(iv) $\lambda_{(2\,6\,3)}(\phi((1\,4\,3\,2)(5\,6))) = (1\,2\,3\,4)(5\,6)$; *and*
(v) $\lambda_{(2\,6\,3)}(\phi((1\,3)(2\,4)(5\,6))) = (5\,6)$.

Proof. (i) By Theorem 5.1,

$$\begin{aligned}\lambda_{(2\,6\,3)}(\phi(1\,2\,3\,4)) &= \lambda_{(2\,6\,3)}(\phi((1\,4)(1\,3)(1\,2)))\\ &= \lambda_{(2\,6\,3)}(\phi(1\,4)\phi(1\,3)\phi(1\,2))\\ &= (2\,3\,6)(1\,3)(2\,5)(4\,6)(1\,6)(2\,4)(3\,5)(1\,2)(3\,6)(4\,5)(2\,6\,3)\\ &= (1\,2\,3\,4).\end{aligned}$$

(ii) Also,

$$\begin{aligned}\lambda_{(2\,6\,3)}(\phi(5\,6)) &= \lambda_{(2\,6\,3)}(\phi((1\,5)(1\,6)(1\,5)))\\ &= \lambda_{(2\,6\,3)}(\phi(1\,5)\phi(1\,6)\phi(1\,5))\\ &= (2\,3\,6)(1\,5)(2\,6)(3\,4)(1\,4)(2\,3)(5\,6)(1\,5)(2\,6)(3\,4)(2\,6\,3)\\ &= (1\,3)(2\,4)(5\,6).\end{aligned}$$

(iii) Using (i) and (ii), we get

$$\begin{aligned}(\lambda_{(2\,6\,3)} \circ \phi)((1\,2\,3\,4)(5\,6)) &= (\lambda_{(2\,6\,3)} \circ \phi)(1\,2\,3\,4)(\lambda_{(2\,6\,3)} \circ \phi)(5\,6)\\ &= (1\,2\,3\,4)(1\,3)(2\,4)(5\,6) = (1\,4\,3\,2)(5\,6).\end{aligned}$$

(iv) From (iii) we conclude that
$(\lambda_{(2\,6\,3)} \circ \phi)((1\,4\,3\,2)(5\,6)) = (\lambda_{(2\,6\,3)} \circ \phi)[((1\,2\,3\,4)(5\,6))^{-1}]$
$= [(\lambda_{(2\,6\,3)} \circ \phi)((1\,2\,3\,4)(5\,6))]^{-1} = [(1\,4\,3\,2)(5\,6)]^{-1}$
$= (1\,2\,3\,4)(5\,6)$.

(v) Again, using (i) and (ii), we get
$\lambda_{(2\,6\,3)}(\phi((1\,3)(2\,4)(5\,6))) = \lambda_{(2\,6\,3)}(\phi((1\,2\,3\,4)^2(5\,6)))$
$= [\lambda_{(2\,6\,3)}(\phi(1\,2\,3\,4))]^2\lambda_{(2\,6\,3)}(\phi(5\,6)) = (1\,2\,3\,4)^2(1\,3)(2\,4)(5\,6)$
$= (1\,3)(2\,4)(1\,3)(2\,4)(5\,6) = (5\,6)$. □

Every automorphism of S_6 is an inner automorphism or the product of an inner automorphism and the outer automorphism ϕ (see [5]). Therefore $M_E(S_6)$ consists of all functions described in Table 4 that commute with ϕ. The next theorem determines such functions.

As mentioned above, $\phi(1\,2) = (1\,2)(3\,6)(4\,5)$. It can be shown that $\phi(1\,2\,3) = (1\,4\,3)(2\,6\,5)$, $\phi(1\,2\,3\,4\,5\,6) = (1\,3)(4\,6\,5)$, and that ϕ preserves the cycle structure of each of the 4-cycles, 5-cycles, the product of two 2-cycles, and the product of a 2-cycle and a 4-cycle. Thus, there are fewer orbits determined by $I \cup \{\phi\}$ than by I alone. As before with $M_I(S_6)$, to describe a function f in $M_E(S_6)$, we only need to define a value for f on each orbit representative. The remaining values can be determined as described at the beginning of Section 2. Therefore, we have eliminated the 3-cycle and the product of a 2-cycle and a 3-cycle from the table in the next theorem. The product of three 2-cycles is not needed as well, but this cycle structure is included in the Y row of the table.

Theorem 5.3. *A function $f \in M_E(S_6)$ if and only if f is a function described in one of the columns of the table below. Furthermore, $|M_E(S_6)| = 720$.*

Table 5. All functions $f \in M_E(S_6)$

$x \in S_6$	$f \in M_E(S_6)$	$f \in M_E(S_6)$
$y \in Y$	(1)	y
$(1\,2\,3\,4)$	(1), $(1\,3)(2\,4)$	$(1\,2\,3\,4)$, $(1\,4\,3\,2)$
$(1\,2\,3\,4\,5)$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$
$(1\,3\,5)(2\,4\,6)$	$\langle(1\,3\,5)(2\,4\,6)\rangle$	$\langle(1\,3\,5)(2\,4\,6)\rangle$
$(1\,2\,3\,4)(5\,6)$	$\langle(1\,2\,3\,4)(5\,6)\rangle$	$\langle(1\,2\,3\,4)(5\,6)\rangle$
$(1\,2\,3\,4\,5\,6)$	$\langle(1\,3\,5)(2\,4\,6)\rangle$	$(1\,2\,3\,4\,5\,6)$, $(1\,6\,5\,4\,3\,2)$, $(1\,4)(2\,5)(3\,6)$

Proof. Let $f \in M_E(S_6)$. Then $f \in M_Q(S_6)$ and is represented in Table 4. We focus on $f(1\,2\,3\,4)$. By Lemma 5.2, $\lambda_{(2\,6\,3)}(\phi(1\,2\,3\,4)) = (1\,2\,3\,4)$ implies that $\lambda_{(2\,6\,3)} \circ \phi \in \mathrm{Stab}(1\,2\,3\,4)$.

Yet for $g \in \{(5\,6), (1\,2\,3\,4)(5\,6), (1\,4\,3\,2)(5\,6), (1\,3)(2\,4)(5\,6)\}$, we see by Lemma 5.2 that $\lambda_{(2\,6\,3)} \circ \phi \notin \mathrm{Stab}(g)$. By Betsch's Lemma (Lemma 2.2), $f(1\,2\,3\,4) \notin \{(5\,6), (1\,2\,3\,4)(5\,6), (1\,4\,3\,2)(5\,6), (1\,3)(2\,4)(5\,6)\}$. Since the only difference between Table 4 and Table 5 is the action of f on the 4-cycles, eliminating these four possibilities for $f(1\,2\,3\,4)$ yields a function represented in Table 5.

Now assume f is a function described in Table 5. Note that each orbit representative g is mapped to a multiple of g. By extension, f restricted to each orbit will be a multiple of the identity function. By Lemma 2.3, $f \in M_A(S_6)$, where $A = \mathrm{Aut}\,S_6$. Functions in Table 5 already satisfy the conditions of Lemma 3.1. Thus $f \in M_E(S_6)$.

There are $2 \cdot 5 \cdot 3 \cdot 4 \cdot 3 = 360$ possible functions represented in each column of Table 5. Thus there are 720 total functions in $M_E(S_6)$. □

Now that we have the functions in $M_E(S_6)$, we next find the functions in $C(M_E(S_6))$. First we must revisit Theorem 4.2 for S_6.

Theorem 5.4. *Let $c \in C(M_E(S_6))$.*

(i) *If $g = (1\,2\,3\,4)(5\,6)$, then $c(g) \in \{(1)\} \cup Ig$.*
(ii) *If $g = (1\,2\,3\,4\,5\,6)$, then $c(g) \in \{(1)\} \cup Ig \cup Y$.*

Proof. We mimic the proof of Theorem 4.2. Let $c \in C(M_E(S_6))$. For (i), let $g = (1\,2\,3\,4)(5\,6)$ and define $f_1(x) = \begin{cases} x & \text{if } x \in Ag \\ (1) & \text{otherwise} \end{cases}$, where $A = \mathrm{Aut}\,S_6$. Using Lemmas 2.3 and 3.1, we get $f_1 \in M_E(S_6)$. Thus $c(g) = c(f_1(g)) = f_1(c(g))$ and $c(g)$ is a fixed point of f_1. Therefore $c(g) \in \{(1)\} \cup Ag$. But $Ag = Ig$, and we have the result.

For condition (ii), let $g = (1\,2\,3\,4\,5\,6)$ and define the function $f_2(x) = \begin{cases} x & \text{if } x \in Ag \cup Y \\ 3x & \text{if } x \in S_6 \backslash (A_6 \cup Ag \cup Y) \\ (1) & \text{if } x \in A_6 \backslash Y \end{cases}$. As in (i), $f_2 \in M_E(S_6)$, and $c(g)$ is a fixed point of f_2. Using the same argument given in Theorem 4.2, we get $c(g) \in \{(1)\} \cup Ag \cup Y$. For $g = (1\,2\,3\,4\,5\,6)$, Ag consists of all 6-cycles and all products of a 2-cycle and a 3-cycle. By Theorem 5.3, $c(g)$ cannot be the product of a 2-cycle and a 3-cycle. Hence $c(g) \in \{(1)\} \cup Ig \cup Y$. □

The next theorem characterizes the center of $M_E(S_6)$.

Theorem 5.5. *A function $c \in C(M_E(S_6))$ if and only if c is a function described in one of the columns of the table below. Furthermore, $|C(M_E(S_6))| = 70$.*

TABLE 6. All functions $c \in C(M_E(S_6))$

$x \in S_6$	$c \in M_E(S_6)$	$c \in M_E(S_6)$
$y \in Y$	(1)	y
$(1\,2\,3\,4)$	(1), $(1\,3)(2\,4)$	$(1\,2\,3\,4)$, $(1\,4\,3\,2)$
$(1\,2\,3\,4\,5)$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$
$(1\,3\,5)(2\,4\,6)$	(1)	$(1\,3\,5)(2\,4\,6)^a$, $(1\,5\,3)(2\,6\,4)^b$, $(1)^c$
$(1\,2\,3\,4)(5\,6)$	(1)	$(1\,2\,3\,4)(5\,6)$, $(1\,4\,3\,2)(5\,6)$
$(1\,2\,3\,4\,5\,6)$	(1)	$(1\,2\,3\,4\,5\,6)^a$, $(1\,6\,5\,4\,3\,2)^b$, $(1\,4)(2\,5)(3\,6)^c$

Proof. Let $c \in C(M_E(S_6))$. Then $c \in M_E(S_6)$ and c must be one of the functions given in Table 5. By Theorem 5.4, $c((1\,2\,3\,4)(5\,6)) \neq (1\,3)(2\,4)$, and $c((1\,2\,3\,4)(5\,6)) \in \{(1), (1\,2\,3\,4)(5\,6), (1\,4\,3\,2)(5\,6)\}$. Likewise, $c(1\,2\,3\,4\,5\,6) \notin \{(1\,3\,5)(2\,4\,6), (1\,5\,3)(2\,6\,4)\}$ by Theorem 5.4. It follows that $c(1\,2\,3\,4\,5\,6) = (1)$ when $c(Y) = (1)$.

Let $a = (1\,2\,3\,4\,5\,6)$. Using Table 5, there exists a function $f \in M_E(S_6)$ such that $f(a) = (1\,3\,5)(2\,4\,6) = 2a$ and $f(Y) = (1)$. Thus if $c(Y) = (1)$, then $c((1\,3\,5)(2\,4\,6)) = c(f(a)) = f(c(a)) = f(1) = (1)$.

If $c(a) = a$, then $c((1\,3\,5)(2\,4\,6)) = c(f(a)) = f(c(a)) = f(a) = (1\,3\,5)(2\,4\,6)$. If $c(a) = -a$, then $c((1\,3\,5)(2\,4\,6)) = c(f(a)) = f(c(a)) = f(-a) = 2(-a) = (1\,5\,3)(2\,6\,4)$ by Lemma 2.1. If $c(a) = (1\,4)(2\,5)(3\,6)$, then $c((1\,3\,5)(2\,4\,6)) = c(f(a)) = f(c(a)) = f((1\,4)(2\,5)(3\,6)) = (1)$.

Now let $a = (1\,2\,3\,4)(5\,6)$. Using Table 5, there exists a function $f \in M_E(S_6)$ such that $f(a) = (1\,3)(2\,4) = 2a$. If $c(a) = a$, then $c((1\,3)(2\,4)) = c(f(a)) = f(c(a)) = f(a) = (1\,3)(2\,4)$, and $c|_Y = id$. If $c(a) = -a$, then $c((1\,3)(2\,4)) = c(f(a)) = f(c(a)) = f(-a) = 2(-a) = (1\,3)(2\,4)$ by Lemma 2.1, and $c|_Y = id$. If $c(a) = (1)$, then $c((1\,3)(2\,4)) = c(f(a)) = f(c(a)) = f(1) = (1)$, and $c(Y) = (1)$. We conclude that c is one of the functions in Table 6.

Now let $\alpha \in M_E(S_6)$ be a function described in Table 6 and let $f \in M_E(S_6)$. By Lemma 3.2, we need to show that $\alpha(f(a)) = f(\alpha(a))$ for all orbit representatives a determined by $I \cup \{\phi\}$.

If $a \in Y \cup \{(1\,2\,3\,4), (1\,2\,3\,4\,5), (1\,3\,5)(2\,4\,6)\}$, then $\alpha(f(a)) = f(\alpha(a))$ by Lemmas 3.3 and 3.4.

Let $a = (1\,2\,3\,4)(5\,6)$. Then $\alpha(a) \in \{(1), a, -a\}$. Assume $f(a) = 2a = (1\,3)(2\,4)$. By Lemma 2.1, $f(-a) = 2(-a) = (1\,3)(2\,4)$.

Assume $\alpha(a) = (1)$. Then $\alpha(Y) = (1)$ and $f(\alpha(a)) = f(1) = (1) = \alpha((1\,3)(2\,4)) = \alpha(f(a))$. Now assume $\alpha(a) = b \in \{a, -a\}$. Then $\alpha|_Y = id$ and $f(\alpha(a)) = f(b) = 2b = (1\,3)(2\,4) = \alpha((1\,3)(2\,4)) = \alpha(f(a))$. Lemma 3.3 gives $\alpha(f(a)) = f(\alpha(a))$ for all other combinations of $\alpha(a)$ and $f(a)$.

Now let $a = (1\,2\,3\,4\,5\,6)$ and $f(a) = b \in \{2a = (1\,3\,5)(2\,4\,6), 4a = (1\,5\,3)(2\,6\,4)\}$. Then $f(-a) = -b$ by Lemma 2.1 and $f((1\,4)(2\,5)(3\,6)) = (1)$ by Table 5.

If $\alpha(a) = (1)$, then $\alpha(b) = (1)$ by Table 6. So $f(\alpha(a)) = f(1) = (1) = \alpha(b) = \alpha(f(a))$. If $\alpha(a) = a$, then $\alpha(b) = b$ by Table 6. So $f(\alpha(a)) = f(a) = b = \alpha(b) = \alpha(f(a))$. If $\alpha(a) = -a$, then $\alpha(b) = -b$ by Table 6. So $f(\alpha(a)) = f(-a) = -b = \alpha(b) = \alpha(f(a))$. Finally, if $\alpha(a) = (1\,4)(2\,5)(3\,6)$, then $\alpha(b) = (1)$ by Table 6. So $f(\alpha(a)) = f((1\,4)(2\,5)(3\,6)) = (1) = \alpha(b) = \alpha(f(a))$. Lemmas 3.3 and 3.4 give $\alpha(f(a)) = f(\alpha(a))$ for all other combinations of $\alpha(a)$ and $f(a)$. Hence $\alpha \in C(M_E(S_6))$.

There are $2 \cdot 5 = 10$ possible functions represented in the first column of Table 6 and $2 \cdot 5 \cdot 3 \cdot 2 = 60$ possible functions represented by the second column. Thus there are 70 total functions in $C(M_E(S_6))$. □

The center of a nearring is not always a subnearring (see [3]). The final result characterizes when $C(M_E(S_n))$ is a subnearring of $M_E(S_n)$.

Theorem 5.6. *Let $n \geq 3$. Then $C(M_E(S_n))$ is a subnearring of $M_E(S_n)$ if and only if $n = 3$ or $n = 4$.*

Proof. In [2], it was shown that $M_E(S_3)$ and $M_E(S_4)$ are commutative rings. Thus their centers are subnearrings. For the direct implication, assume $n \geq 5$. We know $id \in C(M_E(S_n))$. Consider $id + id \in M_E(S_n)$.

For $n \neq 6$, $(id + id)((1\,2\,3)(4\,5)) = (1\,3\,2)$ and $id + id \notin C(M_E(S_n))$ by Theorem 4.2. For $n = 6$, $(id + id)(1\,2\,3\,4\,5\,6) = (1\,3\,5)(2\,4\,6)$ and $id + id \notin C(M_E(S_6))$ by Theorem 5.4. Thus for $n \geq 5$, $C(M_E(S_n))$ is not closed under addition and $C(M_E(S_n))$ is not a subnearring of $M_E(S_n)$. □

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